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DECISION MAKING IN AN INCOMPLETELY KNOWN STOCHASTIC SYSTEM, I

1. Introduction. In the paper a Bayes control policy of the system (6) is determined in the case when the performance index depends on the parameter of disturbances and is measured in terms of their variance. It is assumed that the horizon of control is a bounded random variable with known distribution and that the disturbances have a distribution belonging to the exponential family. The Bayes risk for the optimal control is determined. The paper is also an introduction to the paper [5] in which the exact analytical form of a minimax control policy of the system is found.

The problem of determining a Bayes control policy of parameter adaptive control systems for disturbances with distributions belonging to the exponential family was considered in [2], [3], [4]. Followed by an exposition concerning the exponential class of distributions (Section 2) and by some remarks concerning the problem stated (Section 3), in Section 4 the problem of filtering and in Section 5 that of determining a Bayes control is solved.

2. Exponential class of probability distributions. In probability theory and statistics the exponential class of probability distributions is often considered. In this section we present some of its properties. The exposition is based mainly on the paper [1].

Definition. Let $\Lambda \subseteq R \cup \{\pm\infty\}$ with $\operatorname{card}\Lambda \geqslant 2$ and let $\mathcal{F}_\Lambda = \{P_\lambda\colon \lambda \in \Lambda\}$ be a family of probability distributions on R. The family $\mathcal{F}_\Lambda$ is called an *exponential family of probability distributions* if there are a normed measure μ on R, two real-valued functions $A(\cdot)$, $B(\cdot)$ on Λ, with $B(\lambda') \neq B(\lambda'')$ if $\lambda' \neq \lambda''$, and a constant $q > 0$ such that

$$dP_\lambda(v) = \exp\{A(\lambda)q + B(\lambda)v\}\,d\mu(v). \tag{1}$$

The set of all probability measures μ belonging to the exponential family is called the *exponential class of distributions*.

For each probability measure μ let

$$Z_\mu = \{z \in R,\ \varphi_\mu(z) < \infty\},$$

where

$$\varphi_\mu(z) = \int_R \exp(zv)\,\mu(dv).$$

The function $\varphi_\mu(z)$ is convex, thus Z_μ is an interval which includes zero.

Define

$$dP_z(v) = \frac{\exp(zv)\,d\mu(v)}{\int_R \exp(zv)\,\mu(dv)} = \exp\big(-y_\mu(z)+zv\big)\,d\mu(v), \tag{2}$$

where

$$y_\mu(z) = -\ln \int_R \exp(zv)\,\mu(dv).$$

The family

$$\mathscr{F}_{Z_\mu} = \{P_z\colon\ z \in Z_\mu\}$$

is an exponential family. Representation (2) is called the *canonical representation of the exponential family.*

Sometimes it is convenient to have a representation by a parameter λ which has a statistical interpretation, for example λ is the mean value of P_λ. It can be proved that for any exponential family of distributions having the canonical representation (2) with Z_μ nondegenerate there exists a representation (1) such that

(3) $B(\cdot)$ is strictly increasing on Λ, $B(\Lambda) = Z_\mu$,

(4) $A(\cdot)$ and $B(\cdot)$ are holomorphic on $\Lambda_0 = (\lambda_1, \lambda_2)$, where λ_1 and λ_2 are end points of Λ,

(5) for each $\lambda \in \Lambda_0$

$$\int_R vP_\lambda(dv) = q\lambda = -q\frac{A'(\lambda)}{B'(\lambda)},$$

$$\int_R v^2\,P_\lambda(dv) - \Big(\int_R vP_\lambda(dv)\Big)^2 = \frac{q}{B'(\lambda)}.$$

We assume that the exponential family has a canonical representation with Z_μ nondegenerate.

3. The Bayes control problem. Let us consider the discrete linear system with complete observations and random horizon

$$x_{n+1} = \alpha_n x_n + u_n + \gamma_n v_n, \qquad x_0 = e, \qquad n = 0, 1, \ldots, N, \tag{6}$$

where x_n is the state variable, u_n is the control, $v_0, v_1, \ldots,$ are independent random variables with the same distribution, N is a random variable independent of $v_0, v_1, \ldots$ with given distribution

$$P(N = i) = p_i, \quad i = 1, 2, \ldots, M, \quad p_M > 0, \quad \sum_{i=1}^{M} p_i = 1, \tag{7}$$

α_n, γ_n, e are given constants, $\gamma_n \neq 0$.

