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A NOTE ON MOMENT INEQUALITIES FOR ORDER STATISTICS FROM STAR-SHAPED DISTRIBUTIONS

1. Introduction. Moment inequalities for certain functions of order statistics from ordered families of positive random variables were studied recently by Bartoszewicz [3]. In [3] the distributions have been assumed to be ordered by dispersion (for definitions see Shaked [6]). Under the assumption that the distribution of random variable X is dominated by the distribution of random variable Y in dispersive ordering, it has been proved in Theorem 2 that $E(\varphi(X - EX)) \leq E(\varphi(Y - EY))$ for each convex function φ . Also inequalities for the covariances of order statistics have been given. As corollaries some results for order statistics from IFR and DFR distributions have been stated. The classes of IFR and DFR distributions are of interest in reliability theory. The importance of these and other classes and properties thereof are discussed in the text by Barlow and Proschan [2].

The aim of the present paper is to derive conclusions corresponding to Theorem 2 in [3], but under the different assumption that the distributions are star-shape ordered. Important contributions directed to the objective of classifying and characterizing the star-shape and related ordering relations include works of Barlow and Proschan [1], [2], Lehmann [5], Van Zwet [7] and others.

For reasons of convenience, we have collected the definitions and results needed in this paper together in Section 2, with references where their proofs may be found. In Section 3, we prove a result corresponding to Theorem 2 of the paper [3] already mentioned, namely under the assumption of star-shape ordering of the distributions in spite of the assumption of dispersive ordering; a similar conclusion is obtained (see Theorem 1). In this section we present also some inequalities for variances and some other moments of order statistics from IFRA and DFRA distributions, as Corollary 2 after the above theorem.

2. Preliminaries. To avoid technical complications in the statement of the results we assume throughout that the supports of the underlying distributions are intervals (finite or infinite) and that these distributions have no atoms. Our distributions have strictly increasing and continuous inverses on $(0, 1)$. We assume also that $F(0) = 0$ for each distribution F .

We use the following notation and conventions:

$$F^{-1}(t) = \inf \{x: F(x) > t\}, \quad t \in [0, 1), \quad F^{-1}(1) = \sup \{x: F(x) < 1\},$$

$$\bar{F}(x) = 1 - F(x), \quad m_F^r = \int_0^\infty x^r dF(x), \quad \sigma_F^r = \int_0^\infty (x - m_F^1)^r dF(x), \quad r \geq 1.$$

The stochastic order relation we denote by $F \leq_{st} G$. Let $(X_{1:n}, \dots, X_{n:n})$ be the vector of order statistics from F . We use $F_{k:n}(x)$ to denote the distribution of $X_{k:n}$.

For G strictly increasing, define:

Definition 1. F is *star-shaped with respect to G* (written $F < G$) if $G^{-1}F(x)$ is a star-shaped function (that is, $(1/x)G^{-1}F(x)$ is nondecreasing) in x on the support of F .

Note that $F < G$ is a partial ordering of the scale equivalent classes of distributions, namely $F < G$ is equivalent to $F(\alpha x) < G(\beta x)$ for each $\alpha > 0$, $\beta > 0$; thus we may group into equivalence classes distributions that differ only by a positive scale factor. Moreover, this ordering possesses the single crossing property: if $F < G$ then $\bar{F}(x)$ crosses $\bar{G}(\beta x)$ at most once and from above, as x increases from 0 to ∞ , for each $\beta > 0$ (for details see Barlow and Proschan [2], pp. 106–107).

In order to relate discussed ordering with IFRA and DFRA classes we recall the following lemma (see e.g. Barlow and Proschan [2], p. 107).

LEMMA 1. Let $G(x) = 1 - e^{-\lambda x}$, $\lambda > 0$. Then $F < G$ ($F > G$) is equivalent to F IFRA (DFRA).

3. Basic results. Now we prove the theorem which promises to be useful in other contexts, where the comparison of variances is of interest. We assume that the considered integrals are finite.

THEOREM 1. Let $F < G$ and $m_F^1 \leq m_G^1$. Then

$$\int_0^\infty \varphi(m_G^1(x - m_F^1)) dF(x) \leq \int_0^\infty \varphi(m_F^1(x - m_G^1)) dG(x)$$

for each nondecreasing convex function φ .

