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INTERVAL EDGE-COLORING OF CATERPILLARS WITH HAIRS OF ARBITRARY LENGTH

1. INTRODUCTION

The interval edge-coloring of a weighted graph so that each edge receives a number of consecutive colors and no colors of the edges at any vertex are the same is a generalization of the classical edge-coloring problem to graphs with integer weights on the edges. The problem is clearly NP-hard, as it is already NP-complete to determine the chromatic index of a simple graph [4]. However, unlike the standard problem, the generalized edge-coloring remains NP-hard even for some restrictive families of graphs for which polynomial-time algorithms for the classical problem are known, e.g., bipartite graphs [3], series-parallel graphs [7], and trees [6]. Nevertheless, in view of potential applications in scheduling and timetabling, it would be useful to have efficient algorithms for polynomially solvable subproblems, and such subproblems concerning caterpillars are considered in this paper.

It is well known that deciding interval edge-colorability of a weighted tree is strongly NP-complete [2]. In the paper we consider, at first, a simplified problem of coloring the edges of *caterpillars*, i.e., trees in which the removal of all pendant vertices results in a path. These pendant vertices can be thought of as *hairs* attached to the *body* of the caterpillar, i.e., a path of non-pendant vertices. Then we deal only with *caterpillars with hairs of arbitrary length* (or *generalized caterpillars* as we shall call them). In these graphs, paths of one or more edges can be attached to any vertex in the body of the caterpillar [1].

We begin by showing in the next section that the edges of caterpillars with hairs of length 1 can be colored in linear time. In Section 3 we consider the complexity of coloring the edges of caterpillars with hairs of arbitrary length and prove that interval colorability remains NP-complete even for caterpillars with a hair of length 2. Next, in Section 4 we consider a very restrictive case of *binomial caterpillars*, i.e., ones that contain two kinds of edges, namely: edges of unit weight and edges of weight L, where L is an arbitrary integer greater than 1. We develop two linear-time algorithms for two special cases of generalized binomial caterpillars. Finally, we give a sharp bound on the chromatic index of such a graph.

2. POLYNOMIAL SOLVABILITY IN THE CASE OF CATERPILLARS

We begin with definitions concerning the interval coloring of the edges of a graph. Given a graph $G = (V(G), E(G))$, to each edge $e \in E(G)$ we associate a positive integer $w(e)$. The pair (G, w) is said to be a *weighted graph*. Let $I(v)$ denote the set of edges incident with vertex $v \in V(G)$. We define the *weighted degree* of v to be

$$\Delta_v = \sum_{e \in I(v)} w(e) \quad \text{for any } v \in V(G).$$

The *maximum weighted degree* of (G, w) is the quantity

$$\Delta = \max\{\Delta_v \colon v \in V(G)\}.$$

By an *interval coloring* of the edges of (G, w) we mean a function

$$c\colon E(G) \to \{S \subseteq \{1, \ldots, k\}\}$$

whose values are sets of consecutive integers satisfying $|c(e)| = w(e)$ and $c(e) \cap c(f) = \emptyset$ whenever $e \cap f \neq \emptyset$. The *chromatic index* $\chi'(G, w)$ is the least integer k for which there is an interval k-coloring of (G, w). A *chromatic coloring* of (G, w) is an interval coloring attaining the chromatic index $\chi'(G, w)$.

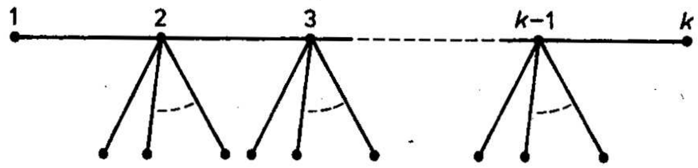

Fig. 1. A caterpillar

Let $C = (V(C), E(C))$ be an n-vertex caterpillar with hairs of length 1. Let P be a k-vertex body of C. The pair (C, w) is said to be a *weighted caterpillar with hairs of length* 1 or, briefly, a *caterpillar*. An example of such a caterpillar is shown in Fig. 1. Consequently, the pair (G, w), where G is a caterpillar with hairs of arbitrary length, is called a *generalized caterpillar*.

