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APPROXIMATE SOLUTION OF THE ONE-DIMENSIONAL HEAT CONDUCTION EQUATION BY THE BOUNDARY ELEMENT METHOD

1. Introduction. The aim of the paper is to find approximate solution of the heat conduction equation

(1) $$u_t = u_{xx}$$

in the plane domain $D_T \subset R^2_{(x,t)}$ defined by the inequalities

$$\gamma_1(t) < x < \gamma_2(t) \quad \text{for } 0 < t < T.$$

We are looking for a solution u of (1) satisfying the boundary conditions

(2) $$u(\gamma_j(t), t) = f_j(t) \quad \text{for } 0 \leqslant t \leqslant T \quad (j = 1, 2)$$

and the initial condition

(3) $$u(x, 0) = 0 \quad \text{for } \gamma_1(0) < x < \gamma_2(0).$$

Representing the function u as a sum of two heat potentials of the second kind, we are led to the system (31) of two integral equations of Volterra type, which has to be satisfied by the unknown densities. The boundary element method now consists in an approximate solution of this system by the Galerkin method in the space of piecewise constant functions.

After proving the unique solvability of the system (31) in suitably chosen Nikolskiĭ spaces we obtain estimates of the error in the $L^\infty(0, T)$-norm.

Our method was influenced by the paper of Graham [3], where similar questions for the integral equation of Fredholm type were considered.

2. Basic notation and assumptions. Throughout this paper we assume that the following conditions hold:

(4) $$\gamma_j \in C^3[0, T] \ (j = 1, 2), \quad \gamma_1(t) < \gamma_2(t) \text{ for } t \in [0, T],$$
$$f_j \in C[0, T] \quad \text{and} \quad f_j(0) = 0 \quad (j = 1, 2).$$

We put

$$\Delta_\delta f(t) = f(t+\delta) - f(t) \quad \text{and} \quad [0, T]_\varepsilon = \{t \in [0, T]:\ t+\varepsilon \in [0, T]\}.$$

For any $n \in N$, let Π_n denote the partition of $[0, T]$ given by

$$\Pi_n\colon\ 0 = t_0 < t_1 < \ldots < t_n = T.$$

Let now $S(\Pi_n)$ be a finite-dimensional space of functions which are constant on each interval (t_{i-1}, t_i) for $i = 1, \ldots, n$ and, to ensure that they are well defined, we assume left continuity at each knot and right continuity at 0.

By P_n we denote the orthogonal projection of $L^2(0, T)$ onto $S(\Pi_n)$ and, when it does not lead to misunderstandings, the matrix operator

$$\begin{bmatrix} P_n & 0 \\ 0 & P_n \end{bmatrix}.$$

In the sequel, as n varies, we assume that the partitions Π_n remain quasiuniform, i.e., there exists a constant c with the property

$$\frac{\max\limits_{i=1,\ldots,n} (t_i - t_{i-1})}{\min\limits_{j=1,\ldots,n} (t_j - t_{j-1})} \leqslant c \tag{5}$$

for each partition Π_n.

We note that condition (5) implies that $h \to 0$ as $n \to \infty$, where $h = \max\limits_i (t_i - t_{i-1})$.

For $x = (x_1, x_2)$ with $x_1, x_2 \in L^\infty(0, T)$ we write

$$\|x\|_\infty = \max(\|x_1\|_\infty, \|x_2\|_\infty).$$

We denote by $[X]^2$ the Cartesian product of the function space X with itself, for instance

$$[C[0, T]]^2 = C[0, T] \times C[0, T].$$

We need also the function

$$V(x, t, z, s) = \frac{x - z}{(t - s)^{3/2}} \exp\left[-\frac{(x - z)^2}{4(t - s)}\right],$$

defined for any $x, z \in R$ and $t > s$.

All derivatives are understood in the weak distributional sense.

Throughout this paper, c always denotes a constant. We permit it to change its value from paragraph to paragraph.

3. Some auxiliary definitions and theorems. We consider now the function spaces we shall need in the sequel.

1° The Sobolev space $W_1^1(0, T)$ is a space with the norm

$$\|u\|_{1,1} = \sum_{|\alpha| \leqslant 1} \|D^\alpha u\|_1.$$

It is known (see [1], Lemma 5.8) that the following theorem holds true:

THEOREM A. *We have*

$$W_1^1(0, T) \subset C[0, T] \tag{6}$$

with continuous imbedding.

2° The Nikolskiĭ space $N_p^\alpha[0, T]$ with noninteger $\alpha > 0$, $1 \leqslant p \leqslant \infty$, is the space of all functions $\varphi \in L^p(0, T)$ satisfying the condition

$$|\varphi|_{\alpha,p} = \sup_{\delta \neq 0} |\delta|^{-\alpha_0} \|\Delta_\delta D^{[\alpha]} \varphi\|_{L^p(0,T)_\delta} < \infty,$$

where $[\alpha]$ is the integer part of α and $\alpha_0 = \alpha - [\alpha]$.