It is assumed that the random variables $v_0, v_1, \ldots$ have the distribution P_λ belonging to the exponential family, i.e. its density with respect to a σ-finite measure μ_0 on R is

$$p(v, \lambda) = S(v, q)\exp[qA(\lambda)+vB(\lambda)], \tag{8}$$

where λ is a parameter. We suppose that the parametrization is chosen so as to satisfy the conditions (3)–(5).

We assume that $p(v, \lambda)$ is known with the only exception of the parameter $\lambda \in \Lambda_0$.

Let us formulate now the problem of Bayes control. Suppose that λ is a random variable. Let the distribution π of the parameter λ have the density

$$g(\lambda; \beta, r) = D(\beta, r)\exp[\beta A(\lambda)+rB(\lambda)], \quad \lambda \in \Lambda_0. \tag{9}$$

Given the initial state e, the a priori distribution π of the parameter λ with the density $g(\lambda; \beta, r)$ and the distribution P of N, $P(N \leqslant M) = 1$, choose controls u_n, $n = 0, 1, \ldots, N$, based on all available data $X_n = (x_0, x_1, \ldots, x_n)$ and $U_{n-1} = (u_0, u_1, \ldots, u_{n-1})$ such that the Bayes risk

$$\begin{aligned} (10)\quad & r(\pi, U) \\ &= \mathrm{E}_N\{\mathrm{E}_\pi\{\mathrm{E}_\lambda[B'(\lambda)\sum_{i=0}^{N}(\xi_i x_i^2+2\eta_i x_i \lambda+\zeta_i \lambda^2+k_i u_i^2) \mid X_0]\}\} \\ &= \mathrm{E}_N\{\mathrm{E}[B'(\lambda)\sum_{i=0}^{N}(\xi_i x_i^2+2\eta_i x_i \lambda+\zeta_i \lambda^2+k_i u_i^2) \mid X_0]\} \end{aligned}$$

attains its minimum.

Here E_λ denotes the expectation with respect to the distribution P_λ of the random variables $v_0, v_1, \ldots, v_N$ (for $X_0 = x_0 = e$ given), E_π the expectation with respect to the distribution π of the parameter λ, E_N the expectation with respect to the distribution (7) of the random variable N, and $U = (u_0, u_1, \ldots, u_M)$ a control policy.

It is assumed that

$$\xi_i > 0, \quad \xi_i \zeta_i - \eta_i^2 \geqslant 0, \quad k_i > 0.$$

The Bayes risk $r(\pi, U)$ attains its minimum if for $n = M, M-1, \ldots, 1, 0$

the expression

(11) $$r_n(\pi, U^{(n)}) = \mathrm{E}_N\{\mathrm{E}[B'(\lambda)\sum_{i=n}^{N}(\xi_i x_i^2+2\eta_i x_i\lambda+\zeta_i\lambda_i^2+k_i u_i^2)\mid X_n, U_{n-1}]\mid N\geqslant n\}$$

where $\mathrm{E}_N\{\cdot\mid N\geqslant n\}$ denotes the conditional expectation under the condition $N\geqslant n$, $\mathrm{E}[\cdot\mid X_n, U_{n-1}]$ denotes the conditional expectation given X_n, U_{n-1}, $U^{(n)}=(u_n, \ldots, u_M)$, attains its minimum.

Since $D_\lambda^2(v_n)=q/B'(\lambda)$, this means that the performance index

$$J=\sum_{i=0}^{N}(\xi_i x_i^2+2\eta_i x_i\lambda+\zeta_i\lambda^2+k_i u_i^2)$$

is measured in terms of the variance of the random variables v_n.

