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ON COINCIDENCES IN RENEWAL STREAMS

1. Introduction. Let us consider two sequences of random variables $\{X_n^{(i)},\ n = 1, 2, \ldots\}$, $i = 1, 2$, which represent the working times of two renewed elements. Assume that these sequences consist of independent, integer-valued positive random variables which are identically distributed for fixed i. The last assumption is noted as $X_j^{(i)} \stackrel{d}{=} X^{(i)}$, where $\stackrel{d}{=}$ denotes the equality of distributions.

The considered sequences of random variables generate the renewal streams $S_0^{(i)} = 0$, $S_n^{(i)} = X_1^{(i)} + X_2^{(i)} + \ldots + X_n^{(i)}$, $n = 1, 2, \ldots$, the renewal processes $N^{(i)}(t) = \max\{n \geqslant 0:\ S_n^{(i)} < t\}$, $t \geqslant 0$, and the residual time processes $\gamma^{(i)}(t) = S_{N^{(i)}(t)+1}^{(i)} - t$, $t \geqslant 0$, $i = 1, 2$.

Let T be the moment of the first coincidence of renewals in the considered renewal streams

$$T = \min\{t > 0:\ \gamma^{(1)}(t) = \gamma^{(2)}(t) = 0\}.$$

In the paper the distribution of the random variable T is considered. We analyze in detail the special case of identically distributed streams for which $X^{(1)} \stackrel{d}{=} X^{(2)}$, and an example of Bernoulli trial stream. In addition, the distribution of the number of renewals in the interval $(0, T)$ and also the generalized problem of coincidences in the alternating processes are considered.

Note that the bivariate stochastic process $(\gamma^{(1)}(t), \gamma^{(2)}(t))$, $t \geqslant 0$, is a Markov process; therefore, the study of the random variable T leads to the first passage problem to the state $(0, 0)$. In the case where the support of the random variables $X^{(1)}$ and $X^{(2)}$ is finite, the distribution of the random variable T is a phase-type distribution in the Neuts sense [7]. The study of the distribution of the number of renewals to the coincidence moment is the problem of the number of entries to the fixed set of states before absorption in a Markov process. The coincidence phenomena in renewal streams have occurred in Kingman's paper [5] for the proof of the semigroup character of an operation on the renewal sequences.

In spite of this simple interpretation of the coincidence problems it appears that the standard methods of studying the mentioned characteristics of Markov processes are not efficient. In the paper we reduce the technical difficulties with regard to practical uses. Our problem follows from an analysis of the safety system used in mining winding shifts (see [2]).

2. Auxiliary random variables. Let us introduce the auxiliary random variables $T(i)$, $i = 0, \pm 1, \pm 2, \ldots$, defined in the following way:

$$T(0) = 0,$$
$$T(i) = \begin{cases} \min\{t > 0\colon \gamma^{(1)}(t) = \gamma^{(2)}(t) = 0 \mid X_1^{(1)} = i\}, & i > 0, \\ \min\{t > 0\colon \gamma^{(1)}(t) = \gamma^{(2)}(t) = 0 \mid X_1^{(2)} = -i\}, & i < 0. \end{cases} \tag{1}$$

The conditions of the model imply the equalities

$$T \stackrel{d}{=} \min(X^{(1)}, X^{(2)}) + T'(X^{(1)} - X^{(2)}) \stackrel{d}{=} T(X^{(1)}) \stackrel{d}{=} T(-X^{(2)}), \tag{2}$$

and

$$T(i) \stackrel{d}{=} \begin{cases} \min(i, X^{(2)}) + T'(i - X^{(2)}), & i > 0, \\ \min(-i, X^{(1)}) + T''(i + X^{(1)}), & i < 0, \end{cases} \tag{3}$$

where $X^{(1)}$, $X^{(2)}$, $T'(j)$, $T''(j)$ ($j = \pm 1, \pm 2, \ldots$) are mutually independent and $T'(j) \stackrel{d}{=} T''(j) \stackrel{d}{=} T(j)$.