It is known (see [7]) that $N_p^\alpha[0, T]$ equipped with the norm

$$\|\varphi\|_{\alpha,p} = \|\varphi\|_p + |\varphi|_{\alpha,p}$$

is a Banach space. Moreover, the following known results hold (see [7]):

THEOREM B. *We have*

$$N_p^{1+\alpha_2}[0, T] \subset N_p^{1+\alpha_1}[0, T] \subset W_1^1(0, T) \tag{7}$$

for $0 < \alpha_1 < \alpha_2 < 1$, $1 \leqslant p \leqslant \infty$, *with continuous imbedding.*

THEOREM C. *We have*

$$N_p^\alpha[0, T] \subset N_q^\beta[0, T] \tag{8}$$

for $\alpha > 0$, $1 \leqslant p \leqslant q \leqslant \infty$ *and* $\beta = \alpha - (1/p - 1/q) > 0$ *with continuous imbedding.*

Remark 1. From [6] it follows that all imbeddings in Theorem B are compact.

We need the following proposition:

PROPOSITION 1. *Let* $m \in C^1[-M, M]$. *Then*

$$\int_{[-M,M]_\delta} \left|\Delta_\delta \frac{m(s)}{\sqrt{|s|}}\right| ds \leqslant c\left(\max_t |m(t)| + \max_t |m'(t)|\right)\sqrt{|\delta|}, \qquad t \in [-M, M]. \tag{9}$$

Proof. Applying the mean value theorem we have

$$\frac{m(t)}{\sqrt{|t|}} = m(t+\delta)\left(\frac{1}{\sqrt{|t+\delta|}} - \frac{1}{\sqrt{|t|}}\right) + \delta m'(t+\theta\delta)\frac{1}{\sqrt{|t|}}$$

with some $\theta \in (0, 1)$. As both functions m and m' are bounded, the calculation of the integrals yields our assertion.

Now we return to the operator P_n. We prove that the operator norm of P_n, considered as an operator on $L^\infty(0, T)$, is uniformly bounded, namely:

PROPOSITION 2. *We have*

$$\|P_n\| \leqslant 1. \tag{10}$$

Proof. For $\varphi \in L^\infty(0, T)$ let us set $P_n \varphi|_{(t_{i-1}, t_i)} = c_i$. Then

$$c_i = \frac{1}{t_i - t_{i-1}} \int_{t_{i-1}}^{t_i} \varphi(s)\, ds, \quad i = 1, \ldots, n,$$

and hence

$$\mathop{\forall}_{n \in N} \|P_n \varphi\|_\infty \leqslant 1 \quad \text{for } \|\varphi\|_\infty \leqslant 1.$$

Remark 2. Note that in the proof of Proposition 2 we do not use the assumption (5).

Let us define

$$L\varphi(t) = \int_0^t \frac{m(t, s)}{\sqrt{t-s}} \varphi(s)\, ds = \int_0^t \frac{m(t, t-s)}{\sqrt{s}} \varphi(t-s)\, ds.$$

We collect the properties of the operator L in the following propositions:

PROPOSITION 3. *Let* $m \in C([0, T] \times [0, T])$. *Then the operator*

(i) L: $L^\infty(0, T) \to C[0, T]$ *is compact,*

(ii) L: $L^1(0, T) \to L^1(0, T)$ *is bounded.*

Proof. (i) Assuming that $\varphi \in L^\infty(0, T)$, we have for $0 \leqslant t$, $t + \delta \leqslant T$

$$\begin{aligned} \Delta_\delta L\varphi(t) = \int_0^t \left(\frac{1}{\sqrt{t+\delta-s}} - \frac{1}{\sqrt{t-s}} \right) m(t+\delta, s)\, \varphi(s)\, ds + \\ + \int_0^t \frac{m(t+\delta, s) - m(t, s)}{\sqrt{t-s}} \varphi(s)\, ds + \int_t^{t+\delta} \frac{m(t+\delta, s)}{\sqrt{t+\delta-s}} \varphi(s)\, ds \end{aligned} \tag{11}$$

and

$$|L\varphi(t)| \leqslant \int_0^t \frac{|m(t, s)|}{\sqrt{t-s}} ds\, \|\varphi\|_\infty. \tag{12}$$

From (11), (12) and the continuity of m it follows that the set $A = \{L\varphi \colon \|\varphi\|_\infty \leqslant 1\}$ is bounded in $C[0, T]$ and the functions belonging to A are equicontinuous on $[0, T]$. By the Ascoli–Arzeli theorem, the set A is precompact in $C[0, T]$.

(ii) For $\varphi \in L^1(0, T)$ we have

$$\|L\varphi\|_1 \leqslant \int_0^T \int_0^t \frac{|m(t, s)|}{\sqrt{t-s}} |\varphi(s)|\, ds\, dt;$$

therefore, changing the order of integration, we obtain

$$\|L\varphi\|_1 \leqslant \int_0^T |\varphi(s)| \int_s^T \frac{|m(t, s)|}{\sqrt{t-s}} dtds.$$

As m is bounded, the last inequality yields $\|L\varphi\|_1 \leqslant c\|\varphi\|_1$ with c not depending on φ.

