

PROBLEMS

1. WALLS OF RECTANGLES (M. CHROBAK)

Consider a set W of rectangles in the plane such that each rectangle has height 1, length $l \in N^+$ ($l \geq 1$), and its bottom is parallel to the x -axis. W is a *wall* if no two rectangles intersect and no rectangle can be placed below its present position without changing its left-edge coordinate. We define $d(W, a)$ to be the number of rectangles which intersect the line $x = a$, the *density* of W

$$d(W) = \max_x d(W, x),$$

and the *height* $h(W)$ of W to be maximal value of a y -coordinate of a bottom of some rectangle in W . Let

$$H(d) = \max \{h(w) : d(W) = d\}.$$

PROBLEMS. (i) Determine the value of $H(d)$.

(ii) Is it true that $\sup_d H(d)/d < \infty$?

The motivation for these problems comes from an approximation algorithm for the dynamic storage allocation problem.

2. PLANE EMBEDDINGS (M. CHROBAK)

It is well known that every planar graph can be drawn in the plane using segments of straight lines as edges. Is the following statement true?

For each n there exists a set V of n points in the plane (called a *universal set*) such that each planar graph G with n vertices can be drawn in the plane using the elements of V as the vertices and straight-line segments as edges.

3. CUBIC HAMILTONIAN GRAPHS (M. CHROBAK)

It is known that each cubic Hamiltonian graph has at least two Hamiltonian cycles. Given a cubic graph G with a Hamiltonian cycle. Is it possible to find another Hamiltonian cycle of G in polynomial time?

4. JUMP NUMBER PROBLEM – 1 (M. HABIB)

The computation of the jump number for a partially ordered set (poset) was proved by Pulleyblank [2] to be NP-hard. However, this problem is polynomially solvable on some particular classes of posets such as N -free (i.e. posets having no subgraphs isomorphic to N in their diagrams; see [3]) or posets having bounded width (see [1]). In order to investigate the boundary between NP-hard and polynomially solvable cases, we propose to study the following particular classes of posets:

- W -free (i.e., posets having no W in their diagrams),
- interval orders,
- 2-dimensional (transitive orientations of comparability graphs).

References

- [1] C. J. Colbourn and W. R. Pulleyblank, *Minimizing setups in ordered sets of fixed width*, Order 1 (1985), pp. 225–229.
- [2] W. R. Pulleyblank, *On minimizing setups in precedence constrained scheduling*, Discrete Appl. Math. (to appear).
- [3] I. Rival, *Optimal linear extensions by interchanging chains*, Proc. Amer. Math. Soc. 89 (1983), pp. 387–394.

5. DOMINO COVERS IN SQUARE CHESSBOARDS (P. JOHN, H. SACHS and H. ZERNITZ)

Let a_n denote the number of different ways in which a (generalized) square chessboard with side length $2n$ (i.e., with $4n^2$ little squares) can be covered with $2n^2$ dominoes each of which covers two little squares ($n = 1, 2, 3, \dots$). Put

$$(1) \quad a_n = 2^n b_n^2, \quad b_n > 0.$$

The following is well known (see, e.g., [2], p. 96, and [1], pp. 245–247):

$$(2) \quad a_n^2 = 2^{4n^2} \prod_{j=1}^n \prod_{k=1}^{2n} \left(\cos^2 \frac{j\pi}{2n+1} + \cos^2 \frac{k\pi}{2n+1} \right),$$

$$b_1 = 1, \quad b_2 = 3, \quad b_3 = 29, \quad b_4 = 901.$$

Applying the method described in our paper (this issue) we found, by simple calculation,

$$b_5 = 89\,893, \quad b_6 = 28\,793\,575.$$

This led us to the conjecture that b_n is an odd integer for any value of n . Meanwhile we proved this conjecture:

THEOREM. $a_n = 2^n b_n^2$, where

$$b_n = 2^{n(n-1)} \prod_{1 \leq j < k \leq n} \left(\cos^2 \frac{j\pi}{2n+1} + \cos^2 \frac{k\pi}{2n+1} \right)$$

is an odd integer, $n = 1, 2, 3, \dots$

Proof. Put

$$(3) \quad \varepsilon_i := 2 \cos \frac{i\pi}{2n+1} \quad (i = 1, 2, \dots, 2n);$$

note that

$$(4) \quad \varepsilon_i = -\varepsilon_{2n+1-i}.$$

From (2)–(4) we obtain

$$a_n^2 = \prod_{j=1}^n \prod_{k=1}^{2n} (\varepsilon_j^2 + \varepsilon_k^2) = \prod_{j=1}^n \prod_{k=1}^n (\varepsilon_j^2 + \varepsilon_k^2)^2;$$

thus

$$(5) \quad a_n = \prod_{j=1}^n \prod_{k=1}^n (\varepsilon_j^2 + \varepsilon_k^2) = \prod_{i=1}^n (2\varepsilon_i^2) \left[\prod_{1 \leq j < k \leq n} (\varepsilon_j^2 + \varepsilon_k^2) \right]^2.$$