4. Filtering. Let us suppose that the a priori density of the parameter λ is given by (9). When such an a priori density is assigned to λ then the object of filtering, to determine the Bayes control, is to produce the a posteriori density for λ after any new observation of x. We change the control after obtaining the new data.

Having observed x_1 and for u_0 given, we determine the value v of the random variable v_0, then the density $f(\lambda\mid X_1)$ of the parameter λ given X_1, U_{n-1} (equal to the density $f(\lambda\mid v_0=v)$ of λ given $v_0=v$) may be calculated according to the Bayes rule

$$f(\lambda\mid X_1)=f(\lambda\mid v_0=v)=\frac{p(v,\lambda)g(\lambda;\beta,r)}{\int_{\Lambda_0}p(v,\lambda)g(\lambda;\beta,r)\,d\lambda}$$
$$=\frac{\exp[(\beta+q)A(\lambda)+(r+v)B(\lambda)]}{\int_{\Lambda_0}\exp[(\beta+q)A(\lambda)+(r+v)B(\lambda)]\,d\lambda}$$
$$=D(\beta+q, r+v)\exp[(\beta+q)A(\lambda)+(r+v)B(\lambda)]=g(\lambda;\beta_1, r_1),$$

where

$$\beta_1=\beta+q, \qquad r_1=r+v_0.$$

Similarly, after x_n is measured and U_{n-1} is given, the a posteriori density of parameter λ is

(12) $$f(\lambda\mid X_n, U_{n-1})=\frac{p(v_{n-1},\lambda)g(\lambda;\beta_{n-1}, r_{n-1})}{\int_{\Lambda_0}p(v_{n-1},\lambda)g(\lambda;\beta_{n-1}, r_{n-1})\,d\lambda}$$
$$=D(\beta_{n-1}+q, r_{n-1}+v_{n-1})\exp[(\beta_{n-1}+q)A(\lambda)+(r_{n-1}+v_{n-1})B(\lambda)]$$
$$=g(\lambda;\beta_n, r_n).$$

where

(13) $$\beta_n = \beta_{n-1} + q, \qquad r_n = r_{n-1} + v_{n-1}.$$

Given X_n and U_{n-1}, the conditional density of the random variable v_n is

(14) $$h(v \mid X_n, U_{n-1}) = \int_{\Lambda_0} p(v, \lambda) g(\lambda; \beta_n, r_n) d\lambda$$
$$\overset{(12)}{=} \frac{p(v, \lambda) g(\lambda; \beta_n, r_n)}{g(\lambda; \beta_{n+1}, r_{n+1})} = \frac{S(v, q) D(\beta_n, r_n)}{D(\beta_n + q, r_n + v)}.$$

If the a priori density of the parameter λ is

(15) $$\bar{g}(\lambda; \beta, r) = C(\beta, r) B'(\lambda) \exp[\beta A(\lambda) + r B(\lambda)],$$

then

$$\bar{f}(\lambda \mid X_n, U_{n-1})$$
$$= C(\beta_{n-1} + q, r_{n-1} + v_{n-1}) B'(\lambda) \exp[(\beta_{n-1} + q) A(\lambda) + (r_{n-1} + v_{n-1}) B(\lambda)]$$
$$= \bar{q}(\lambda; \beta_n, r_n),$$

where equations (13) hold and, moreover,

(16) $$\bar{h}(v \mid X_n, U_{n-1}) = \frac{S(v, q) C(\beta_n, r_n)}{C(\beta_n + q, r_n + v)}.$$

Let, for the family (8) $\Lambda_0 = (\lambda_1, \lambda_2)$. Suppose that there is a nonempty set S of all points $(\beta, r) \in R^2$ for which the following condition holds:

(17) $$\int_{\Lambda_0} \exp[\beta A(\lambda) + r B(\lambda)] d\lambda = \frac{1}{S(r, \beta)(\beta - \alpha)}$$

for some $\alpha \in R$.