Let us introduce the following notation for the distributions:

$$p_k^{(1)} = \Pr(X^{(1)} = k), \quad p_k^{(2)} = \Pr(X^{(2)} = k),$$
$$P_k = \Pr(T = k), \quad P_k(i) = \Pr(T(i) = k), \quad i = 0, \pm 1, \pm 2, \ldots, k = 1, 2, \ldots$$

3. Time of coincidence. In this section we consider in two ways the distribution of the random variable T. In the first way, using renewal theory we analyze the distribution and the expected value. The second way, more complicated, enables us to study the higher order moments.

Let S_n, $n = 1, 2, \ldots$, be the sequence of coincidence moments in the considered renewal streams. It is easy to see that this is a renewal stream with inter-renewal time distributed as the random variable T. It is obvious that for $i = 1, 2$ the renewal probabilities

$$u_k^{(i)} = \Pr(\gamma^{(i)}(k) = 0), \quad k = 1, 2, \ldots,$$

satisfy the renewal equation

$$u_k^{(i)} = p_k^{(i)} + \sum_{j=1}^{k-1} u_{k-j}^{(i)} p_j^{(i)}, \quad k = 1, 2, \ldots$$

We assume that the sum over an empty set of indices is equal to zero. It is easy to note that

$$U_k = \Pr(\gamma^{(1)}(k) = \gamma^{(2)}(k) = 0) = u_k^{(1)} u_k^{(2)} \tag{4}$$

and

$$U_k = P_k + \sum_{j=1}^{k-1} U_{k-j} P_j.$$

Hence

$$P_k = u_k^{(1)} u_k^{(2)} - \sum_{j=1}^{k-1} u_{k-j}^{(1)} u_{k-j}^{(2)} P_j, \qquad k = 1, 2, \dots$$

This formula enables us to evaluate the probabilities P_k, $k = 1, 2, \dots$, term by term.

Since $T(i) \geqslant |i|$, from (1) and (3) we obtain the equalities

$P_k(0) = 1_{0k}$,

$$(5) \quad P_k(i) = \begin{cases} \sum_{j=1}^{i-1} P_{k-j}(i-j)\, p_j^{(2)} + 1_{ik}\, p_i^{(2)} + \sum_{j=i+1}^{k} P_{k-i}(i-j)\, p_j^{(2)}, \\ \qquad\qquad i > 0,\ k = i, i+1, \dots, \\ \sum_{j=1}^{-i-1} P_{k-j}(i+j)\, p_j^{(1)} + 1_{-ik}\, p_{-i}^{(1)} + \sum_{j=-i+1}^{k} P_{k+i}(i+j)\, p_j^{(1)}, \\ \qquad\qquad i < 0,\ k = -i, -i+1, \dots, \end{cases}$$

$P_k(i) = 0, \quad i = \pm 1, \pm 2, \dots, \; k = 0, 1, \dots, |i|-1,$

where

$$1_{ik} = \begin{cases} 1, & i = k, \\ 0, & i \neq k. \end{cases}$$

Equality (5) enables us to evaluate the probabilities $P_k(i)$ term by term. This system of equations may be reordered and solved recurrently. We have

$$\begin{aligned}
P_1(1) &= p_1^{(2)}, \\
P_1(-1) &= p_1^{(1)}, \\
P_2(1) &= P_1(-1)\, p_2^{(2)}, \\
P_2(2) &= P_1(1)\, p_1^{(2)} + p_2^{(2)}, \\
P_2(-1) &= P_1(1)\, p_2^{(1)}, \\
P_2(-2) &= P_1(-1)\, p_1^{(1)} + p_2^{(1)}, \\
P_3(1) &= P_2(-1)\, p_2^{(2)} + P_2(-2)\, p_3^{(2)}, \\
P_3(2) &= P_2(1)\, p_1^{(2)} + P_1(-1)\, p_3^{(2)}, \\
P_3(3) &= P_2(2)\, p_1^{(2)} + P_1(1)\, p_2^{(2)} + p_3^{(2)}, \\
P_3(-1) &= P_2(1)\, p_2^{(1)} + P_2(2)\, p_3^{(1)}, \\
P_3(-2) &= P_2(-1)\, p_1^{(1)} + P_1(1)\, p_3^{(1)}, \\
P_3(-3) &= P_2(-2)\, p_1^{(1)} + P_1(-1)\, p_2^{(1)} + p_3^{(1)},
\end{aligned}$$

and so on.