The ε_i are the eigenvalues of a path on $2n$ vertices, i.e., they are the roots of the polynomial

$$(6) \quad f_{2n}(\lambda) = U_{2n}\left(\frac{\lambda}{2}\right) = g_n(\lambda^2),$$

where $f_m(\lambda)$ is the characteristic polynomial of a path on m vertices, $U_m(x)$ denotes the Chebyshev polynomial of the second kind defined by

$$\sin(m+1)\varphi = (\sin \varphi) U_m(\cos \varphi),$$

and

$$(7) \quad g_n(x) = \sum_{v=0}^n (-1)^v \binom{2n-v}{v} x^{n-v} = x^n + c_1 x^{n-1} + \dots + (-1)^n$$

with integral coefficients (see, e.g., [1], pp. 73 and 246). Thus, by (4), (6) and (7), we have

$$(8) \quad \prod_{i=1}^n \varepsilon_i^2 = (-1)^n \prod_{i=1}^{2n} \varepsilon_i = (-1)^n f_{2n}(0) = (-1)^n g_n(0) = 1.$$

Inserting (8) into (5), we obtain

$$a_n = 2^n \left[\prod_{1 \leq j < k \leq n} (\varepsilon_j^2 + \varepsilon_k^2) \right]^2$$

or, equivalently (according to (1)),

$$(9) \quad b_n = \prod_{1 \leq j < k \leq n} (\varepsilon_j^2 + \varepsilon_k^2) = 2^{n(n-1)} \prod_{1 \leq j < k \leq n} \left(\cos^2 \frac{j\pi}{2n+1} + \cos^2 \frac{k\pi}{2n+1} \right).$$

Note that the ε_i^2 ($i = 1, 2, \dots, n$) are the roots of the polynomial $g_n(x)$. The second expression in (9) being an integral symmetric function of the ε_i^2 , b_n must be rational as well as integral.

Next we shall establish that, in the field generated by the ε_i , each term $\varepsilon_j^2 + \varepsilon_k^2$ ($1 \leq j < k \leq n$) is prime with respect to 2: by (9), this implies immediately that b_n is odd.

We have

$$\varepsilon_i^2 = 4 \cos^2 \frac{i\pi}{2n+1} = 2 \left(1 + \cos \frac{2i\pi}{2n+1} \right);$$

thus

$$\begin{aligned} \varepsilon_j^2 + \varepsilon_k^2 &= 4 + 2 \left(\cos \frac{2j\pi}{2n+1} + \cos \frac{2k\pi}{2n+1} \right) \\ &= 4 + 2 \cos \frac{(j+k)\pi}{2n+1} \left(2 \cos \frac{(j-k)\pi}{2n+1} \right) \\ &= 4 + \varepsilon_{k+j} \varepsilon_{k-j}. \end{aligned}$$

By (8), ε_{k+j} and ε_{k-j} are divisors of unity; thus $\varepsilon_j^2 + \varepsilon_k^2$ and 2 cannot have a divisor (as an integral ideal) in common. This proves the theorem.

However, the following problem remains open:

Find a (simple) direct, purely combinatorial proof of the statement that $a_n = 2^n b_n^2$, where b_n is an odd integer. Do the numbers b_n satisfy a simple recurrence relation?

References

- [1] D. M. Cvetković, M. Doob and H. Sachs, *Spectra of Graphs — Theory and Applications*, Deutscher Verlag der Wissenschaften, Berlin 1980.
- [2] P. W. Kasteleyn, *Graph theory and crystal physics* in: F. Harary (ed.), *Graph Theory and Theoretical Physics*, Academic Press, London-New York 1967.

6. INTERIORS OF UNIFORM SIZE IN STEINITZ'S THEOREM (J. R. REAY)

Suppose the unit ball $B_1(0)$ is the largest ball at the origin 0 which is contained in $\text{conv } S$ for some set $S \subset \mathbb{R}^d$. Steinitz's theorem asserts that there exist $T \subset S$, $|T| \leq 2d$, and a largest $\varepsilon > 0$ for which $B_\varepsilon(0) \subset \text{int conv } T$.

(a) Determine

$$r(d) = \min_{S \subset \mathbb{R}^d} \max_{T \subset S} \{\varepsilon > 0: B_\varepsilon(0) \subset \text{conv } T, |T| \leq 2d\}.$$

(b) Prove that $r(d) \not\geq 0$.

Both problems are open if $d \geq 3$.

References

- [1] W. E. Bonnice and J. R. Reay, *Relative interiors of convex hulls*, Proc. Amer. Math. Soc. 20 (1969), pp. 246–250.
- [2] J. R. Reay and T. Zamfirescu, *Interiors of uniform size in Steinitz's Theorem*, Proc. Kolloquium über Diskrete Geometrie, Salzburg, Mai 1985, pp. 319–328.

7. AREA OF LATTICE POLYGONS – THE \$ 10 PROBLEM (J. R. REAY)

A *lattice polygon* has all vertices at lattice points of the tiling of \mathbb{R}^2 by regular hexagons of unit area. Characterize the family of lattice polygons P whose area is given by

$$(*) \quad A = (3n + c - 6)/12.$$

Let $n = b + 2i - 2$ denote Pick's number, where b is the number of lattice points on ∂P , the boundary of P , and i is the number of lattice points in the interior of P . Let us define the *boundary characteristic* c of P to be [the number of edges extending into exterior of P from ∂P] – [the number of edges extending into interior of P from ∂P].