Proof. From the single crossing property, $\bar{F}(m_F^1 x)$ crosses $\bar{G}(m_G^1 x)$ at most once, and from above, as x increases from 0 to ∞ . Now $\bar{F}(m_F^1 x)$ crosses $\bar{G}(m_G^1 x)$ exactly once, because

$$\int_0^\infty x dF(m_F^1 x) = \int_0^\infty x dG(m_G^1 x) = 1.$$

Hence from the single crossing property of Karlin and Novikoff (see [4]) we have for each convex function ψ

$$\int_0^{\infty} \psi(x) dF(m_F^1 x) \leq \int_0^{\infty} \psi(x) dG(m_G^1 x),$$

which yields

$$\int_0^{\infty} \psi(x/m_F^1) dF(x) \leq \int_0^{\infty} \psi(x/m_G^1) dG(x).$$

In order to complete the proof, take $\psi(x) = \varphi(m_F^1 m_G^1 x - m_F^1 m_G^1)$, where φ is nondecreasing convex.

From Theorem 1 the monotonicity of the variance and other moments can be obtained.

COROLLARY 1. *Under the assumptions of Theorem 1*

$$m_F^r \leq m_G^r \quad \text{and} \quad \sigma_F^r \leq \sigma_G^r$$

for $r \geq 1$.

Note that from Theorem 2 in [3] it immediately follows that Corollary 1 is not valid if $m_F^1 > m_G^1$. In order to complete the discussion on the monotonicity of variance, we ask whether $F < G$ and $m_F^1 > m_G^1$ yield $\sigma_F^2 > \sigma_G^2$. The following example gives the answer.

Example 1. Let $F(x) = 1 - e^{-x/3}$ (exponential distribution with $\lambda = 1/3$) and $G(x) = 1 - e^{-\sqrt{x}}$ (Weibull distribution with $\lambda = 1$, $\alpha = 1/2$). It is easily verified that $F < G$, $m_F^1 > m_G^1$ and $\sigma_F^2 < \sigma_G^2$.

To state the next corollary we need a lemma.

LEMMA 2. *If $F < G$ and $m_F^1 \leq m_G^1$ then there exists a k , $1 \leq k \leq n$, such that $m_{F_{kn}}^1 \leq m_{G_{kn}}^1$ for $k \leq i \leq n$ (if $k > 1$ then $m_{F_{kn}}^1 > m_{G_{kn}}^1$ for $1 \leq i < k$).*

Proof. Let $U(x) = G^{-1}F(x)$. Under our assumptions $F \leq_{st} G$ or F crosses G exactly once. If $F \leq_{st} G$ then

$$\begin{aligned} F_{k:n}(x) &= n \binom{n-k}{k-1} \int_0^{F(x)} t^{k-1} (1-t)^{n-k} dt \\ &\geq n \binom{n-k}{k-1} \int_0^{G(x)} t^{k-1} (1-t)^{n-k} dt = G_{k:n}(x). \end{aligned}$$

Hence

$$F_{k:n} \leq_{st} G_{k:n} \quad \text{and} \quad m_{F_{k:n}}^1 \leq m_{G_{k:n}}^1 \quad \text{for } k = 1, \dots, n.$$

In the case of the single crossing, $x - U(x)$ changes sign exactly once and from positive to negative values. Denote

$$h(i, n) = \int_0^{\infty} (x - U(x)) dF_{in}(x) = m_{F_{in}}^1 - m_{G_{in}}^1.$$

Since F is continuous we may write (see Barlow and Proschan [1])

$$\begin{aligned} h(i, n) &= n \binom{n}{i-1} \int_{-\infty}^{\infty} (x - U(x)) F^{i-1}(x) \bar{F}^{n-i}(x) dF(x) \\ &= \int_{-\infty}^{\infty} (x - U(x)) K(i, n, x) dF(x). \end{aligned}$$

It is easily verified that $K(i, n, x)$ is totally positive of order ∞ (TP_{∞}) in $i = 1, 2, \dots$, and $-\infty < x < \infty$. From the variation diminishing property of totally positive functions (see Barlow and Proschan [2], p. 93), $h(i, n)$ changes sign at most once as a function of i for fixed n , and from positive to negative, if at all. Hence, from the assumption that $m_F^1 \leq m_G^1$, our lemma is easily deduced.

From Theorem 1, Lemma 1 and Lemma 2 we have

COROLLARY 2. *If F is IFRA distribution, G is DFRA distribution and $m_F^1 \leq m_G^1$ then there exists a k , $1 \leq k \leq n$, such that*

$$m_{F_{i:n}}^r \leq m_{G_{i:n}}^r, \quad \sigma_{F_{i:n}}^r \leq \sigma_{G_{i:n}}^r$$

for $k \leq i \leq n$ and $r \geq 1$.

(If additionally $F \leq_{st} G$ then $k = 1$.)

References

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