A chromatic coloring of (C, w) can be found efficiently as stated in the following

THEOREM 1. *A chromatic coloring of a caterpillar* (C, w) *can be found in time* $O(n)$.

Proof. It is obvious that $\Delta \leqslant \chi'(C, w)$. The following algorithm uses Δ colors, so $\chi'(C, w) = \Delta$:

Let the colors be $1, 2, \ldots, \Delta$ and let $w(i, i+1)$ be the weight of the edge $\{i, i+1\}$ of the body P (see Fig. 1). At first we color the edges

$\{i, i+1\}$ with colors $1, 2, \ldots, w(i, i+1)$ if i is odd,

$\{i, i+1\}$ with colors $\Delta, \ldots, \Delta+1-w(i, i+1)$ if i is even.

Next, for each $i = 2, \ldots, k-1$ we color the hairs incident with i with consecutive colors chosen greedily in the interval

$$\begin{array}{ll} [w(i-1, i)+1, \Delta-w(i, i+1)] & \text{if } i \text{ is even}, \\ [w(i, i+1)+1, \Delta-w(i-1, i)] & \text{if } i \text{ is odd}. \end{array}$$

Since our algorithm clearly has a linear running time, the theorem is proved.

3. NP-COMPLETENESS OF COLORING GENERALIZED CATERPILLARS

In the previous section we have shown that the problem of finding a chromatic coloring of caterpillars can be solved efficiently, i.e., by a polynomial-time algorithm. However, the same problem but in the case of generalized caterpillars becomes NP-hard, and hence it is unlikely to be solvable efficiently.

Given a generalized caterpillar (G, w) and a positive integer k. By CHROMATIC INDEX we mean the question: "is $\chi'(G, w) \leqslant k$?".

THEOREM 2. *CHROMATIC INDEX is NP-complete even when (G, w) is a generalized caterpillar with a hair of length 2.*

Proof. We transform the following NP-complete PARTITION problem [5]: "Given a set of positive integers $A = \{a_1, \ldots, a_n\}$ such that $\sum a_i = 2b$, $1 \leqslant i \leqslant n$, is there a partition P of A such that

$$\sum_{i \in P} a_i = \sum_{i \notin P} a_i = b?\text{"}$$

to CHROMATIC INDEX.

Given an instance of PARTITION, the generalized caterpillar (G, w) is constructed from a path of three edges d_1, b_1 and c with weights $w(d_1) = b+1$, $w(b_1) = b$ and $w(c) = 1$. The end point of the path which belongs to c is the centre of a star with n edges of weights a_1 through a_n. To the other end point of edge c there is attached a hair consisting of two edges b_2 and d_2, as shown in Fig. 2. The weights of these edges are $w(b_2) = b$ and $w(d_2) = b+1$.

Suppose that $\chi'(G, w) \leqslant 2b+1$. In other words, there is an interval coloring of (G, w) with $k = 2b+1$ colors. Since $w(d_1)+w(b_1) = k$, b_1 must be assigned either colors $c(b_1) = \{1, \ldots, b\}$ or $c(b_1) = \{b+2, \ldots, k\}$. By symmetry, this must also hold for the edge b_2. Thus one must be assigned the first interval of colors, and the other the second one. This implies that the unit-weight edge c must be colored with $b+1$. Thus some of the edges a_i must be colored within the first interval $[1, b]$ and the others within the second interval $[b+2, k]$. Now it is easy to see that (G, w) can be colored with k colors if and only if there is a partition P of A.

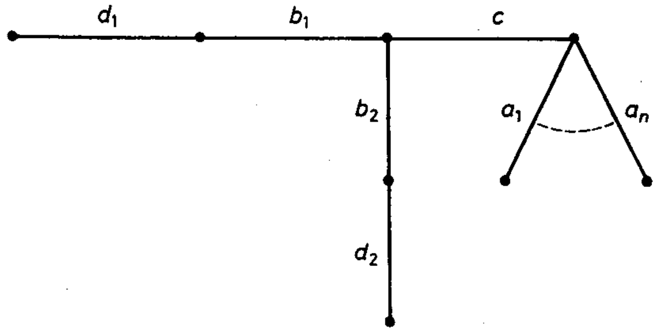

Fig. 2. A generalized caterpillar with a hair of length 2

The NP-completeness of CHROMATIC INDEX provides strong evidence that the general optimization problem cannot be solved by a polynomial-time algorithm. However, by restricting the weights of a generalized caterpillar to just two integers it is possible to obtain polynomial-time algorithms for special cases of the problem.