All below considered limits are finite and

$$\lim_{\lambda \to \lambda_1^+} \exp[\beta A(\lambda) + r B(\lambda)] = \lim_{\lambda \to \lambda_2^-} \exp[\beta A(\lambda) + r B(\lambda)],$$

(18)

$$\lim_{\lambda \to \lambda_1^+} \lambda \exp[\beta A(\lambda) + r B(\lambda)] = \lim_{\lambda \to \lambda_2^-} \lambda \exp[\beta A(\lambda) + r B(\lambda)],$$

(19) $$\int_{\Lambda_0} B'(\lambda) \exp[\beta A(\lambda) + r B(\lambda)] d\lambda < \infty,$$

(20) if $(\beta, r) \in S$ then $(\beta + q, r + v) \in S$ for $q = \mathrm{E}_\lambda(v_n)/\lambda$ and each $v \in \operatorname{supp} P_\lambda$.

All the above assumptions hold for distributions that are most frequently involved in probability theory, for example, for the binomial, normal, gamma, Poisson and negative binomial distributions.

For the binomial distribution

$$p(v, \lambda) = \binom{q}{v} \lambda^v (1-\lambda)^{q-v},$$

$\alpha = -1$, the set S is defined by the conditions $r > 0$, $\beta - r > 0$, $S(r, \beta) = \Gamma(\beta + 1)/\Gamma(r+1)\Gamma(\beta - r + 1)$.

For the normal distribution (with variance equal to 1)

$$p(v, \lambda) = \frac{1}{\sqrt{2\pi}} e^{-(v-\lambda)^2/2},$$

$\alpha = 0$, $\beta > 0$, $-\infty < r < \infty$, $S(r, \beta) = (1/\sqrt{2\pi\beta})\, e^{-r^2/2\beta}$.

For the Poisson distribution

$$p(v, \lambda) = \frac{\lambda^v}{v!} e^{-\lambda},$$

$\alpha = 0$, $\beta > 0$, $r > 0$, $S(r, \beta) = \beta^r/\Gamma(r+1)$.

For the gamma distribution

$$p(v, \lambda) = \frac{1}{\Gamma(q)\lambda^q} v^{q-1} e^{-v/\lambda},$$

$\alpha = 1$, $\beta > 1$, $r > 0$, $S(r, \beta) = r^{\beta-1}/\Gamma(\beta)$.

For the negative binomial distribution

$$p(v, \lambda) = \frac{\Gamma(q+v)}{\Gamma(q)\, v!} \cdot \frac{\lambda^v}{(1+\lambda)^{q+v}},$$

$\alpha = 1$, $\beta > 1$, $r > 0$, $S(r, \beta) = \Gamma(\beta + r)/\Gamma(\beta)\Gamma(r+1)$.

Integrating by parts and taking into account the conditions (17)–(19) we obtain

$$\beta \int_{\Lambda_0} \lambda B'(\lambda) \exp[\beta A(\lambda) + rB(\lambda)]\, d\lambda = r \int_{\Lambda_0} B'(\lambda) \exp[\beta A(\lambda) + rB(\lambda)]\, d\lambda, \tag{21}$$

$$\int_{\Lambda_0} (r - \lambda\beta)^2 B'(\lambda) \exp[\beta A(\lambda) + rB(\lambda)]\, d\lambda = \beta/S(r, \beta)(\beta - \alpha) \tag{22}$$

for each $(\beta, r) \in S$.

Moreover, from (9) and (17) it follows that

$$D(\beta, r) = S(r, \beta)(\beta - \alpha) \tag{23}$$

for each $(\beta, r) \in S$.

Let the a priori distribution of the parameter λ be given by (9) when $(\beta, r) \in S$. Then from (12) and (15) we have

(24) $$\mathrm{E}(B'(\lambda) | X_n, U_{n-1}) = D(\beta_n, r_n) \int_{\Lambda_0} B'(\lambda) \exp[\beta_n A(\lambda) + r_n B(\lambda)] \, d\lambda = \frac{D(\beta_n, r_n)}{C(\beta_n, r_n)}.$$

Moreover, from (21)