PROPOSITION 4. *Let* $m \in C^2([0, T] \times [0, T])$. *Then the operator*

$$L: W_1^1(0, T) \to N_1^{3/2}[0, T]$$

is bounded.

Proof. We start with calculating the derivative of the function $L\varphi$ in the interval $(0, T]$. Assuming first that φ is smooth and putting $\psi_\tau(t) = m(t, t-\tau)\varphi(t-\tau)$, for $0 < t \leqslant T$ we have

$$L\varphi(t+\delta) - L\varphi(t) = \int_0^t \frac{1}{\sqrt{\tau}}(\psi_\tau(t+\delta) - \psi_\tau(t))d\tau + \int_t^{t+\delta} \frac{\psi_\tau(t+\delta)}{\sqrt{\tau}} d\tau, \tag{13}$$

and therefore

$$\frac{1}{\delta}(L\varphi(t+\delta) - L\varphi(t)) = \int_0^t \frac{1}{\sqrt{\tau}} \psi_\tau'(t+\theta\delta)d\tau + \frac{1}{\delta} \int_t^{t+\delta} \frac{\psi_\tau(t+\delta)}{\sqrt{\tau}} d\tau.$$

In the first integral on the right-hand side we may pass to the limit with $\delta \to 0$ according to the Lebesgue theorem and the second integral may be estimated as

$$\bar{m}(t, \delta) \int_t^{t+\delta} \frac{d\tau}{\sqrt{\tau}} \leqslant \int_t^{t+\delta} \frac{\psi_\tau(t+\delta)}{\sqrt{\tau}} d\tau \leqslant M(t, \delta) \int_t^{t+\delta} \frac{d\tau}{\sqrt{\tau}},$$

where $\bar{m}(t, \delta)$ and $M(t, \delta)$ denote $\inf_\tau \psi_\tau$ and $\sup_\tau \psi_\tau$, respectively, for τ in $[t, t+\delta]$. Hence $L\varphi(t)$ is differentiable at any point $t \in (0, T]$ and

$$(L\varphi)'(t) = \int_0^t \frac{\psi_\tau'(t)}{\sqrt{\tau}} d\tau + \frac{m(t, 0)}{\sqrt{t}} \varphi(0). \tag{14}$$

From (13) it follows that

$$|L\varphi(t+\delta) - L\varphi(t)| \leqslant c\left(2\delta\sqrt{t} + \int_t^{t+\delta} \frac{d\tau}{\sqrt{\tau}}\right),$$

where c is the upper bound for ψ_τ and ψ_τ'. Hence the function $L\varphi$ is absolutely continuous on $[0, T]$, and therefore the derivative on the left-hand side of (14) may be understood in the distributional sense.

Let now $\varphi \in W_1^1(0, T)$ be an arbitrary function and let $\varphi_n \to \varphi$ in $W_1^1(0, T)$, where φ_n are smooth. Then $L\varphi_n \to L\varphi$ in $L^1(0, T)$ according to part (ii) of Proposition 3. Thus

$$L\varphi_n \to L\varphi \quad \text{in } \mathscr{D}'(0, T). \tag{15}$$

Formula (14) is valid with φ replaced by φ_n. Using the imbedding (6) and applying once more part (ii) of Proposition 3 to the integral operator on the right-hand side of (14) we state in view of (15) that (14) remains valid for $\varphi \in W_1^1(0, T)$ as well.

To estimate $|L\varphi|_{3/2,1}$ let us consider $\Delta_\delta(L\varphi)'$. We assume $\delta > 0$; the case $\delta < 0$ may be treated similarly. Denoting by m_j $(j = 1, 2)$ the partial derivative of m with respect to the j-th argument we have

$$\psi_\tau'(t) = \sum_{j=1}^{2} m_j(t, t-\tau)\varphi(t-\tau) + m(t, t-\tau)\varphi'(t-\tau).$$

Therefore, after the substitution of $\tau = s+\delta$ in the integral occurring in $(L\varphi)'(t+\delta)$, we obtain

$$\Delta_\delta(L\varphi)'(t) = \sum_{j=0}^{2} J_j(t, \delta) + \sum_{j=0}^{2} K_j(t, \delta) + R(t, \delta), \tag{16}$$

where

$$J_0(t, \delta) = \int_0^t j(t, s, \delta)\varphi'(t-s)\,ds,$$

with

$$j(t, s, \delta) = \frac{m(t+\delta, t-s)}{\sqrt{s+\delta}} - \frac{m(t, t-s)}{\sqrt{s}},$$

$$K_0(t, \delta) = \int_{-\delta}^{0} \frac{m(t+\delta, t-s)}{\sqrt{s+\delta}}\varphi'(t-s)\,ds,$$

$$R(t, \delta) = \varphi(0)\Delta_\delta \frac{m(t, 0)}{\sqrt{t}}$$

and for $j = 1, 2$ we have

$$J_j(t, \delta) = \int_0^t \left(\frac{m_j(t+\delta, t-s)}{\sqrt{s+\delta}} - \frac{m_j(t, t-s)}{\sqrt{s}}\right)\varphi(t-s)\,ds,$$

$$K_j(t, \delta) = \int_{-\delta}^{0} \frac{m_j(t+\delta, t-s)}{\sqrt{s+\delta}}\varphi(t-s)\,ds.$$