4. Special cases.

Case 1. In [2], the case $X^{(1)} \overset{d}{=} X^{(2)} \overset{d}{=} X$, important for practical uses, is considered. Now it is assumed that the random variables which generate the renewal processes are identically distributed: $\Pr(X = k) = p_k$, $k = 1, 2, \ldots$ In this case the relations for the auxiliary random variables $T(i)$ may be simplified by a new definition:

$$T(0) = 0,$$

$$T(i) = \min\{t > 0\colon \gamma^{(1)}(t) = \gamma^{(2)}(t) = 0 \mid X_1^{(1)} = i\}, \qquad i > 0.$$

Now, from the conditions of the model we get

$$T \overset{d}{=} T(X), \tag{6}$$

and the relations for the distributions of the random variables $T(\cdot)$ in terms of the random variables are

$$T(i) \overset{d}{=} \min(i, X) + T'(|i - X|), \tag{7}$$

where X and $T'(j)$, $j = 1, 2, \ldots$, are mutually independent, and $T'(j) \overset{d}{=} T(j)$.

These relations, in terms of distribution functions, form the system of equations

$$P_k(0) = 1_{k0}, \quad k = 0, 1, \ldots,$$

$$P_k(i) = \sum_{j=1}^{i-1} P_{k-j}(i-j)\,p_j + 1_{ik}\,p_i + \sum_{j=i+1}^{k} P_{k-i}(j-i)\,p_j, \quad k = i, i+1, \ldots,$$

$$P_k(i) = 0, \quad k = 0, 1, \ldots, i-1, \; i = 1, 2, \ldots$$

Also this system of equations may be solved by the evaluation of probabilities recurrently term by term.

Case 2. If $X^{(i)}$, $i = 1, 2$, are geometrically distributed, i.e., $p_k^{(i)} = p^{(i)}(q^{(i)})^{k-1}$, $k = 1, 2, \ldots, 0 < p^{(i)} < 1$, $q^{(i)} = 1 - p^{(i)}$, $i = 1, 2$, then using the lack-of-memory property of the geometric distribution we may introduce a new definition

$$T \overset{d}{=} \begin{cases} X_1^{(1)} & \text{if } \gamma^{(2)}(X_1^{(1)}) = 0, \\ X_1^{(1)} + T' & \text{otherwise,} \end{cases} \tag{8}$$

where $X_1^{(1)}$ and T' are independent, and $T' \overset{d}{=} T$.

In this case $\Pr(\gamma^{(2)}(X_1^{(1)}) = 0) = p^{(2)}$, whence

$$\Pr(T = k) = P_k = p^{(2)}p^{(1)}(q^{(1)})^{k-1} + \\ + q^{(2)} \sum_{j=1}^{k-1} p^{(1)}(q^{(1)})^{j-1} P_{k-j}, \quad k = 1, 2, \ldots \tag{9}$$

The solution of (9) has the obvious form

$$P_k = p^{(1)} p^{(2)} (1-p^{(1)} p^{(2)})^{k-1}, \quad k = 1, 2, \ldots$$

Case 3. Formula (8) holds in the more general case where $\{p_k^{(2)}\}$ has a geometric distribution and $\{p_k^{(1)}\}$ has a general one. Now formula (9) has the form

$$(10) \qquad P_k = p^{(2)} p_k^{(1)} + q^{(2)} \sum_{j=1}^{k-1} p_j^{(1)} P_{k-j}, \quad k = 1, 2, \ldots$$

In this case it is easy to evaluate the generating function

$$\Pi(s) = \sum_{k=1}^{\infty} s^k P_k$$

as the function of

$$\varphi^{(1)}(s) = \sum_{k=1}^{\infty} s^k p_k^{(1)}.$$

From (10) we get

$$\Pi(s) = p^{(2)} \varphi^{(1)}(s) + q^{(2)} \varphi^{(1)}(s) \Pi(s),$$

and hence

$$\Pi(s) = \sum_{j=0}^{\infty} p^{(2)} (q^{(2)})^j (\varphi^{(1)}(s))^{j+1} = \frac{p^{(2)} \varphi^{(1)}(s)}{1-q^{(2)} \varphi^{(1)}(s)}.$$

This is the generating function of the inter-event time rarified renewal stream (see [3], p. 174). From this we can obtain the distribution of the random variable T and of the moments and also the limiting distributions under $p^{(2)} \to 0$ may be proved.