(25) $$\begin{aligned} \mathrm{E}(\lambda B'(\lambda) | X_n, U_{n-1}) &= D(\beta_n, r_n) \int_{\Lambda_0} \lambda B'(\lambda) \exp[\beta_n A(\lambda) + r_n B(\lambda)] \, d\lambda \\ &= D(\beta_n, r_n) \frac{r_n}{\beta_n} \int_{\Lambda_0} B'(\lambda) \exp[\beta_n A(\lambda) + r_n B(\lambda)] \, d\lambda \\ &= \frac{D(\beta_n, r_n)}{C(\beta_n, r_n)} \cdot \frac{r_n}{\beta_n} \end{aligned}$$

and from (22) and (23)

(26) $$\begin{aligned} &\mathrm{E}(\lambda^2 B'(\lambda) | X_n, U_{n-1}) \\ &= D(\beta_n, r_n) \int_{\Lambda_0} \lambda^2 B'(\lambda) \exp[\beta_n A(\lambda) + r_n B(\lambda)] \, d\lambda \\ &= D(\beta_n, r_n) \int_{\Lambda_0} \left[\frac{1}{\beta_n^2} (r_n - \lambda \beta_n)^2 + 2 \frac{r_n}{\beta_n} \lambda - \frac{r_n^2}{\beta_n^2} \right] \exp[\beta_n A(\lambda) + r_n B(\lambda)] \, d\lambda \\ &= \frac{1}{\beta_n} + \frac{D(\beta_n, r_n)}{C(\beta_n, r_n)} \cdot \frac{r_n^2}{\beta_n^2}. \end{aligned}$$

The existence of the integrals in (24)–(26) follows from the condition (20). Suppose that

(27) $$\mathrm{E}_\lambda(v_n^2) = q_1 \lambda^2 + q_2 \lambda + q_3.$$

Since $\mathrm{E}_\lambda(v_n) = q\lambda$ and $D_\lambda^2(v_n) = q/B'(\lambda)$, we have

$$B'(\lambda) = \frac{q}{(q_1 - q^2)\lambda^2 + q_2 \lambda + q_3},$$

and

$$\begin{aligned} q &= \mathrm{E}\left[\left((q_1 - q^2)\lambda^2 + q_2 \lambda + q_3\right) B'(\lambda) \,|\, X_n, U_{n-1}\right] \\ &= \left[(q_1 - q^2) \frac{r_n^2}{\beta_n^2} + q_2 \frac{r_n}{\beta_n} + q_3 \right] \frac{D(\beta_n, r_n)}{C(\beta_n, r_n)} + \frac{q_1 - q^2}{\beta_n}. \end{aligned}$$

Thus

$$\frac{D(\beta_n, r_n)}{C(\beta_n, r_n)} = \frac{\beta_n q - q_1 + q^2}{\beta_n} \cdot \frac{1}{(q_1 - q^2)\frac{r_n^2}{\beta_n^2} + q_2 \frac{r_n}{\beta_n} + q_3}, \tag{28}$$

$$(n = 0, 1, 2, \ldots).$$

By (26) and (28) we may put

$$\mathrm{E}\left(\lambda^2 B'(\lambda) \mid X_n, U_{n-1}\right) = \frac{D(\beta_n, r_n)}{C(\beta_n, r_n)} (T_1^{(n)} r_n^2 + T_2^{(n)} r_n + T_3^{(n)}),$$

where the coefficients $T_i^{(n)}$ do not depend on r_n (but depend on β_n).

Moreover, as in [3], we assume that

$$\begin{aligned} \bar{\mathrm{E}}(v_n \mid X_n, U_{n-1}) &= Q^{(n)} r_n = q \frac{r_n}{\beta_n}, \\ \bar{\mathrm{E}}(v_n^2 \mid X_n, U_{n-1}) &= Q_1^{(n)} r_n^2 + Q_2^{(n)} r_n + Q_3^{(n)}, \end{aligned} \tag{29}$$

where $\bar{\mathrm{E}}(\cdot \mid X_n, U_{n-1})$ is the expectation with respect to the distribution (16).

Suppositions (27) and (29) also hold for all special distributions mentioned before.

