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BIAS-ROBUST ESTIMATES BASED ON ORDER STATISTICS AND SPACINGS IN THE EXPONENTIAL MODEL

1. Preliminaries. Let F, G, H, K, W denote distribution functions. We identify the distributions with the distribution functions. We assume that all considered distributions are strictly increasing and continuous on their common support $[0, \infty)$ and that all expectations in consideration exist and are finite. Let E_F denote the expectation under the distribution F .

We use the stochastic order relation $\stackrel{st}{\leq}$ for distributions: i.e. $F \stackrel{st}{\leq} G$ if and only if $F(x) \geq G(x)$ for every x . If X and Y are random variables which are distributed according to F and G , respectively, then we also write $X \stackrel{st}{\leq} Y$ if and only if $F \stackrel{st}{\leq} G$ holds.

Let $X_{1:n}, X_{2:n}, \dots, X_{n:n}$ be order statistics of a sample from a distribution F . The random variables $V_{1:n} = X_{1:n}$, $V_{i:n} = X_{i:n} - X_{i-1:n}$, $i = 2, 3, \dots, n$, are called *spacings from the distribution F* . Let W_λ be the exponential distribution, i.e. $W_\lambda(x) = W(x/\lambda) = 1 - e^{-x/\lambda}$, $x > 0$, $\lambda > 0$. It is well known that the normalizing spacings $(n-i+1)V_{i:n}$ from the distribution W_λ are independent identically distributed random variables with the distribution W_λ . Thus $E_{W_\lambda} V_{i:n} = \lambda/(n-i+1)$, $i = 1, 2, \dots, n$.

We use the following property of IFR and DFR distributions.

LEMMA 1 (Barlow and Proschan [1]). *If F is an IFR (DFR) distribution and $F(0) = 0$, then $(n-i+1)V_{i:n}$ is stochastically decreasing (increasing) in $i = 1, 2, \dots, n$, i.e.*

$$(n-i+1)V_{i:n} \stackrel{st}{\leq} (n-i)V_{i-1:n}, \quad i = 1, 2, \dots, n-1.$$

We use also the notion of dispersive ordering of distributions which has been introduced by Saunders and Moran [9] and detailed studied by Shaked

[10]. We say that the distribution F has a smaller dispersion than G ($F \stackrel{\text{disp}}{\leq} G$) if $F^{-1}(\beta) - F^{-1}(\alpha) \leq G^{-1}(\beta) - G^{-1}(\alpha)$ for each $0 < \alpha < \beta < 1$, where $F^{-1}(u) = \inf\{x: F(x) \geq u\}$, $u \in (0, 1)$ (for G^{-1} analogously). It is easy to notice that the dispersive ordering is a partial order relation in the class of distributions. The following lemmas give properties of the dispersive ordering which are used in the sequel.

LEMMA 2 (Shaked [10]). Let F and G be strictly increasing and continuous on their support $[0, \infty)$.

- (i) If $F \stackrel{\text{disp}}{\leq} G$ then $F \stackrel{\text{st}}{\leq} G$.
- (ii) Let F and G be absolutely continuous with corresponding densities f and g . If $F \stackrel{\text{st}}{\leq} G$ and if $f(x-c) - g(x)$ has for every $c > 0$ at most two sign changes with the sign sequence being $-, +, -$ in case of two changes, then $F \stackrel{\text{disp}}{\leq} G$.

LEMMA 3 (Bartoszewicz [3]). (i) Let $V_{1:n}, V_{2:n}, \dots, V_{n:n}$ and let $U_{1:n}, U_{2:n}, \dots, U_{n:n}$ be spacings from the distributions F and G respectively.

If $F \stackrel{\text{disp}}{\leq} G$, then $V_{i:n} \stackrel{\text{st}}{\leq} U_{i:n}$, $i = 1, 2, \dots, n$.

- (ii) If $W \stackrel{\text{st}}{\leq} F$ and F is an IFR or a DFR distribution, then $W \stackrel{\text{disp}}{\leq} F$.