As

$$j(t, s, \delta) = \left(\frac{1}{\sqrt{s+\delta}} - \frac{1}{\sqrt{s}}\right) m(t+\delta, t-s) +$$

$$+ \frac{1}{\sqrt{s}}(m(t+\delta, t-s) - m(t, t-s)),$$

after changing the order of integration we have

$$\int_0^{T-\delta} |J_0(t, \delta)|\, dt \leqslant \bar{J}_0(\delta) + \bar{\bar{J}}_0(\delta),$$

where

$$\bar{J}_0(\delta) = \int_0^{T-\delta} \left|\frac{1}{\sqrt{s+\delta}} - \frac{1}{\sqrt{s}}\right| \int_s^{T-\delta} |m(t+\delta, t-s)|\, |\varphi'(t-s)|\, dtds, \tag{17}$$

$$\bar{\bar{J}}_0(\delta) = \int_0^{T-\delta} \frac{1}{\sqrt{s}} \int_s^{T-\delta} |m(t+\delta, t-s) - m(t, t-s)|\, |\varphi'(t-s)|\, dtds. \tag{18}$$

The integral with respect to t in (17) does not exceed const $\|\varphi\|_{1,1}$, and calculating the integral with respect to s, we obtain for $\delta \leqslant T$ the following inequality:

$$|\bar{J}_0(\delta)| \leqslant \text{const} \sqrt{\delta}\, \|\varphi\|_{1,1}.$$

Similarly, applying the mean value theorem to the difference on the right-hand side of (18), we obtain

$$|\bar{\bar{J}}_0(\delta)| \leqslant \text{const}\, \delta\, \|\varphi\|_{1,1},$$

and so

$$\int_0^{T-\delta} |J_j(t, \delta)|\, dt \leqslant \text{const} \sqrt{\delta}\, \|\varphi\|_{1,1} \tag{19}$$

for $j = 0$. Using quite the same arguments as above we may prove that (19) is valid for $j = 1, 2$ as well.

Considering now the integral K_0 we have

$$\int_0^{T-\delta} |K_0(t, \delta)|\, dt = \int_{-\delta}^{0} \frac{1}{\sqrt{s+\delta}} \int_0^{T-\delta} |m(t+\delta, t-s)|\, |\varphi'(t-s)|\, dtds.$$

As the integral with respect to t may be estimated by const $\|\varphi\|_{1,1}$, after evaluating the integral with respect to s, we obtain

$$\int_0^{T-\delta} |K_j(t, \delta)|\, dt \leqslant \text{const} \sqrt{\delta}\, \|\varphi\|_{1,1} \tag{20}$$

for $j = 0$ and in a similar way for $j = 1, 2$.

It remains to estimate the last member on the right-hand side of (16). From (9) we get

$$\int_0^{T-\delta} \left|\Delta_\delta \frac{m(t,0)}{\sqrt{t}}\right| dt \leqslant \text{const}\sqrt{\delta}$$

and therefore, in view of (6), we obtain

$$\int_0^{T-\delta} |R(t,\delta)|\, dt \leqslant \text{const}\sqrt{\delta}\,\|\varphi\|_{1,1}. \tag{21}$$

The estimates (19)–(21) complete the proof.

PROPOSITION 5. (i) *Let* $m \in C^2([0, T]\times[0, T])$. *Then for each* $t \in [0, T]$ *and* $n \in N$ *there exists a function* $u^t \in S(\Pi_n)$ *such that*

$$\int_0^t \left|\frac{m(t,s)}{\sqrt{t-s}} - u^t(s)\right| ds \leqslant ch^{1/2},$$

where c *is independent of* t *and* n.

(ii) *Let* $\varphi \in N^\eta_\gamma[0, T] \cap C[0, T]$ *for some* η $(0<\eta<1)$. *Then there exists a function* $v \in S(\Pi_n)$ *such that* $\|\varphi - v\|_\infty \leqslant ch^\eta$ *and* c *is independent of* n.

Proof. Munteanu and Schumaker obtained in [5], Lemma 5.5, estimates of the error of spline approximations to functions defined on a rectangle $[a, b]\times[c, d]$. We can apply their result in the case of functions of one variable. Namely, we can prove the following

LEMMA. *Let* $f \in L^p(0, T)$, $1 \leqslant p < \infty$, *or* $f \in C[0, T]$, $p=\infty$. *Then*

$$\inf_{g\in S(\Pi_n)} \|f-g\|_p \leqslant c\big(h\|f\|_p + \omega_1(h, f, p)\big), \tag{22}$$

where

$$\omega_1(h, f, p) = \sup_{|\delta|\leqslant h} \|\Delta_\delta f\|_{L^p(0,T)_\delta}$$

and c *is independent of* n.