From (13), (14) and (16) we obtain

$$\begin{aligned} &\mathrm{E}\left(\frac{D(\beta_{n+1}, r_{n+1})}{C(\beta_{n+1}, r_{n+1})} x_{n+1} \mid X_n, U_{n-1}\right) \\ &= \int_{-\infty}^{\infty} \frac{D(\beta_{n+1}, r_{n+1})}{C(\beta_{n+1}, r_{n+1})} \cdot \frac{S(v, q) D(\beta_n, r_n)}{D(\beta_{n+1}, r_{n+1})} x_{n+1}\, \mu_0(dv) \\ &= \frac{D(\beta_n, r_n)}{C(\beta_n, r_n)} \int_{-\infty}^{\infty} \frac{S(v, q) C(\beta_n, r_n)}{C(\beta_n + q, r_n + v_n)} (\alpha_n x_n + u_n + \gamma_n v)\, \mu_0(dv) \\ &= \frac{D(\beta_n, r_n)}{C(\beta_n, r_n)} [\alpha_n x_n + u_n + \gamma_n \bar{\mathrm{E}}(v_n \mid X_n, U_{n-1})] \\ &= \frac{D(\beta_n, r_n)}{C(\beta_n, r_n)} [\alpha_n x_n + u_n + \gamma_n Q^{(n)} r_n]. \end{aligned} \tag{30}$$

Since $r_{n+1} = r_n + v_n$, we obtain also

$$\mathrm{E}\left(\frac{D(\beta_{n+1}, r_{n+1})}{C(\beta_{n+1}, r_{n+1})} r_{n+1} \mid X_n, U_{n-1}\right) = \frac{D(\beta_n, r_n)}{C(\beta_n, r_n)} (1 + Q^{(n)}) r_n, \tag{31}$$

$$\begin{aligned} &\mathrm{E}\left(\frac{D(\beta_{n+1}, r_{n+1})}{C(\beta_{n+1}, r_{n+1})} x_{n+1}^2 \mid X_n, U_{n-1}\right) \\ &= \frac{D(\beta_n, r_n)}{C(\beta_n, r_n)} [(\alpha_n x_n + u_n)^2 + 2\gamma_n (\alpha_n x_n + u_n) Q^{(n)} r_n + \gamma_n^2 (Q_1^{(n)} r_n^2 + Q_2^{(n)} r_n + Q_3^{(n)})], \end{aligned} \tag{32}$$

(33) $$\mathrm{E}\left(\frac{D(\beta_{n+1}, r_{n+1})}{C(\beta_{n+1}, r_{n+1})} r_{n+1}^2 \mid X_n, U_{n-1}\right) = \frac{D(\beta_n, r_n)}{C(\beta_n, r_n)}\left[(1+2Q^{(n)}+Q_1^{(n)}) r_n^2 + Q_2^{(n)} r_n + Q_3^{(n)}\right],$$

(34) $$\mathrm{E}\left(\frac{D(\beta_{n+1}, r_{n+1})}{C(\beta_{n+1}, r_{n+1})} x_{n+1} r_{n+1} \mid X_n, U_{n-1}\right)$$
$$= \frac{D(\beta_n, r_n)}{C(\beta_n, r_n)}\left[\gamma_n (Q^{(n)}+Q_1^{(n)}) r_n^2 + \left((\alpha_n x_n + u_n)(1+Q^{(n)}) + \gamma_n Q_2^{(n)}\right) r_n + \gamma_n Q_3^{(n)}\right].$$

5. Solution of the Bayes control problem. Suppose that the disturbances v_n have the distributions given by (8), that the a priori distribution π of the parameter λ is given by (9), and that the conditions (3)–(5), (17)–(20), (27) and (29) hold. Let the distribution of the random horizon N be given by (7). Consider the problem of determining the Bayes control policy $U = (u_0, u_1, \ldots, u_M)$ which minimizes the Bayes risk $r(\pi, U)$ defined by (10). It is sufficient to minimize, for $n = M, \ldots, 1, 0$, the function $r_n(\pi, U^{(n)})$ defined by (11) which can be presented in the form