2. Problem. Consider the problem of unbiased estimation of the scale parameter λ of the distribution W_λ based on a sample of size n . The appropriate statistical model is

$$M_0 = (R_1^+, \mathcal{B}_1^+, \{W_\lambda, \lambda > 0\})^n.$$

Suppose that the model M_0 is violated in such a way that the underlying random variables have instead of W_λ an unknown distribution F_λ with the scale parameter λ from the set of distributions $\Pi_{H,K}(W_\lambda)$ satisfying the following conditions:

- (i) $H_\lambda \stackrel{\text{disp}}{\leq} F_\lambda \leq K_\lambda$ for every $F_\lambda \in \Pi_{H,K}(W_\lambda)$, $\lambda > 0$, where H_λ and K_λ are some fixed continuous distributions with scale parameter λ , i.e., $H_\lambda(x) = H(x/\lambda)$ and $K_\lambda(x) = K(x/\lambda)$, and $H(0) = 0 = K(0)$;
- (ii) $H_\lambda \in \Pi_{H,K}(W_\lambda)$ and $K_\lambda \in \Pi_{H,K}(W_\lambda)$;
- (iii) $\Pi_{H,K}(W_{\lambda'}) \cap \{W_\lambda, \lambda > 0\} = \{W_{\lambda'}\}$ for every $\lambda' > 0$.

Let T be an unbiased estimate of λ in the model M_0 . If F_λ runs over the set $\Pi_{H,K}(W_\lambda)$, then

$$b_T(\lambda) = \sup_{F_\lambda \in \Pi_{H,K}(W_\lambda)} (E_{F_\lambda} T - \lambda) - \inf_{F_\lambda \in \Pi_{H,K}(W_\lambda)} (E_{F_\lambda} T - \lambda)$$

is the oscillation of the bias of T over $\Pi_{H,K}(W_\lambda)$ and gives us a measure of robustness of the estimate T with respect to its bias under violation Π . The function $\lambda \rightarrow b_T(\lambda)$, $\lambda > 0$, is called the *bias-robustness* of T (see Zieliński

[11]). The problem is to find T^* such that

$$1) \quad b_{T^*}(\lambda) \leq b_T(\lambda) \quad \text{for every } \lambda > 0$$

and every T in a given class of statistics. The estimate T^* for which (1) holds is called the *uniformly most bias-robust estimate* in the given class of statistics.

In the series of papers Zieliński [12], Bartoszewicz [2], [4], Zieliński and Zieliński [13] and Bartoszewicz and Zieliński [5] studied the problem of robust estimation of λ under some violations in the class of statistics

$$\mathcal{T}^+ = \{T(\alpha) = \sum_{i=1}^n \alpha_i X_{i:n}, \alpha_i \geq 0, E_{W_\lambda} T = \lambda, \lambda > 0\},$$

i.e. in the class of nonnegative linear combinations of order statistics which are unbiased estimates of λ in M_0 . Now we consider this problem under the violation $\Pi_{H,K}(W_\lambda)$ in the class \mathcal{T}^+ as well as in the class \mathcal{S}^+ of statistics defined as follows

$$\mathcal{S}^+ = \{S(\alpha) = \sum_{i=1}^n \alpha_i V_{i:n}, \alpha_i \geq 0, E_{W_\lambda} S = \lambda, \lambda > 0\}.$$

Thus \mathcal{S}^+ is the class of nonnegative linear combinations of spacings, unbiased estimates of λ in M_0 . It is easy to notice that $\mathcal{T}^+ \subset \mathcal{S}^+$, so \mathcal{S}^+ contains the sample mean

$$\bar{X} = \frac{1}{n} \sum_{i=1}^n (n-i+1) V_{i:n} = \frac{1}{n} \sum_{i=1}^n X_{i:n},$$

the minimum variance unbiased estimate of λ in M_0 based on the censored sample of r first observations

$$\frac{1}{n} \sum_{i=1}^r (n-i+1) V_{i:n} = \frac{1}{n} \sum_{i=1}^r X_{i:n} + (n-r) X_{r:n}, \quad r = 1, 2, \dots, n,$$

and also the unbiased estimate of λ based on the range

$$(X_{n:n} - X_{n-1:n}) / (1 + 1/2 + \dots + 1/(n-1)).$$

Observe that if $S \in \mathcal{S}^+$, then $b_S(\lambda) = \lambda b_S(1)$, so that the problem reduces to that of finding S^* minimizing $b_S(1)$. Such S^* would be the uniformly most bias-robust estimate of λ in the class \mathcal{S}^+ .