5. Moments. In the case of renewal streams generated by the random variables defined on the finite support, the moments $\mathrm{E}T^r$, $r = 1, 2, \ldots$ are finite (see [7]). Now we prove that if the distributions of random variables $X^{(1)}$ and $X^{(2)}$ are nonlattice and the expected values $\mathrm{E}X^{(i)} = \mu_1^{(i)}$, $i = 1, 2$, are finite, then

$$(11) \qquad \vartheta_1 = \mathrm{E}T = \mu_1^{(1)} \mu_1^{(2)}.$$

Note for the proof that the coincidence stream is in this case also a nonlattice renewal stream. From the renewal theory it follows that $u_n^{(i)} \to 1/\mu_1^{(i)}$, $i = 1, 2$, and $u_n \to 1/\vartheta_1$ if $n \to \infty$. Hence and from (4) we obtain (11).

From the relation (3) or (7) it is possible to write the system of equations for the moments of the random variables $T(i)$. From (2) or (6) we have immediately the expected values $\mathrm{E}T^r$. We write now in detail the case of identically distributed renewal processes and an example.

Put $\vartheta_1(i) = \mathrm{E}T(i)$ and $\vartheta_2(i) = \mathrm{E}T^2(i)$, $i = 1, 2, \ldots$ In the case of identically distributed processes, from (7) we get the system of equations

$$
\begin{aligned}
\vartheta_1(i) &= \mu_1(i) + \sum_{j=1}^{i-1} \vartheta_1(i-j)\,p_j + \sum_{j=i+1}^{\infty} \vartheta_1(j-i)\,p_j, \\
\vartheta_2(i) &= \mu_2(i) + \sum_{j=1}^{i-1} \vartheta_2(i-j)\,p_j + \sum_{j=i+1}^{\infty} \vartheta_2(j-i)\,p_j,
\end{aligned}
\tag{12}
$$

where

$$\mu_1(i) = \sum_{j=1}^{\infty} \min(i, j)\,p_j,$$

$$\mu_2(i) = \sum_{j=1}^{\infty} \min^2(i, j)\,p_j + 2\sum_{j=1}^{i-1} j\vartheta_1(i-j)\,p_j + 2\sum_{j=i+1}^{\infty} i\vartheta_1(j-i)\,p_j.$$

It is easy to see that

$$\vartheta_1 = \mathrm{E}T = \mathrm{E}T(X) = \sum_{j=1}^{\infty} p_j \mathrm{E}T(j), \qquad \vartheta_2 = \mathrm{E}T^2 = \sum_{j=1}^{\infty} p_j \mathrm{E}T^2(j).$$

Example 1. If $p_i = 1/n$, $i = 1, 2, \ldots, n$, then

$$\mu_1(i) = \frac{i(i+1)}{2n} + \frac{i(n-i)}{n},$$

$$\mu_2(i) = \frac{1}{4} i(n+1)(2n+1-i)$$

and

$$
\begin{aligned}
\vartheta_1(i) &= \tfrac{1}{4} n(n+1) + \tfrac{1}{2} i, \\
\vartheta_2(i) &= \tfrac{1}{8} n^2(n+1)^2 + \tfrac{1}{4} in(n+1), \qquad i = 1, 2, \ldots, n,
\end{aligned}
$$

whence

$$\vartheta_1 = \tfrac{1}{4}(n+1)^2, \qquad \vartheta_2 = \tfrac{1}{8} n(n+1)^3.$$

6. Number of renewals up to the coincidence. Now we consider the random variable $M = N^{(1)}(T) + N^{(2)}(T)$, e.g., the number of renewals in both renewal streams up to the coincidence moment. Taking into account the primary Markov process $(\gamma^{(1)}(t), \gamma^{(2)}(t))$, $t \geqslant 0$, it is easy to note that M is the number of entries to the set of states $\{(0, k), (k, 0);\ k = 1, 2, \ldots\}$ until the entry to the state $(0, 0)$. Now we define the univariate Markov chain in which the absorption time in the state (0) has the same distribution as the random variable M.