4. TWO LINEAR ALGORITHMS FOR GENERALIZED BINOMIAL CATERPILLARS

Let (B, w) be a generalized caterpillar with two kinds of edges: light edges with weight 1 and heavy edges with weight L, where L is an arbitrary integer greater than 1. From here on, such a graph will be called a *generalized binomial caterpillar*. Furthermore, any edge $e \in E(B)$ with weight L will be called an *L-edge* and any vertex $v \in V(B)$ belonging to an L-edge an *L-vertex*. Similarly, any edge $e \in E(B)$ with weight 1 will be called *1-edge*. For clarity, we shall use the term "1-edge" whenever we wish to exclude L-edges. Maximal subtrees of B induced by L-edges will be called *L-trees*. For any vertex $v \in V(B)$ we distinguish the following degrees: the ordinary *degree* d_v, the *L-degree* D_v defined to be the number of L-edges incident with vertex v, and the *weighted degree* $\Delta_v = d_v + (L-1) D_v$ (defined in Section 2). We use d, D and Δ to denote the maximum degree, L-degree and weighted degree of any vertex of (B, w), respectively. In this section we give two linear algorithms for interval coloring the edges of (B, w), depending on the value of its maximum L-degree D.

4.1. Algorithm for generalized binomial caterpillars with $D \geq 2$. It is well known that any unweighted tree can be d-colored by a linear-time algorithm which traverses that tree in a *depth-first* (DF) manner and assigns colors to the edges in a "greedy" way [6]. For the purpose of efficient coloring we assume that the vertices of a generalized binomial caterpillar (B, w) are numbered in a DF order starting with any L-vertex v with $D_v = D$ to be vertex 1 and searching for succeeding vertices in favor of those which are adjacent by L-edges to already numbered vertices. If, however, there is no such an L-vertex, then ties are broken by choosing a vertex v with $D_v = D$. Our algorithm colors the edges of (B, w) greedily in the DF order. The coloring of any edge is usually done by assigning the first available interval of colors except two cases where an edge $e = \{u, v\}$ contains u or v as the lowest numbered vertex of an L-tree. First, if v is the root of L-tree and $D_v = D$, to avoid introducing an additional color, the color for 1-edge e is chosen to be one of the following:

$$1,\ L+1,\ 2L+1, \ldots, DL+1, \ldots, \Delta-1,\ \Delta. \tag{1}$$

Second, if u is the root of L-tree such that $D_u < D$ and e is an L-edge, then it is colored with the first available interval of colors among $[1, L], \ldots, [(D-1)L+1, DL]$.

A control abstraction for the algorithm is as follows:

procedure *GBC1* (B, w);
begin
  number the vertices of (B, w) in a DF order;
  for $v \leftarrow 2$ **to** n **do**
    begin
      $u \leftarrow$ immediate predecessor of v in the DF ordering;
      case
        : v is the root of an L-tree **and** $D_v = D$:
          color $\{u, v\}$ with the first available color in order (1)
        : u is the root of an L-tree **and** $D_u < D$ **and** $\{u, v\}$ is an L-edge:
          color $\{u, v\}$ with the first available interval of colors among
          $[1, L], \ldots, [(D-1)L+1, DL]$
        : **else**: color $\{u, v\}$ in the greedy way
      endcase
    end
end

THEOREM 3. *If $D \geqslant 2$, then algorithm GBC1 produces a Δ-coloring of (B, w) in time $O(n)$.*

Proof. We begin by proving the chromaticity of coloring produced by *GBC1*. First, observe that no 1-edge requires the use of a color $\Delta+1$. Also, no edge belonging to a path of L-edges contained in a hair of (B, w) needs introducing the color $\Delta+1$ because every such L-edge requires the interval $[1, L]$ or $[L+1, 2L]$. Therefore, we can focus our attention on coloring the edges of L-trees rooted at the body P of the caterpillar. We shall show that if there are at least two L-edges having a vertex in common, then no L-edge requires introducing the color $\Delta+1$. This is particularly obvious for the first L-tree rooted at $v=1$, so assume $v \neq 1$ to be the root of any L-tree attached to P and let $e=\{u, v\}$. We shall consider two cases depending on the L-degree of vertex v.