3. Results.

THEOREM 1. (i) If H is an IFRA distribution and K is a DFRA distribution, then the uniformly most bias-robust estimate of λ in the class \mathcal{T}^+ is $T^* = nX_{1:n}$.

(ii) If H is a DFRA distribution and K is an IFRA distribution, then the uniformly most bias-robust estimate of λ in the class \mathcal{T}^+ is

$$T^* = X_{n:n} / (1 + 1/2 + \dots + 1/n).$$

Proof. Since from Lemma 2 (i) we have $H \stackrel{st}{\leq} F \stackrel{st}{\leq} K$ for $F \in \Pi_{H,K}(W)$, the proof runs similarly as the proof of Theorems 1 and 2 in [5].

THEOREM 2. (i) *If H is an IFR distribution and K is a DFR distribution, then the uniformly most bias-robust estimate of λ in the class \mathcal{S}^+ is $S^* = nV_{1:n}$.*

(ii) *If H is a DFR distribution and K is an IFR distribution, then the uniformly most bias-robust estimate of λ in the class \mathcal{S}^+ is $S^* = V_{n:n}$.*

Proof. It follows from Lemma 3 (i) and the property of stochastic ordering that if $F \in \Pi_{H,K}(W)$, then $E_H V_{i:n} \leq E_F V_{i:n} \leq E_K V_{i:n}$. Hence for each $S \in \mathcal{S}^+$ we have

$$\sup_{F \in \Pi_{H,K}(W)} E_F S(\alpha) = \sum_{i=1}^n \alpha_i E_K V_{i:n}$$

and

$$\inf_{F \in \Pi_{H,K}(W)} E_F S(\alpha) = \sum_{i=1}^n \alpha_i E_H V_{i:n}.$$

The problem of finding S^* reduces to that of finding nonnegative $\alpha_1, \alpha_2, \dots, \alpha_n$ which minimize

$$\sum_{i=1}^n \alpha_i (E_K V_{i:n} - E_H V_{i:n})$$

under the condition of S^* being unbiased in M_0 , i.e.

$$\sum_{i=1}^n \alpha_i E_W V_{i:n} = 1.$$

This is a linear programming problem and the solution is $(\alpha_1, \alpha_2, \dots, \alpha_n)$ with exactly one non-zero coordinate. Hence $S^* = V_{i^*:n}/E_W V_{i^*:n}$ where i^* minimizes

$$(E_K V_{i:n} - E_H V_{i:n})/E_W V_{i:n} = (n-i+1)(E_K V_{i:n} - E_H V_{i:n}).$$

It follows from Lemma 1 that if H is an IFR distribution and K is a DFR distribution, then $i^* = 1$, which ends the proof of Theorem 2 (i). Under the hypothesis of Theorem 2 (ii) we have $i^* = n$.

4. Examples. 1. Zieliński [12] has considered the model M_0 with the following violation

$$\Pi_1(W_\lambda) = \{W_{\lambda,p}: p_1 \leq p \leq p_2, 0 < p_1 \leq 1 \leq p_2 \leq 2.16\}, \quad \lambda > 0,$$

where $W_{\lambda,p}$ is the exponential-power distribution with the probability density function

$$f_{\lambda,p}(x) = \frac{e^{-(x/\lambda)^p}}{\lambda \Gamma(1+1/p)}, \quad x > 0.$$

It is easy to prove that $W_{\lambda,p}$ is a DFR distribution for $0 < p \leq 1$ and an IFR distribution for $p \geq 1$. Bartoszewicz [4] has proved that if $0 < p < p' \leq 1$,

then $W_{1,p'} \stackrel{\text{disp}}{\leq} W_{1,p}$. Now let $1 \leq p < p' \leq 2.16$. Consider for every $c > 0$

$$(2) \quad f_{1,p'}(x-c) - f_{1,p}(x) = \frac{e^{-(x-c)p'}}{\Gamma(1+1/p)} \left[\frac{\Gamma(1+1/p)}{\Gamma(1+1/p')} - e^{-x^p + (x-c)p'} \right], \quad x \geq c.$$