Let us characterize the state of one element at the moment of renewal of the second element. Define the random chain as

(13) $$Z_1 = X_1^{(1)\prime} - X_1^{(2)\prime},$$

(14) $$Z_{n+1} = \begin{cases} Z_n - X_n^{(2)\prime} & \text{if } Z_n > 0, \\ Z_n + X_n^{(1)\prime} & \text{if } Z_n < 0, \\ 0 & \text{if } Z_n = 0, \ n = 1, 2, \ldots, \end{cases}$$

where $X_n^{(i)\prime}$, $i = 1, 2$, $n = 1, 2, \ldots$, are mutually independent, and $X_n^{(i)\prime} \stackrel{d}{=} X^{(i)}$. Note that $Z_1, Z_2, \ldots$ is a Markov chain with the absorbing state (0). We have $Z_n < 0$ if the n-th renewal moment occurs in the first renewal stream, $Z_n > 0$ if the n-th renewal moment occurs in the second renewal stream. We have $\Pr(Z_n = 0) = \Pr(M = n-1)$.

Let us introduce the following notation for the probability distributions of the chain and for the transition probabilities:

$$D_n(k) = \Pr(Z_n = k), \qquad D(k, l) = \Pr(Z_{n+1} = l \mid Z_n = k).$$

From (13) we have

$$D_1(k) = \Pr(X_1^{(1)} - X_1^{(2)} = k) = \sum_{j=1}^{\infty} p_j^{(2)} p_{k+j}^{(1)}, \qquad k = 0, \pm 1, \pm 2, \ldots,$$

and from (14) we obtain

$$D(k, l) = \begin{cases} p_{k-l}^{(2)}, & k = 1, 2, \ldots, \\ p_{l-k}^{(1)}, & k = -1, -2, \ldots, \\ 1_{kl}, & k = 0, \ l = 0, \pm 1, \pm 2, \ldots, \end{cases}$$

where $p_j^{(i)} = 0$ for $j \leqslant 0$, $i = 1, 2$.

In the case $X^{(1)} \stackrel{d}{=} X^{(2)} \stackrel{d}{=} X$ we define the Markov chain as

$$Z_1 = \max(X_0, X_1) - \min(X_0, X_1) = |X_0 - X_1|,$$

$$Z_{n+1} = \begin{cases} |Z_n - X_{n+1}| & \text{if } Z_n > 0, \\ 0 & \text{if } Z_n = 0, \ n = 1, 2, \ldots, \end{cases}$$

where $X_0, X_1, \ldots$ are mutually independent and distributed as X. In this case we have

$$D_1(k) = \Pr(Z_1 = k) = (2 - 1_{0k}) \sum_{i=1}^{\infty} p_i p_{i+k}, \qquad k = 0, 1, \ldots,$$

$$D(k, l) = \begin{cases} p_{k+l}, & k > 0, \ l = 0 \text{ or } 0 < k \leqslant l, \\ p_{k+l} + p_{k-l}, & 0 < l < k, \\ 1_{kl}, & k = 0, \ l = 0, 1, \ldots \end{cases}$$

The distribution of the random variable M can be found from the general formulas

$$\Pr(M = n) = \sum_{k=1}^{\infty} D_1(k) D^{(n)}(k, 0),$$

where $D^{(n)}(k, l)$ is the n-fold convolution of the matrix $D(k, l)$.