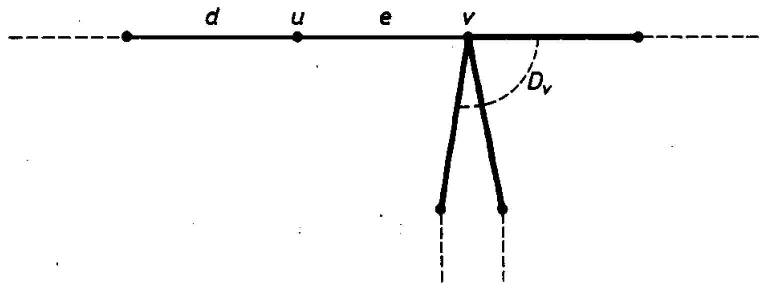

Fig. 3. Illustration for Theorem 3; heavy lines denote L-edges

Case 1. $D_v = D$. If u is not an L-vertex, then there is a 1-edge d such that both d and e belong to the body P. This situation is shown in Fig. 3, where the remaining hairs attached to the end points of edges d and e are left out. According to algorithm *GBC1*, edge e receives either color 1 if $c(d) \neq \{1\}$, or $L+1$ if $c(d)=\{1\}$. Since all the edges of L-tree rooted at v are colored in a greedy way, the maximum color required for this tree is $DL+1 \leqslant \Delta_v \leqslant \Delta$. If u is an L-vertex, then, by the fact that L-edges incident with u are colored with consecutive intervals, it follows that either D_u or D_u+1 colors specified in (1) are not available for e, depending on the number of colors used for edge d. Hence, if $D_u < D$, then at least one of the first $D+1$ colors from (1) is available to e. Also, if $D_u = D$ and $d_u \leqslant D+1$, at least one of the first $D+1$ colors is available to e. If, however, $D_u = D$ and $d_u > D+1$, then color $DL+2 \leqslant \Delta_u \leqslant \Delta$ is feasible for e. By greedy coloring, in all the cases the maximum color required by the L-tree rooted at v is clearly $DL+1 \leqslant \Delta$.

Case 2. $D_v < D$. Regardless of the color assigned to edge e the L-edges incident with v require $(D_v+1)L \leqslant DL$ colors. By greedy coloring, the remaining edges of the L-tree rooted at v require at most $DL \leqslant \Delta$ colors.

Now we estimate the time complexity of *GBC1*. First observe that the preparatory numbering of the vertices can be done in linear time. In the coloring phase, determining a feasible interval for any edge in $E(B)$ can be done in constant time provided that intervals of unavailable colors at each vertex are stored as one, two or at most three pairs of integers. With this data structure, $O(1)$ time is used to insert a set of colors assigned to an edge into the lists of tight intervals of unavailable colors associated with end points of that edge. Thus the overall running time for *GBC1* is $O(n)$.

4.2. Algorithm for generalized binomial caterpillars with $D = 1$. Unexpectedly, the problem of optimal coloring the generalized binomial caterpillars with $D = 1$ is much more complicated than that of coloring (B, w) with $D \geqslant 2$. First of all notice that these caterpillars may require more than Δ colors. An illustration of this fact is a generalized binomial caterpillar shown in Fig. 8. We shall call such a graph a *superstar*. For each vertex $v \in V(B)$, by the *superdegree* s_v we mean the quantity

$$s_v = \begin{cases} \text{number of } L\text{-vertices adjacent to } v & \text{if } D_v = 0, \\ 0 & \text{if } D_v = 1. \end{cases}$$

S is used to denote the maximum superdegree of any vertex in $V(B)$. By an *S-vertex* we mean any vertex v with $s_v = S$.