$$r(\pi, U^{(n)}) = \mathrm{E}_N \left\{\mathrm{E}\left[B'(\lambda) \sum_{i=n}^{N} (\xi_i x_i^2 + 2\eta_i x_i \lambda + \zeta_i \lambda^2 + k_i u_i^2) \mid X_n, U_{n-1}\right] \mid N \geqslant n\right\}$$
$$= \sum_{k=n}^{M} \mathrm{E}\left[B'(\lambda) \sum_{i=n}^{k} (\xi_i x_i^2 + 2\eta_i x_i \lambda + \zeta_i \lambda^2 + k_i u_i^2) \mid X_n, U_{n-1}\right] \frac{p_k}{\pi_k}$$
$$= \mathrm{E}\left[B'(\lambda) \sum_{i=n}^{M} \frac{\pi_i}{\pi_n} (\xi_i x_i^2 + 2\eta_i x_i \lambda + \zeta_i \lambda^2 + k_i u_i^2) \mid X_n, U_{n-1}\right],$$

where

$$\pi_k = \sum_{i=k}^{M} p_i.$$

Let

$$W_n = \min_{U^{(n)}} r_n(\pi, U^{(n)}) = r_n(\pi, U^{(n)*}),$$

where $U^{(n)*} = (u_n^*, u_{n+1}^*, \ldots, u_M^*)$. From Bellman's dynamic programming optimality principle we obtain

(35) $$W_n = \min_{u_n} \left\{\mathrm{E}\left[B'(\lambda)(\xi_n x_n^2 + 2\eta_n x_n \lambda + \zeta_n \lambda^2 + k_n u_n^2) \mid X_n, U_{n-1}\right]\right.$$
$$\left. + \min_{u_{n+1}, \ldots, u_M} \mathrm{E}\left[B'(\lambda) \sum_{i=n+1}^{M} \frac{\pi_i}{\pi_n} (\xi_i x_i^2 + 2\eta_i x_i \lambda + \zeta_i \lambda^2 + k_i u_i^2) \mid X_n, U_{n-1}\right]\right\}$$

$$= \min_{u_n} \left\{ \frac{D(\beta_n, r_n)}{C(\beta_n, r_n)} \left[\xi_n x_n^2 + \frac{2\eta_n}{\beta_n} x_n r_n + \zeta_n (T_1^{(n)} r_n^2 + T_2^{(n)} r_n + T_3^{(n)}) + k_n u_n^2 \right] \right.$$

$$+ \min_{u_{n+1},\ldots,u_M} \mathrm{E} \left[\frac{\pi_{n+1}}{\pi_n} \mathrm{E} \left[B'(\lambda) \sum_{i=n+1}^{M} \frac{\pi_i}{\pi_{n+1}} (\xi_i x_i^2 \right. \right.$$

$$\left. \left. \left. + 2\eta_i x_i \lambda + \zeta_i \lambda^2 + k_i u_i^2) \,|\, X_{n+1}, U_n \right] \middle| X_n, U_{n-1} \right] \right\}$$

$$= \min_{u_n} \left\{ \frac{D(\beta_n, r_n)}{C(\beta_n, r_n)} \left[\xi_n x_n^2 + \frac{2\eta_n}{\beta_n} x_n r_n + \zeta_n (T_1^{(n)} r_n^2 + T_2^{(n)} r_n + T_3) + k_n u_n^2 \right] \right.$$

$$\left. + \frac{\pi_{n+1}}{\pi_n} \mathrm{E} [W_{n+1} \,|\, X_n, U_{n-1}] \right\}.$$

Then for the Bayes control u_n^* we have

$$2 \frac{D(\beta_n, r_n)}{C(\beta_n, r_n)} k_n u_n^* + \frac{\pi_{n+1}}{\pi_n} \mathrm{E} \left(\frac{\partial W_{n+1}}{\partial u_n^*} \middle| X_n, U_{n-1} \right) = 0,$$

or, since $x_{n+1} = \alpha_n x_n + u_n^* + \gamma_n v_n$,

$$(36) \qquad 2 \frac{D(\beta_n, r_n)}{C(\beta_n, r_n)} k_n u_n^* + \frac{\pi_{n+1}}{\pi_n} \mathrm{E} \left(\frac{\partial W_{n+1}}{\partial x_{n+1}} \middle| X_n, U_{n-1} \right) = 0.$$