Proof of the Lemma. We consider first the extension of the function $f \in L^p(0, T)$, $1 \leqslant p < \infty$, on $H = [0, T]\times[0, T]$ defined by $\bar{f}(x, y) = f(x)$ for $x, y \in [0, T]$. By Lemma 5.5 from [5] we have

$$\inf_{g\in S^2(\Pi_n)} \|\bar{f}-g\|_p \leqslant c\big(h\|\bar{f}\|_p + \omega_1(h, \bar{f}, p)\big), \tag{23}$$

where

$$S^2(\Pi_n) = \{s_1(x)s_2(y):\ s_1, s_2 \in S(\Pi_n)\}$$

and

$$\omega_1(h, \bar{f}, p) = \sup_{|h_1|,|h_2| \leqslant h} \|\bar{f}(x+h_1, y+h_2) - \bar{f}(x, y)\|_{L^p(H_{h_1,h_2})}$$

with $H_{h_1,h_2} = \{(x, y) \in H\colon (x+h_1, y+h_2) \in H\}$.

We note that

$$\|\bar{f}(x+h_1, y+h_2) - \bar{f}(x, y)\|^p_{L^p(H_{h_1,h_2})} \leqslant T\|f(x+h_1) - f(x)\|^p_{L^p(0,T)_{h_1}}$$

and hence we obtain

(24)
$$\omega_1(h, \bar{f}, p) \leqslant T^{1/p}\omega_1(h, f, p).$$

Now, for any $g(x, y) = s_1(x)s_2(y)$ with $s_1, s_2 \in S(\Pi_n)$ we have

(25)
$$\|\bar{f} - g\|^p_p = \int_0^T \int_0^T |\bar{f}(x, y) - s_1(x)s_2(y)|^p\,dx\,dy$$
$$= \sum_{i=1}^n \int_0^T |f(x) - s_1(x)c_i|^p\,dx\,(y_i - y_{i-1}),$$

where $c_i = s_2|_{(y_{i-1},y_i)}$ $(i = 1, \ldots, n)$ are constants. Thus from (25) we obtain

$$\|f - g\|_p \geqslant T^{1/p} \min_{1 \leqslant i \leqslant n} \|f - s_1 c_i\|_p.$$

Therefore we have

(26)
$$\inf_{g \in S^2(\Pi_n)} \|f - g\|_p \geqslant c \inf_{g \in S(\Pi_n)} \|f - g\|_p.$$

Since $\|\bar{f}\|_{L^p(H)} = T^{1/p}\|f\|_{L^p(0,T)}$, our assertion follows from (23), (24), and (26).

We deal similarly in the case $f \in C[0, T]$, $p = \infty$.

Now we may pass to the proof of part (i) of Proposition 5. Note that (22) implies that there exists a $g \in S(\Pi_n)$ such that

(27)
$$\|f - g\|_p \leqslant 2c(h\|f\|_p + \omega_1(h, f, p)),$$

where c is as in (22).

For fixed $t \in [0, T]$ for the function

$$f_t(s) = \frac{m(t, s)}{\sqrt{|t-s|}}$$

by Proposition 1 we have

(28)
$$\omega_1(h, f_t, 1) \leqslant ch^{1/2},$$

where c is independent of t.

As m is bounded, $\|f_t\|_1 \leqslant c$ is valid uniformly in t and our assertion follows from (22), (27), and (28).

To obtain (ii) we note that $\omega_1(h, \varphi, \eta) \leqslant ch^\eta$ and our assertion follows from (27).

PROPOSITION 6. *Let* $m \in C^2([0, T] \times [0, T])$ *and* $\varphi \in C[0, T]$. *Then*

$$\|(L - LP_n)\varphi\|_\infty \leqslant ch^{1/2}\|\varphi - P$$

where c *is independent of* n *and* φ.

Proof. For any $\varphi \in C[0, T]$ and $t \in [t_{i-1}, t_i]$ we have

$$\int_0^t \frac{m(t, s)}{\sqrt{t-s}}(\varphi(s) - P_n\varphi(s))ds = \Big(\int_0^{t_{i-1}} + \int_{t_{i-1}}^t\Big)\frac{m(t, s)}{\sqrt{t-s}}(\varphi(s) - P_n\varphi(s))ds.$$

By Proposition 5 and by the L^2-orthogonality of P_n we estimate the first integral as follows:

$$\left|\int_0^{t_{i-1}} \frac{m(t, s)}{\sqrt{t-s}}(\varphi(s) - P_n\varphi(s))ds\right| \tag{29}$$

$$= \left|\int_0^{t_{i-1}} \left(\frac{m(t, s)}{\sqrt{t-s}} - \bar{u}^t(s)\right)(\varphi(s) - P_n\varphi(s))ds\right|$$

$$\leqslant ch^{1/2}\|\varphi - P_n\varphi\|_\infty,$$

where $\bar{u}^t(s) = u^t(s)$ for $s \in [0, t_{i-1}]$, and $\bar{u}^t(s) = 0$ for $s \in (t_{i-1}, T]$.