It is not so hard to see that $\chi'(B, w) = \Delta$ if (B, w) is a generalized caterpillar with $D = 1$ and such that $S = 0$ or $S = 1$. However, if $S \geqslant 2$, then the chromatic index of (B, w) can be greater than Δ. In what follows we develop a procedure for optimal coloring the edges of (B, w) in the case $S \geqslant 3$. (The possibility of the efficient coloring of binomial caterpillars with $S \leqslant 2$ is discussed in concluding of this section.) A crucial point of this approach is a careful coloring of 1-edges incident with S-vertices, since their colors affect the colors to be used for L-edges belonging to such superstars and, consequently, the total number of colors required by a caterpillar. For this purpose, we preserve certain colors for 1-edges adjacent to L-edges of the superstars and use them in a greedy way with respect to the interval coloring of the succeeding L-edges. Hence, the remaining 1-edges incident with S-vertices can be colored with non-preserved colors only.

To design a procedure for optimal coloring the edges of (B, w) we change slightly the coloring order of Section 4.1. Namely, we first look for the set of all vertices with the superdegree equal to S. Then, starting from the first S-vertex (say, the leftmost S-vertex in the body P) to be vertex 1 we number the vertices in a DF order searching for vertices in favor of those which are adjacent by L-edges to already numbered vertices. If, however, there are no such vertices, then ties are broken first by choosing an L-vertex belonging to the body, second by choosing an L-vertex belonging to a hair. Results of this preparatory ordering are stored in the form of a *FATHER* array in which the i-th entry indicates the immediate predecessor of vertex i, $2 \leqslant i \leqslant n$. For example, such an ordering was used to number the vertices of caterpillars shown in Figs. 5, 7 and 8.

After numbering the vertices, we proceed to the second stage of the algorithm, that is, the partial coloring of the paths joining consecutive S-vertices. Starting with the last S-vertex in P, say v, we color the path

$$P_v = \big(\{v, FATHER(v)\}, \{FATHER(v), FATHER\big(FATHER(v)\big)\}, \ldots\big)$$

up the body with colors L and $L-1$ alternately, until an S-vertex or a vertex adjacent to an L-vertex in P is encountered. In either case we continue the alternate coloring starting with the S-vertex followed by v, and so on until vertex 1 is reached. More precisely, in most of the cases the colors used for consecutive edges of P_v are $L, L-1, L \ldots$, respectively, or $L-1, L, L-1, \ldots$, if $\{v, FATHER(v)\}$ cannot be colored with L. If, however, $\Delta = S+1$ and P_v is a path of length $2k$, $k \geq 2$, passing through $2k-1$ vertices other than L-vertices, then we first check whether there is a vertex $u \in P_v$ such that $s_u \leq S-2$ and $d(u, v) = 2j$, $1 \leq j < k$, where $d(u, v)$ is the length of the path between u and v. If so, we color consecutive edges of P_v with $L, L-1, \ldots, L, \Delta, L-1, \ldots, L$, respectively, where Δ and $L-1$ are the colors used for the edges incident with vertex u. Otherwise, we apply the following coloring scheme: $L, L-1, \ldots, L, L-1$.

In the third stage, our algorithm colors all the remaining edges greedily in the order prespecified by the DF-numbering of vertices. The coloring of any edge is usually done by assigning the lowest possible interval of colors except when 1-edge $\{FATHER(v), v\}$ containing v as an L-vertex is encountered. Namely, if v is the root of an L-edge, to avoid introducing an additional color as far as possible, the color for $\{FATHER(v), v\}$ is chosen to be the first feasible color among

$$1, L+1, L+2, \ldots, k-1, k, 2, \ldots, L-1, L, \tag{2}$$

where $k \leq \chi'(B, w)$ is a current lower bound on the chromatic index of the caterpillar (initially $k = \Delta$).