7. Coincidences in the alternating processes. Let us consider two alternating processes $\alpha^{(i)}(t)$, $t \geqslant 0$, $i = 1, 2$, generated by the sequences of random variables $X_j^{(i)}$, $Y_j^{(i)}$, $j = 1, 2, \ldots$, $i = 1, 2$. It is assumed that these random variables are mutually independent and $X_j^{(i)} \stackrel{d}{=} X^{(i)}$, $Y_j^{(i)} \stackrel{d}{=} Y^{(i)}$, $X^{(i)} > 0$, $Y^{(i)} > 0$. They generate two alternating processes (see [1], [6], p. 283)

$$\alpha^{(i)}(t) = \begin{cases} 1 & \text{if } Z_n^{(i)''} < t \leqslant Z_{n+1}^{(i)'}, \\ 0 & \text{if } Z_{n+1}^{(i)'} < t \leqslant Z_{n+1}^{(i)''}, \ n = 0, 1, \ldots, \end{cases}$$

where $Z_0^{(i)''} = 0$, $Z_n^{(i)'} = X_1^{(i)} + Y_1^{(i)} + \ldots + Y_{n-1}^{(i)} + X_n^{(i)}$ and $Z_n^{(i)''} = Z_n^{(i)'} + Y_n^{(i)}$, $n = 1, 2, \ldots$, $i = 1, 2$.

Let T be the moment of the first coincidence of zero in the considered processes

$$T = \inf\{t > 0\colon \alpha^{(1)}(t) = \alpha^{(2)}(t) = 0\}.$$

The problem of coincidence of zero in two identically distributed alternating processes is studied as the two-lift problem (see [4]).

Let us introduce the auxiliary random variables $T(u)$, $-\infty < u < \infty$:

$$T(0) = 0,$$

$$T(u) = \begin{cases} \inf\{t > 0\colon \alpha^{(1)}(t) = \alpha^{(2)}(t) = 0 \mid X_1^{(1)} = u,\ X_1^{(2)} = 0\}, & u > 0, \\ \inf\{t > 0\colon \alpha^{(1)}(t) = \alpha^{(2)}(t) = 0 \mid X_1^{(1)} = 0,\ X_1^{(2)} = -u\}, & u < 0. \end{cases}$$

From the conditions of the model we get

$$T \stackrel{d}{=} \min(X^{(1)}, X^{(2)}) + T'(X^{(1)} - X^{(2)}),$$

where $X^{(1)}$, $X^{(2)}$, $T'(v)$, $-\infty < v < \infty$, are mutually independent, and $T'(v) \stackrel{d}{=} T(v)$.

The distributions of the random variables $T(u)$ satisfy the equations: for $u > 0$

$$T(u) \stackrel{d}{=} \begin{cases} u & \text{if } u < Y^{(2)}, \\ u + T'(u - Y^{(2)} - X^{(2)}) & \text{if } u \geqslant Y^{(2)},\ Y^{(2)} + X^{(2)} > u, \\ Y^{(2)} + X^{(2)} + T''(u - Y^{(2)} - X^{(2)}) & \text{if } u \geqslant Y^{(2)},\ Y^{(2)} + X^{(2)} \leqslant u, \end{cases}$$

for $u < 0$

$$T(u) \stackrel{d}{=} \begin{cases} -u & \text{if } -u < Y^{(1)}, \\ -u + T'''(u + Y^{(1)} + X^{(1)}) & \text{if } -u \geqslant Y^{(1)},\ Y^{(1)} + X^{(1)} > -u, \\ Y^{(1)} + X^{(1)} + T''''(u + Y^{(1)} + X^{(1)}) & \text{if } -u \geqslant Y^{(1)},\ Y^{(1)} + X^{(1)} \leqslant -u, \end{cases}$$

where $X^{(i)}$, $Y^{(i)}$, $T'(v), \ldots, T''''(v)$, $-\infty < v < \infty$, $i = 1, 2$, are mutually independent, and $T'(v) \stackrel{d}{=} \ldots \stackrel{d}{=} T''''(v) \stackrel{d}{=} T(v)$.

These equations are analogous to (3) and enable us to write the system of equations for

$$R(u, t) = \Pr(T(u) > t), \quad -\infty < u < \infty,\ t \geqslant 0.$$

We do not consider the general case and we pass to the special case.