The sign of (2) is the same as the sign of

$$\xi(x) = \frac{\Gamma(1+1/p)}{\Gamma(1+1/p')} - e^{-x^p + (x-c)p'}.$$

Notice that $\xi(c) > 0$ and

$$(3) \quad \frac{d}{dx} \xi(x) = \exp \{ -x^p + (x-c)p' [px^{p-1} - p'(x-c)^{p'-1}] \}$$

From (3) it follows that ξ has one maximum and hence $f_{1,p'}(x-c) - f_{1,p}(x)$ has at most one sign change for $x \geq c$. Since $W_{1,p'} \stackrel{\text{st}}{\leq} W_{1,p}$ for $1 \leq p < p' \leq 2.16$ (see [12]), then from Lemma 2 (ii) it follows that $W_{1,p'} \stackrel{\text{disp}}{\leq} W_{1,p}$.

From the transitivity of the dispersive ordering we obtain that $W_{1,p'} \stackrel{\text{disp}}{\leq} W_{1,p}$ for $0 < p < p' \leq 2.16$. Thus from Theorems 1 (i) and 2 (i) it follows that the statistic $nX_{1:n}$ is the uniformly most bias-robust estimate of λ in the class \mathcal{T}^+ as well as in \mathcal{S}^+ .

2. Bartoszewicz [2] has considered the gamma violation of the model M_0 :

$$\Pi_2(W_\lambda) = \{W_{\lambda,p}^* : p_1 \leq p \leq p_2, 0 < p_1 \leq 1 \leq p_2 < \infty\}, \quad \lambda > 0,$$

where $W_{\lambda,p}^*$ is the gamma distribution with the probability density function

$$f_{\lambda,p}^*(x) = \frac{x^{p-1} e^{-x/\lambda}}{\lambda^p \Gamma(p)}, \quad x > 0.$$

Saunders and Moran [9] have proved that if $p_1 < p < p_2$, then

$$W_{\lambda,p_1}^* \stackrel{\text{disp}}{\leq} W_{\lambda,p}^* \stackrel{\text{disp}}{\leq} W_{\lambda,p_2}^* \quad \text{for each } \lambda > 0.$$

It is also well known that W_{λ,p_1}^* is a DFR distribution and W_{λ,p_2}^* is an IFR distribution. Thus the statistic $T^* = X_{n:n}/(1+1/2+\dots+1/n)$ is the uniformly most bias-robust estimate of λ in the class \mathcal{T}^+ and the estimate $S^* = V_{n:n}$ is the uniformly most bias-robust in the class \mathcal{S}^+ . Since $\mathcal{T}^+ \subset \mathcal{S}^+$, we have

$$(4) \quad b_{S^*}(1) \leq b_{T^*}(1).$$

Using the tables [6] and [8] of moments of order statistics from the gamma distribution, we compare the values of bias-robustness of estimates \bar{X} , T^* and S^* for some values of p_1 and p_2 and small n . Let us denote $b_T^{(p_1, p_2)} = E_{W_{1, p_2}^*} T - E_{W_{1, p_1}^*} T$. It is easy to notice that $b_{\bar{X}}^{(p_1, p_2)} = p_2 - p_1$ and it does not depend on n . Gurland [7] has proved that if $X_{n:n}$ is the order statistic of a sample from the gamma distribution $W_{1, p}^*$, then

$$\lim_{n \rightarrow \infty} P \left\{ X_{n:n} - \log \frac{n}{\Gamma(p)} - (p-1) \log \log n \leq x \right\} = e^{-e^{-x}}, \quad -\infty < x < \infty.$$

Hence (see [8], Chapter 7)

$$\lim_{n \rightarrow \infty} \left[E_{W_{1, p}^*} X_{n:n} - \log \frac{n}{\Gamma(p)} - (p-1) \log \log n \right] = C,$$

where $C = 0.57721566\dots$ is Euler's constant. Therefore

$$\lim_{n \rightarrow \infty} b_T^{(p_1, p_2)} = \lim_{n \rightarrow \infty} \frac{\log \Gamma(p_1) - \log \Gamma(p_2) + (p_2 - p_1) \log \log n}{1 + 1/2 + \dots + 1/n} = 0$$

for every $0 < p_1 < p_2$. From (4) we have also $\lim_{n \rightarrow \infty} b_{S^*}^{(p_1, p_2)} = 0$. The values of $b_{\bar{X}}$, b_{T^*} and b_{S^*} for some p_1 , p_2 and n are given in Table 1.