A control abstraction for the algorithm is as follows:

```
procedure GBC2(B, w);
begin
  number the vertices of (B, w) in a DF order;
  if (B, w) has at least two S-vertices then
    for v ← n downto 2 do
      if v is an S-vertex and FATHER(v) is not an L-vertex
        then color path P_v up the body in the above-described way
      endif
  endif of the preparatory coloring;
  k ← Δ;
  for v ← 2 to n do
  begin
    u ← FATHER(v);
    if {v, v+1} is an L-edge
      then c ← first color available to {u, v} in order (2);
           if c ≤ L and c+L > k
             then color {u, v} with k+1; k ← k+1
             else color {u, v} with c
           endif
```

else if $\{u, v\}$ is uncolored
then color $\{u, v\}$ in a greedy way
endif
endif
end of the main coloring
end

THEOREM 4. *If $S \geqslant 3$, then algorithm GBC2 uses $O(n)$ time to produce a chromatic coloring of (B, w).*

Proof. Let (B, w) be a generalized caterpillar with no two L-edges having a vertex in common and such that $S \geqslant 3$. If no additional color is introduced, algorithm *GBC2* produces clearly a legal Δ-coloring. So suppose that for some edge an additional color $k+1$ was used. Note that 1-edges incident with a vertex v satisfying $1 \leqslant s_v < S$ can always be colored within the same S colors that were used for the first superstar. Also, due to the way in which ties are resolved, no additional color is introduced when coloring an edge incident with an L-vertex, even if this is a 1-edge adjacent to another 1-edge with a color assigned in the preparatory coloring of P. Thus, a new color $k+1$ can be introduced only for a 1-edge incident with an S-vertex. Let u be any S-vertex of the caterpillar (B, w). When for an L-vertex v adjacent to u a 1-edge $\{u, v\}$ gets a color $c \leqslant L$ such that $c+L \leqslant k$, no additional color will be used. However, if $c+L > k$, an additional color $k+1$ for edge $\{u, v\}$ will be introduced. We shall consider this case in full detail.

If vertex $u = 1$ is the only S-vertex in $V(B)$, then because of the way the edges incident with u are colored and the fact that they are all adjacent to each other, it follows that no interchange of colors makes it possible to avoid color $k+1$. Since this holds for all additional colors, the number of colors used for the neighbourhood of $u = 1$ is chromatic.

Now suppose that (B, w) contains at least two S-vertices. Let u be any S-vertex and let $\{p, u\}$ and $\{u, q\}$, where $p = FATHER(u)$ and $u = FATHER(q)$, be two 1-edges of the body P that are incident with u. Furthermore, let w be the next vertex of P such that $s_w = S$. This situation is shown in Fig. 4, where the remaining hairs attached to vertices p, u, q, and w are left out. In order to show that a new color is necessary, we shall consider

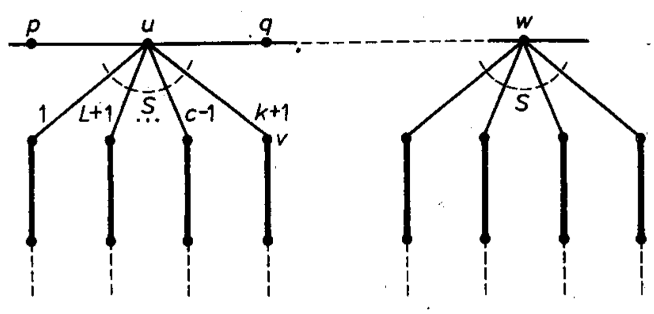

Fig. 4. Illustration for Theorem 4

four cases depending on the colors used for edges $\{p, u\}$ and $\{u, q\}$ in the preparatory coloring of P.

Case 1. Neither of the edges was precolored. This case is analogous to that where $u = 1$ is the only S-vertex of the caterpillar. By a similar argument to that used above, the number of colors assigned to edges incident with u cannot be diminished.

Case 2. Both of the edges were precolored. In this case one edge is colored with L and the other with $L-1$. Since $S+2 \leqslant d_u \leqslant \Delta \leqslant k$, we have $S \leqslant k-2$. Because all colors $1, L+1, \ldots, c-1$ used for the edges incident with u are chosen greedily in order (2) and precede $L-1$, no interchange of the colors including $L-1$ and L results in a better coloring of the superstar. Thus, color $k+1$ is necessary.