Let $X^{(1)} \stackrel{d}{=} X^{(2)} \stackrel{d}{=} X$ and $Y^{(1)} \stackrel{d}{=} Y^{(2)} \stackrel{d}{=} Y$. Then we may define new auxiliary random variables as

$$T(0) = 0,$$

$$T(u) = \inf\{t > 0\colon \alpha^{(1)}(t) = \alpha^{(2)}(t) = 0 \mid X_1^{(1)} = u,\ X_1^{(2)} = 0\}, \qquad u > 0.$$

From the conditions of the model we get

$$T \stackrel{d}{=} \min(X', X'') + T'(|X' - X''|),$$

and for $u > 0$ we obtain

$$T(u) \stackrel{d}{=} \begin{cases} u & \text{if } u < Y, \\ u + T'(Y + X - u) & \text{if } u \geqslant Y,\ Y + X > u, \\ Y + X + T''(u - Y - X) & \text{if } u \geqslant Y,\ Y + X \leqslant u, \end{cases} \tag{15}$$

where X, X', X'', Y, $T'(v)$, $T''(v)$, $v > 0$, are mutually independent, and $X' \stackrel{d}{=} X'' \stackrel{d}{=} X$, $T'(v) \stackrel{d}{=} T''(v) \stackrel{d}{=} T(v)$.

From (15) we get the equation

$$\Pr(T(u) > t) = \int_u^{\infty} \Pr(u > t)\, dF_Y(y) + \int_0^u \Big(\int_{u-y}^{\infty} \Pr\big(u + T(y + x - u) > t\big)\, dF_X(x) + \int_0^{u-y} \Pr\big(y + x + T(u - y - x) > t\big)\, dF_X(x)\Big)\, dF_Y(y),$$

where $F_X(x) = \Pr(X \leqslant x)$ and $F_Y(y) = \Pr(Y \leqslant y)$.

Since $T(u) \geqslant u$, we have $R(u, t) = 1$ for $t \leqslant u$. For $t > u$ we get the integral equation

$$R(u, t) = \int_0^u \Big(\int_{u-y}^{\infty} R(y + x - u,\ t - u)\, dF_X(x) + \int_0^{u-y} R(u - y - x,\ t - y - x)\, dF_X(x)\Big)\, dF_Y(y).$$

Equation (15) enables us to write the relations for the expected values $ET(u)$ which have a form similar to that of the relations (12). We do not consider it because, similarly as before, no solutions suitable to practical uses can be found here. Now we pass to an important special case.

Case 4. If the random variables $X^{(i)}$ are exponentially distributed with parameters $\lambda^{(i)}$ and the random variables $Y^{(i)}$ are distributed without restrictions, then

$$T \stackrel{d}{=} \min(X^{(1)}, X^{(2)}) + 1_{X^{(1)} > X^{(2)}} T_1 + 1_{X^{(1)} < X^{(2)}} T_2, \tag{16}$$

and

$$T_1 \stackrel{d}{=} \min(X^{(1)}, Y^{(2)}) + 1_{X^{(1)} > Y^{(2)}} T', \tag{17}$$

$$T_2 \stackrel{d}{=} \min(X^{(2)}, Y^{(1)}) + 1_{X^{(2)} > Y^{(1)}} T'', \tag{18}$$

where $X^{(i)}$, $Y^{(i)}$, T_i $(i = 1, 2)$, T' and T'' are mutually independent, and $T' \stackrel{d}{=} T'' \stackrel{d}{=} T$.

That enables us to evaluate the distributions and moments.

8. Discrete alternating processes. Hitherto we did not assume that the random variables which generate the alternating processes are integer-valued. Now we assume this and also that they are identically distributed. Let us write

$$\Pr(X = k) = p_k^{(X)}, \quad \Pr(Y = k) = p_k^{(Y)}, \quad k = 1, 2, \ldots,$$
$$\Pr(T(u) = k) = P(u, k), \quad u = 1, 2, \ldots, \ k = 1, 2, \ldots$$

Since $T(u) \geqslant u$, we obtain from (15) the equalities

$$P(0, k) = 1_{0k},$$
$$P(u, k) = 1_{uk} \sum_{j=u}^{\infty} p_j^{(Y)} + \sum_{j=1}^{u-1} \Big(\sum_{i=u-j+1}^{k-j} P(j+i-u, k-u) p_i^{(X)} + \sum_{i=1}^{u-j-1} P(u-j-i, k-j-i) p_i^{(X)} \Big) p_j^{(Y)}, \quad k = u, u+1, \ldots,$$
$$P(u, k) = 0, \quad k = 1, 2, \ldots, u-1, \ u = 1, 2, \ldots$$

Also this system of equations may be solved recurrently by the evaluation of the probabilities term by term.