As m is bounded, calculating the integral we have

$$\left|\int_{t_{i-1}}^t \frac{m(t, s)}{\sqrt{t-s}}(\varphi(s) - P_n\varphi(s))ds\right| \leqslant ch^{1/2}\|\varphi - P_n\varphi\|_\infty. \tag{30}$$

Now the required result follows from (29) and (30).

An elementary calculation shows that the following proposition holds:

PROPOSITION 7. *Let* $\gamma \in C^3[0, T]$. *Then the function* φ *defined by*

$$\varphi(t, s) = \frac{\gamma(t) - \gamma(s)}{t - s} \quad \text{for } t \neq s \quad \text{and} \quad \varphi(t, t) = \gamma'(t)$$

belongs to $C^2([0, T] \times [0, T])$.

In our considerations we apply the collectively compact approximation theory of Anselone [2].

Let $[\mathscr{X}]$ denote the Banach space of bounded linear operators $T: \mathscr{X} \to \mathscr{X}$, where $\mathscr{X}$ is a Banach space. Then a set $\mathscr{K} \subset [\mathscr{X}]$ is *collectively compact* provided that the set $\mathscr{K}\mathscr{B} = \{Kx: K \in \mathscr{K}, \|x\|_{\mathscr{X}} \leqslant 1\}$ is relatively compact.

We need the following theorem ([2], Theorem 1.6):

THEOREM D. *Let* $K, K_n \in [\mathscr{X}]$, $n = 1, \ldots$ *Assume that* $K_n \to K$ *pointwise,* K_n *is collectively compact, and* K *is compact. Then* $(I - K)^{-1}$ *exists if and only* *if for some N and all $n > N$ the operators $(I-K_n)^{-1}$ exist and are uniformly bounded, in which case $(I-K_n)^{-1} \to (I-K)^{-1}$ pointwise.*

4. The exact problem. We seek a solution of the problem (1)–(3) in the form of a sum of two heat potentials (see [4]):

$$u(x, t) = \frac{1}{2\pi}\Big(\int_0^t V(x, t, \gamma_1(s), s) y_1(s)\,ds + \int_0^t V(x, t, \gamma_2(s), s) y_2(s)\,ds\Big) \quad \text{for } (x, t) \in D_T.$$

Now, applying the properties of the heat potentials (see [4]) and using (2) we obtain for the unknown densities y_1 and y_2 the system of Volterra type integral equations

$$\text{(31)} \qquad \begin{aligned} y_1(t) - \Big(\int_0^t K_{11}(t, s) y_1(s)\,ds + \int_0^t K_{12}(t, s) y_2(s)\,ds\Big) &= f_1(t), \\ y_2(t) - \Big(\int_0^t K_{21}(t, s) y_1(s)\,ds + \int_0^t K_{22}(t, s) y_2(s)\,ds\Big) &= -f_2(t), \end{aligned}$$

where

$$K_{ij}(t, s) = \frac{1}{\sqrt{t-s}} m_{ij}(t, s), \qquad s < t \ (i, j = 1, 2),$$

with

$$m_{ij}(t, s) = \frac{1}{2\pi}(-1)^i \frac{\gamma_i(t) - \gamma_j(s)}{t-s} \exp\Big[-\frac{(\gamma_i(t) - \gamma_j(s))^2}{4(t-s)}\Big]$$

for $s < t$ $(i, j = 1, 2)$.

We abbreviate (31), using the standard operator notation, to the form

$$\text{(32)} \qquad (I-K)y = f,$$

where $K = [K_{ij}]$ $(i, j = 1, 2)$ with

$$(K_{ij}\varphi)(t) = \int_0^t K_{ij}(t, s)\varphi(s)\,ds \qquad (i, j = 1, 2).$$

Assumptions (4) yield the following

PROPOSITION 8. *There exists an extension of m_{ij} on $[0, T] \times [0, T]$ such that $m_{ij} \in C^2([0, T] \times [0, T])$ $(i, j = 1, 2)$.*

Proof. In the case $i \neq j$ we put $m_{ij}(t, s) = 0$ for $t \leqslant s$. Then there are no singularities and m_{ij} belongs even to $C^3[0, T]$.

In the case $i = j$ our assertion follows from Proposition 7. This completes the proof.