Case 3. One edge was precolored with L and the other was uncolored. Since $S+1 \leqslant d_u \leqslant \Delta \leqslant k$, we have $S \leqslant k-1$. Because all the colors precede L in order (2), by a similar argument to that used in Case 2, the additional color assigned to $\{u, v\}$ cannot be avoided.

Case 4. One edge was precolored with $L-1$ and the other was uncolored. In this case either $S+2 \leqslant \Delta \leqslant k$ or $\Delta = S+1$ and path P_w consists of odd vertices different from L-vertices. In the former case we have $S \leqslant k-2$ and, by the same argument as used above, color $k+1$ cannot be avoided. In the latter case, by the way P_w is colored it follows that either the caterpillar contains a 2-superstar (see Fig. 5) or P_w is of length $2k$, $k \geqslant 2$, and to every second vertex $r \in P_w$ there are attached $S-1$ hairs such that $s_r = S-1$ (see Fig. 6). If $c < L-1$, no interchange of the colors $1, L+1, \ldots, c-1$

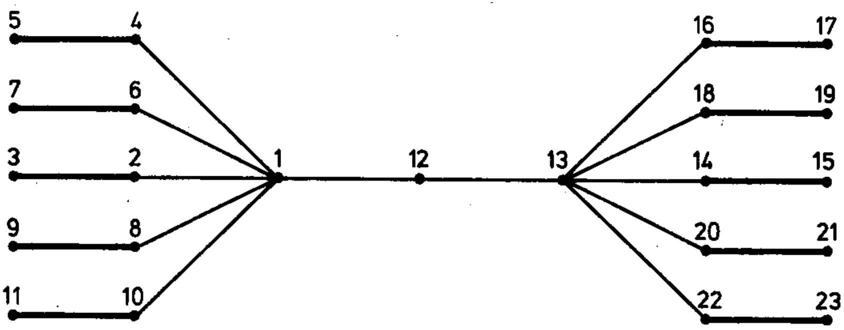

Fig. 5. 2-superstar with $S = 5$

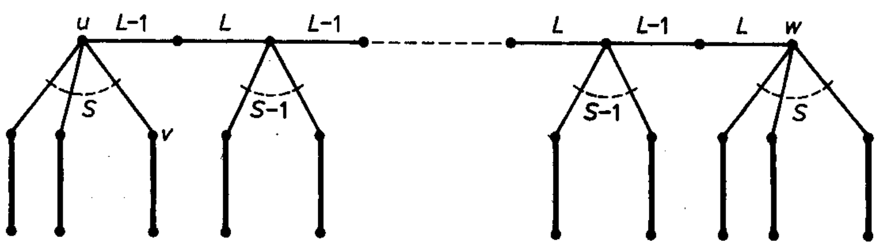

Fig. 6. Caterpillar used in the proof of Case 4

makes it possible to avoid the color $k+1$. If $c = L$, then, by symmetry of the induced caterpillar, any recoloring of path P_w would block the same colors preceding L in (2) either in the neighbourhood of a vertex r with $s_r = S-1$ or in the superstar attached to the S-vertex w. Thus in all the cases a new color is necessary.

Now we proceed to estimate the time complexity of *GBC2*. The preparatory ordering of vertices takes $O(n)$ time. Also, the preparatory coloring of the body can be done in linear time. In the main coloring, the most time consuming operation is the coloring of an edge $\{u, v\}$ with $D_v = 1$. Nevertheless, this can be done in constant time because there is a bounded number of tight intervals of unavailable colors at vertex u. Since the coloring of all the edges of (B, w) can be accomplished in linear time, the complexity of *GBC2* is $O(n)$, as required.