Case 5. Let us assume that $p_k^{(X)} = pq^{k-1}$, $k = 1, 2, \ldots$ Then analogously to equations (16)–(18) we have

$$T \stackrel{d}{=} \min(X', X'') + 1_{X' \neq X''} T_1, \quad T_1 \stackrel{d}{=} \min(X, Y) + 1_{X > Y} T',$$

where X', X'', Y, T_1 and T' are mutually independent, and $X' \stackrel{d}{=} X'' \stackrel{d}{=} X$, $T' \stackrel{d}{=} T$.

Let us consider the equation

$$T_1 \stackrel{d}{=} \min(X, Y) + 1_{X>Y}(\min(X', X'') + 1_{X' \neq X''} T_1'),$$

where $T_1' \stackrel{d}{=} T_1$.

We have

$$(19) \qquad \Pr(T_1 = k) = \alpha_k + \sum_{j=1}^{k-1} \beta_j \Pr(T_1 = k-j), \quad k = 1, 2, \ldots,$$

where

$$\alpha_k = \Pr(X = k, Y \geqslant k) + \Pr(Y + X' = k, X' = X'', X > Y),$$
$$\beta_j = \Pr(Y + \min(X', X'') = j, X' \neq X'', X > Y).$$

In this case, analogously to (10), it is easy to evaluate the generating function

$$\Pi_1(s) = \sum_{k=1}^{\infty} s^k \Pr(T_1 = k)$$

as the function of

$$A(s) = \sum_{k=1}^{\infty} s^k \alpha_k \quad \text{and} \quad B(s) = \sum_{k=1}^{\infty} s^k \beta_k.$$

From (19) we have

$$\Pi_1(s) = \frac{A(s)}{1 - B(s)}.$$

For the expected values we have

$$\mathrm{E}T = \mathrm{E}\min(X', X'') + \Pr(X' \neq X'')\mathrm{E}T_1,$$
$$\mathrm{E}T_1 = \mathrm{E}\min(X, Y) + \Pr(X > Y)\mathrm{E}T.$$

9. Limiting properties. The analysis of some numerical experiments suggests that in the series of distributions $\{p_{k,n}^{(i)}, k = 1, 2, \ldots\}$, $n = 1, 2, \ldots$, if $\sup p_{k,n}^{(i)} \to 0$ as $n \to \infty$, $i = 1$ and (or) $i = 2$, then the distribution of the random variable T may be estimated by the exponential distribution. Note that in Example 1 we have $\mathrm{E}T^2/(\mathrm{E}T)^2 = \vartheta_2/\vartheta_1^2 \to 2$ as $n \to \infty$, which confirms the limit exponential conjecture. Under the assumption of Case 3 the problem is solved by Rényi's theorem (see [3], p. 177) on the convergence of rarified renewal streams to the Poisson stream. The problem of exponential asymptoticity in the two-lift problem was considered, among others, by Kaplan [4]. It is an open problem to describe the class of limiting distributions and the domain of attraction in general coincidence streams.

References

[1] D. R. Cox, *Renewal theory*, New York 1963.
[2] J. M. Czaplicki and B. Kopociński, *On reliability of certain elementary safety system*, Microelectronics and Reliability 5 (2) (1986), pp. 1–3.
[3] B. W. Gniedenko and I. N. Kowalenko, *Wstęp do teorii obsługi masowej* (*An introduction to queueing theory*), Warszawa 1971.
[4] N. L. Kaplan, *Another look at the two-lift problem*, J. Appl. Probab. 18 (1981), pp. 697–706.
[5] J. F. C. Kingman, *An approach to the study of Markov processes*, J. Roy. Statist. Soc. B 28 (1966), pp. 417–447.
[6] B. Kopociński, *Zarys teorii odnowy i niezawodności* (*An introduction to renewal theory and reliability*), Warszawa 1973.
[7] M. F. Neuts, *Probability distributions of a phase type*, pp. 173–206 in: *Liber Amicorum Professor Emeritus Dr. H. Florin*, Katolieke Universiteit Leuven, Leuven 1975.

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