We deal furthermore with the system (31). When the functions f_1 and f_2 belong to a suitable Nikolskiĭ space we have the following

THEOREM 1. *Let $f_1, f_2 \in N_1^{1+\beta}[0, T]$ for some β $(0 < \beta < 1)$. Then there exists a unique solution $y_1, y_2 \in N_1^{1+\gamma}[0, T]$ of* (31), *where $\gamma = \min\{1/2, \beta\}$.*

Proof. It is a well-known fact (see [8]) that for any $f_1, f_2 \in C[0, T]$ the system (31) has a unique solution $y = (y_1, y_2)$ with $y_1, y_2 \in C[0, T]$. Hence and by (6) the homogeneous equation $y = Ky$ has no nontrivial solutions in $[W_1^1(0, T)]^2$.

From Proposition 4 it follows that the operator

$$K\colon [W_1^1(0, T)]^2 \to [N_1^{3/2}[0, T]]^2 \tag{33}$$

is bounded and by (7) we have $[N_1^{3/2}[0, T]]^2 \subset [W_1^1(0, T)]^2$ with compact inclusion. Therefore, the operator $K\colon [W_1^1(0, T)]^2 \to [W_1^1(0, T)]^2$ is compact. Now from the Fredholm Alternative it follows that for any functions f_1, $f_2 \in N_1^{1+\beta}[0, T] \subset W_1^1(0, T)$ there uniquely exist $y_1, y_2 \in W_1^1(0, T)$ such that equation (31) is satisfied. Since $y = f + Ky$, the result follows by (33) and (7).

5. Approximate problem. We are going to approximate the solution of (31). We formulate the approximate problem as follows:

(34) Find $y_n = (y_{1n}, y_{2n})$ with $y_{1n}, y_{2n} \in S(\Pi_n)$ satisfying the equation

$$y_n = P_n f + P_n K y_n. \tag{35}$$

We consider also the iterated Galerkin solution $\bar{y}_n$ defined by

$$\bar{y}_n = f + K y_n.$$

It is easy to see that y_n satisfies the equations

$$y_n = P_n \bar{y}_n \tag{36}$$

and

$$\bar{y}_n = f + K P_n \bar{y}_n. \tag{37}$$

We now pass to the error estimation of the approximations. The first step in proving the required estimates is given in the following

THEOREM 2. *Let $f_1, f_2 \in C[0, T]$. Then for sufficiently large n we have*

(i) *the problem* (34) *has a unique solution $y_n \in [L^\infty(0, T)]^2$ and*

$$c_1 \|y - P_n y\|_\infty \leqslant \|y - y_n\|_\infty \leqslant c_2 \|y - P_n y\|_\infty, \tag{38}$$

(ii) *the problem* (37) *has a unique solution $\bar{y}_n \in [C[0, T]]^2$ and*

$$\|y - \bar{y}_n\|_\infty \leqslant c_3 \|Ky - KP_n y\|_\infty, \tag{39}$$

where c_1, c_2 and c_3 are independent of n.

Proof. We first consider $\bar{y}_n$. From Proposition 3 it follows immediately that the operator $K: [L^\infty(0, T)]^2 \to [C[0, T]]^2$ is compact, and so the operator KP_n is compact on $[C[0, T]]^2$. From the estimate given in Proposition 6 we infer that $KP_n \to K$ pointwise on $[C[0, T]]^2$. By Proposition 2 it is straightforward to show that the set

$$\{KP_n \varphi: n \in N, \varphi \in [C[0, T]]^2, \|\varphi\|_\infty \leq 1\}$$

has a compact closure in $[C[0, T]]^2$, and hence $\{KP_n: n \in N\}$ is a collectively compact set of operators on $[C[0, T]]^2$. Since $(I-K)^{-1}$ exists on $[C[0, T]]^2$ (see [8]), it follows from Theorem D that $(I-KP_n)^{-1}$ also exists on $[C[0, T]]^2$ for sufficiently large n and

$$\|(I-KP_n)^{-1}\| \leq c, \quad n \in N, \tag{40}$$

where c is independent of n.

Since $y_n = (I-KP_n)^{-1} f$, y_n exists for sufficiently large n, and by (37) and (32) we have

$$y - y_n = (I-KP_n)^{-1}(K-KP_n) y. \tag{41}$$

Hence by (40) we obtain inequality (39).

Now, returning to y_n, by (41) and (36) we may write

$$y - y_n = (y - P_n y) + P_n (I-KP_n)^{-1}(K-KP_n) y,$$

whence

$$\|y - y_n\|_\infty \leq (1 + \|P_n\| \cdot \|(I-KP_n)^{-1}\| \cdot \|K\|) \|y - P_n y\|_\infty. \tag{42}$$

The estimates (42) and (40) yield now the right-hand side of (38).

To obtain the left-hand side of (38) we note, in view of (32) and (35), that

$$(I - P_n K)(y - y_n) = y - P_n y.$$

Hence by (10) we have

$$\|y - P_n y\|_\infty \leq (1 + \|K\|) \|y - y_n\|_\infty.$$

This completes the proof of Theorem 2.