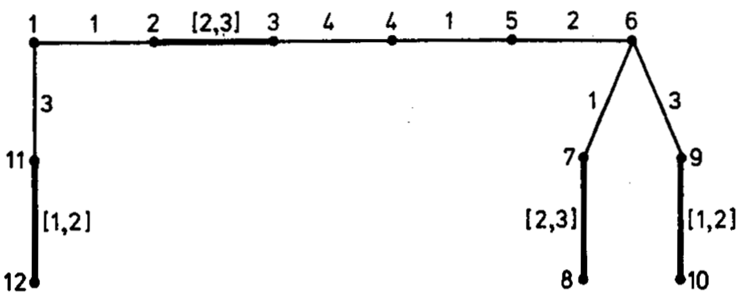

Fig. 7. A counterexample to algorithm *GBC2*

In concluding of this section we comment on the possibility of the efficient coloring of generalized binomial caterpillars with $D = 1$ in the case $S \leqslant 2$. First of all notice that binomial caterpillars with $S \leqslant 1$ can be Δ-colored by algorithm *GBC1*. If $S = 2$, the same algorithm produces a chromatic coloring of caterpillars without S-vertices in the body P. If, however, there is only one such S-vertex in P, we can successfully use algorithm *GBC2*. Finally, if P contains at least two S-vertices, $S = 2$, then *GBC2* fails to guarantee an optimal coloring. A simple example is shown in Fig. 7. In this caterpillar, after the vertex numbering and precoloring of the body, namely $\{4, 5\}$ with 1 and $\{5, 6\}$ with 2, edge $\{3, 4\}$ requires color 4 despite that 3 colors suffice.

5. BOUNDS ON THE CHROMATIC INDEX OF (B, w)

It is obvious that $\Delta \leqslant \chi'(B, w)$ for any binomial caterpillar (B, w). From Section 4.1 it follows that $\chi'(B, w) = \Delta$ if a caterpillar contains at least two L-edges having a vertex in common. The following theorem establishes a general upper bound on the chromatic index of (B, w).

THEOREM 5. *For any generalized binomial caterpillar* (B, w),

$$\chi'(B, w) = \Delta \quad \textit{if } D \geq 2, \tag{3}$$

$$\chi'(B, w) \leq \Delta + \lfloor S/2 \rfloor \quad \textit{if } D = 1. \tag{4}$$

Proof. Equality (3) is an immediate consequence of Theorem 3. To prove inequality (4) suppose that (B, w) is a generalized binomial caterpillar whose chromatic index is greater than Δ. If $S \leq 2$, then clearly $\chi'(B, w) \leq \Delta + 1$. Therefore, assume that $S \geq 3$. In this case (4) can be proved by estimating the number of colors used by *GBC2* when applied to (B, w). Let v be any L-vertex adjacent to an S-vertex u and let c be the first color in (2) which is available to 1-edge $\{u, v\}$ considered in the proof of Theorem 4 (see Fig. 4). If $c > L$, then no additional color is introduced for that edge, so suppose that $c \leq L$. By the way the neighbourhood of u is colored, edge $\{u, v\}$ is assigned a color not greater than $c + L$. Since a new color can be introduced for at most every two 1-edges incident with u and there are S such edges, we have $c \leq \lfloor S/2 \rfloor + 1$. Since no edge of the caterpillar is colored with a color greater than $L + \lfloor S/2 \rfloor + 1$, we obtain

$$\chi'(B, w) \leq \Delta - 1 + \lfloor S/2 \rfloor + 1 = \Delta + \lfloor S/2 \rfloor,$$

and the upper bound follows.

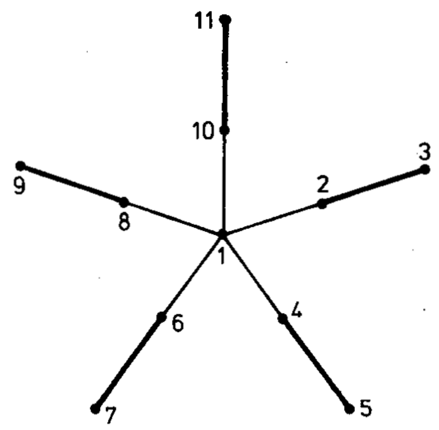

Fig. 8. Superstar with $S = 5$

The bound (4) is tight in the sense that there are caterpillars (B, w) such that $\chi'(B, w)$ equals the upper bound. Upper bound examples are superstars with $L \geq S - 1$ for every $S = 3, 5, \ldots$ An example of such a superstar is shown in Fig. 8.

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