Suppose that W_n is of the form

$$(37) \qquad W_n = \frac{D(\beta_n, r_n)}{C(\beta_n, r_n)} (A_n x_n^2 + 2B_n x_n r_n + C_n r_n^2 + D_n r_n + E_n).$$

For $n = M$ this is fulfilled with

(38)

$$A_M = \xi_M, \quad B_M = \eta_M / \beta_M, \quad C_M = \zeta_M T_1^{(M)}, \quad D_M = \zeta_M T_2^{(M)}, \quad E_M = \zeta_M T_3^{(M)}$$

and the optimal $u_M^* = 0$. Assuming (37) to be true for $n+1$ we obtain, using (36),

$$\frac{D(\beta_n, r_n)}{C(\beta_n, r_n)} k_n u_n^* + \frac{\pi_{n+1}}{\pi_n} \mathrm{E} \left[\frac{D(\beta_{n+1}, r_{n+1})}{C(\beta_{n+1}, r_{n+1})} (A_{n+1} x_{n+1} + B_{n+1} r_{n+1}) \,|\, X_n, U_{n-1} \right]$$

$$= \frac{D(\beta_n, r_n)}{C(\beta_n, r_n)} \left[k_n u_n^* + \frac{\pi_{n+1}}{\pi_n} [A_{n+1} (\alpha_n x_n + u_n^* + \gamma_n Q^{(n)} r_n) + B_{n+1} (1 + Q^{(n)}) r_n] \right] = 0,$$

or

$$u_n^* = - \frac{\frac{\pi_{n+1}}{\pi_n} \alpha_n A_{n+1}}{k_n + \frac{\pi_{n+1}}{\pi_n} A_{n+1}} x_n - \frac{\frac{\pi_{n+1}}{\pi_n} [\gamma_n Q^{(n)} A_{n+1} + (1 + Q^{(n)}) B_{n+1}]}{k_n + \frac{\pi_{n+1}}{\pi_n} A_{n+1}} r_n.$$

Moreover, by inductive argument, using (35) with the help of (30)–(34) we prove (37) with A_n, B_n, C_n, D_n, E_n satisfying the equations

$$A_n = \xi_n + \frac{\frac{\pi_{n+1}}{\pi_n} k_n \alpha_n^2 A_{n+1}}{k_n + \frac{\pi_{n+1}}{\pi_n} A_{n+1}},$$

$$B_n = \frac{\eta_n}{\beta_n} + \frac{\pi_{n+1}}{\pi_n} \cdot \frac{\alpha_n k_n [\gamma_n Q^{(n)} A_{n+1} + (1+Q^{(n)}) B_{n+1}]}{k_n + \frac{\pi_{n+1}}{\pi_n} A_{n+1}},$$

$$C_n = \zeta_n T_1^{(n)} + \frac{\pi_{n+1}}{\pi_n} \left[\gamma_n^2 Q_1^{(n)} A_{n+1} + 2\gamma_n (Q^{(n)} + Q_1^{(n)}) B_{n+1} + (1 + 2Q^{(n)} + Q_1^{(n)}) C_{n+1} \right]$$

$$- \frac{k_n + \frac{\pi_{n+1}}{\pi_n} A_{n+1}}{\alpha_n^2 k_n^2} \left(B_n - \frac{\eta_n}{\beta_n} \right)^2,$$

$$D_n = \zeta_n T_2^{(n)} + \frac{\pi_{n+1}}{\pi_n} [\gamma_n^2 Q_2^{(n)} A_{n+1} + 2\gamma_n Q_2^{(n)} B_{n+1} + Q_2^{(n)} C_{n+1} + (1+Q^{(n)}) D_{n+1}],$$

$$E_n = \zeta_n T_3^{(n)} + \frac{\pi_{n+1}}{\pi_n} [\gamma_n^2 Q_3^{(n)} A_{n+1} + 2\gamma_n Q_3^{(n)} B_{n+1} + Q_3^{(n)} C_{n+1} + E_{n+1}]$$

and the boundary conditions (38).

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