Note that $(*)$ is true if ∂P consists of edges or diagonals of hexagons and $(*)$ is false in general.

References

- [1] Ding Ren and J. R. Reay, *The boundary characteristic and Pick's Theorem in the Archimedean planar tilings* (to appear).
- [2] – *Areas of lattice polygons, applied to computer graphics*, this issue, pp. 547–556.

8. REGULAR GRAPHS (G. SIERKSMA)

Let $N(d_1, \dots, d_p)$ be the number of all simple graphs with degree sequence d_1, \dots, d_p . Let $N(p; k)$ denote the number of k -regular simple graphs on p vertices with $k \geq 1$ and $p \geq 2$. Moreover, $N^*(d_1, \dots, d_p)$ is the number of "locally restricted" simple graphs with degree sequence d_1, \dots, d_p (i.e., $N(d_1, \dots, d_p)$ is the number of all configurations on p fixed vertices with degree sequence d_1, \dots, d_p ; in $N^*(d_1, \dots, d_p)$ all vertices have fixed degree). If s_i ($i = 1, \dots, q$ and $s_1 + \dots + s_q = p$) is the number of vertices with degree d_i , then one can show that

$$N(d_1, \dots, d_p) = \frac{p!}{s_1! \dots s_q!} N^*(d_1, \dots, d_p).$$

We have the following two conjectures:

$$(a) \quad N^*(d_1, \dots, d_p) \leq N(p; \lceil \tfrac{1}{2}p \rceil)$$

and

$$(b) \quad N(d_1, \dots, d_p) \leq \binom{p}{\lceil \tfrac{1}{2}p \rceil} N(p; \lceil \tfrac{1}{2}p \rceil).$$

References

- [1] G. Sierksma and S. Attema, *Labeled regular graphs and subgraphs*, Report 84-14-WS, Econometric Inst., Univ. of Groningen, 1984.
- [2] G. Sierksma, H. Fluks and J. Blaakmeer, *An efficient algorithm for the number of labeled graphs*, Report 85-09-WS, Econometric Inst., Univ. of Groningen, 1985.

9. JUMP NUMBER PROBLEM - 2 (M. M. SYSŁO)

The jump number problem for partially-ordered sets is NP-hard (see Problem 4). Recently, the author proved in [1] a bound for the performance of strongly greedy linear extensions in terms of the number of dummy arcs in arc diagrams of posets. However, the existence of an approximation algorithm for the problem with a constant performance guarantee is still an open problem.

Reference

- [1] M. M. Sysłó, *Minimizing the jump number for partially-ordered sets: a graph-theoretic approach. II*, Discrete Math. 63 (1987), pp. 279-295.

10. HAMILTONIAN CYCLES IN G^2 (T. TRACZYK)

Let G be a 2-connected graph without loops and multiple edges. H. Fleischner proved in 1976 that for every two vertices u and v of G there is a Hamiltonian cycle C in G^2 containing three edges of G , say e_1, e_2, e_3 , such that $u \in e_1, u \in e_2$ and $v \in e_3$.

Is it true that for every three vertices v_1, v_2, v_3 of G there is a Hamiltonian cycle C in G^2 containing three edges f_1, f_2, f_3 of G such that $v_i \in f_i$ ($i = 1, 2, 3$)?

11. THICKNESS OF GRAPHS (W. WESSEL)

The *thickness* $\Theta(G)$ of a graph G is the minimal number of planar subgraphs whose union is G . Let $\delta(G)$ and $\Delta(G)$ denote the minimal and maximal degrees of the vertices of G , respectively. One can easily prove that

$$\left\lceil \frac{\delta(G)+1}{6} \right\rceil \leq \Theta(G) \leq \left\lceil \frac{\Delta(G)}{2} \right\rceil.$$

I could show [2] that the lower bound is sharp for every value of δ .
CONJECTURE.

$$\Theta(G) \leq \left\lceil \frac{\Delta(G)+2}{4} \right\rceil.$$

Since $\Theta(K_{n,n}) = \lceil (n+2)/4 \rceil$ (see [1]), this upper bound would be sharp.

RESTRICTED PROBLEMS. Does there exist a 6- (or even a 5-) regular graph of thickness 3? Does there exist a 10- (9-, 8-, 7-) regular graph of thickness 4? etc.

References

- [1] L. W. Beineke, F. Harary and J. W. Moon, *On the thickness of the complete bipartite graph*, Proc. Cambridge Philos. Soc. 60 (1964), pp. 1-5.
- [2] W. Wessel, *Über die Abhängigkeit der Dicke eines Graphen von seinen Knotenpunktvalenzen*, 2. Kolloq. Geometrie und Kombinatorik, Karl-Marx-Stadt 1983 (Techn. Hochsch. Karl-Marx-Stadt, 1984), pp. 235-238.