THEOREM 3. *Let $f_1, f_2 \in N_1^{1+\beta}[0, T]$ for some β $(0 < \beta < 1)$. Then*

$$\|y - y_n\|_\infty \leq c_1 h^\gamma \quad \textit{and} \quad \|y - y_n\| \leq c_2 h^{1/2+\gamma},$$

where $\gamma = \min\{1/2, \beta\}$, and c_1 and c_2 are independent of n.

Proof. From Theorem 1 and (8) it follows that

$$y \in [N_1^{1+\gamma}[0, T]]^2 \subset [N_\infty^\gamma[0, T]]^2.$$

Since, in view of (6) and (7), $y \in [C[0, T]]^2$, we have

$$\|y - P_n y\|_\infty \leq \|(I - P_n)(y - \xi_n)\|_\infty \leq 2\|y - \xi_n\|_\infty \leq ch^\gamma,$$

where ξ_n is, in view of Proposition 5 (ii), an element of $[S(\Pi_n)]^2$ suitably chosen for y. Now the required result follows from Theorem 2 and Proposition 6.

We can estimate now the error of approximation of a solution of the problem (1)–(3). We define u_n and $\bar{u}_n$ by

$$u_n(x, t) = \frac{1}{2\pi}\left(\int_0^t V(x, t, \gamma_1(s), s)\, y_{1n}(s)\, ds + \int_0^t V(x, t, \gamma_2(s), s)\, y_{2n}(s)\, ds\right)$$

and

$$\bar{u}_n(x, t) = \frac{1}{2\pi}\left(\int_0^t V(x, t, \gamma_1(s), s)\, \bar{y}_{1n}(s)\, ds + \int_0^t V(x, t, \gamma_2(s), s)\, \bar{y}_{2n}(s)\, ds\right) \quad \text{for } (x, t) \in D_T.$$

To estimate $\sup_{D_T} |u - u_n|$ and $\sup_{D_T} |u - \bar{u}_n|$ note that we may represent V as the following sum:

$$V(x, t, \gamma_i(s), s) = V_{1i}(x, t, s) + \frac{1}{\sqrt{t-s}} V_{2i}(x, t, s) \qquad (i = 1, 2),$$

where

$$V_{1i}(x, t, s) = \frac{x - \gamma_i(t)}{(t-s)^{3/2}} m_i(x, t, s) \exp\left[-\frac{(x - \gamma_i(t))^2}{4(t-s)}\right]$$

and

$$V_{2i}(x, t, s) = \frac{\gamma_i(t) - \gamma_i(s)}{t-s} m_i(x, t, s) \exp\left[-\frac{(x - \gamma_i(t))^2}{4(t-s)}\right]$$

with

$$m_i(x, t, s) = \exp\left[-\frac{(x - \gamma_i(t))(\gamma_i(t) - \gamma_i(s))}{2(t-s)}\right] \exp\left[-\frac{(\gamma_i(t) - \gamma_i(s))^2}{4(t-s)}\right] \qquad (i = 1, 2).$$

As m_i and V_{2i} $(i = 1, 2)$ by Proposition 7 are uniformly bounded for $(x, t) \in D_T$, $t > s > 0$, we have

$$\int_0^t V(x, t, \gamma_i(s), s)\,ds \leqslant \int_0^t |V_{1i}|\,ds + \int_0^t \frac{1}{\sqrt{t-s}}|V_{2i}|\,ds$$

$$\leqslant c\left(\int_0^t \frac{x-\gamma_i(t)}{(t-s)^{3/2}} \exp\left[-\frac{(x-\gamma_i(t))^2}{4(t-s)}\right] ds + \int_0^t \frac{ds}{\sqrt{t-s}}\right) \quad (i = 1, 2),$$

where c is independent of $(x, t) \in D_T$.

Substituting

$$v = \frac{x-\gamma_i(t)}{2\sqrt{t-s}}$$

in the first integral on the right-hand side and integrating, we note that the expression is bounded by a constant not depending on $(x, t) \in D_T$. Adding the second integral we obtain

$$\sup_{(x,t)} \int_0^t \left|V(x, t, \gamma_i(s), s)\right| ds < \infty \qquad (i = 1, 2),$$

and hence

(43)
$$\sup_{(x,t)} |u - u_n| \leqslant c_1 \|y - y_n\|_\infty, \qquad (x, t) \in D_T,$$

and

(44)
$$\sup_{(x,t)} |u - \bar{u}_n| \leqslant c_2 \|y - \bar{y}_n\|_\infty, \qquad (x, t) \in D_T,$$

where c_1 and c_2 are independent of n.

We have thus

THEOREM 4. *Let $f_1, f_2 \in N_1^{1+\beta}[0, T]$ for some β $(0 < \beta < 1)$. Then*

$$\sup |u - u_n| = O(h^\gamma) \quad \textit{and} \quad \sup |u - \bar{u}_n| = O(h^{1/2+\gamma}), \qquad (x, t) \in D_T,$$

where $\gamma = \min\{1/2, \beta\}$.

Proof. The required result follows from Theorem 3 and the estimates (43) and (44).

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