TABLE 1

n	$b_T^{(p_1, p_2)}$									
	$p_1 = 0.5, p_2 = 1$		$p_1 = 1, p_2 = 1.5$		$p_1 = 0.5, p_2 = 1.5$		$p_1 = 0.5, p_2 = 2$		$p_1 = 1, p_2 = 2$	
	T^*	S^*	T^*	S^*	T^*	S^*	T^*	S^*	T^*	S^*
2	0.454	0.363	0.424	0.273	0.879	0.637	1.288	0.863	0.833	0.500
3	0.427	0.301	0.386	0.241	0.813	0.515	1.179	0.690	0.753	0.389
4	0.407	0.265	0.362	0.186	0.769	0.415	1.110	0.602	0.703	0.337
5	0.393	0.241	0.345	0.168	0.738	0.428	1.092	0.546	0.668	0.305
9	0.357	0.193	0.307	0.136	0.664	0.326	1.060	0.440	0.590	0.247
∞	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
$b_{\bar{X}}$	0.500		0.500		1.000		1.500		1.000	

From Table 1 we can conclude that bias-robustness of S^* is considerably smaller than that of T^* and \bar{X} . It is also worth noticing that b_{S^*} converges to zero very rapidly. Unfortunately, $\text{Var}_W S^* = 1$ for every n and hence the estimate S^* is inconsistent in the model M_0 , contrary to the estimates \bar{X} and T^* . Therefore using S^* we gain in robustness but loose in efficiency.

References

- [1] R. E. Barlow and F. Proschan, *Inequalities for linear combinations of order statistics from restricted families*, Ann. Math. Statist. 37 (1966), p. 1574-1592.
- [2] J. Bartoszewicz, *On the most bias-robust linear estimates of the scale parameter of the exponential distribution*, Zastos. Mat. 18 (1984), p. 251-255.
- [3] —, *Moment inequalities for order statistics from ordered families of distributions*, Metrika 32 (1985), p. 383-389.
- [4] —, *Mean-square-error-robustness of linear estimates in the exponential model*, Zastos. Mat. 18 (1985), p. 597-608.
- [5] — and R. Zieliński, *A bias-robust estimate of the scale parameter of the exponential distribution under violation of the hazard function*, ibid. 18 (1985), p. 609-612.
- [6] M. C. Breiter and P. R. Krishnaiah, *Tables for the moments of gamma order statistics*, Sankhyā B 30 (1968), p. 59-72.
- [7] J. Gurland, *Distribution of the maxima of the arithmetic mean of correlated random variables*, Ann. Math. Statist. 26 (1955), p. 294-300.
- [8] A. E. Sarhan and B. G. Greenberg (Eds.), *Contributions to Order Statistics*, New York 1962.
- [9] I. W. Saunders and P. A. P. Moran, *On the quantiles of the gamma and F distribution*, J. Appl. Prob. 15 (1978), p. 426-432.
- [10] M. Shaked, *Dispersive ordering of distributions*, ibid. 19 (1982), p. 310-320.
- [11] R. Zieliński, *Robust statistical procedures: a general approach*; in: *Stability problems for stochastic models*, Lecture Notes in Mathematics 982 (edited by V. V. Kalashnikov and V. M. Zolotarev), Springer Verlag, Berlin 1983.
- [12] —, *A most bias-robust linear estimate of the scale parameter of the exponential distribution*, Zastos. Mat. 18 (1983), p. 73-77.
- [13] — and W. Zieliński, *On robust estimation in the simplest exponential model*, ibid. 18 (1984), p. 387-401.

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