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A NOISY DUEL UNDER ARBITRARY MOVING. I

1. Definitions and suppositions. In the paper an m versus 1 bullet noisy duel is solved in which duelists can move at will.

Let us define the game which will be called the *game* $(m, 1)$. Two Players I and II fight in a duel. They can move as they like. Maximal velocity of Player I is v_1, maximal velocity of Player II is v_2, and let $v_1 > v_2 \geqslant 0$. Player I has m bullets (or rockets), Player II has one bullet (rocket).

Assume that at the moment $s = 0$ the players are in the distance 1 off and that $v_1 + v_2 = 1$.

Denote by $P(s)$ the probability that Player I (II) achieves a success (destroys the opponent) if he fires in the distance $1-s$. We assume that the function $P(s)$ is increasing and continuous in the interval $[0, 1]$ and has a continuous second derivative inside this interval, $P(s) = 0$ for $s \leqslant 0$, $P(1) = 1$.

Player I gains 1 if he only achieves the success, gains -1 if Player II only achieves the success, and gains 0 in the remaining cases. It is assumed that the duel is a zero-sum game.

The duel is *noisy* – the player hears the shot of his opponent.

Without loss of generality we can assume that Player II is motionless. Having a solution of the duel $(m, 1)$, in this case the determining of optimal strategies in the general situation is easy.

When Player I has fired all his bullets, his motion in the direction of the opponent loses sense. Then we shall always assume that Player I evades with maximal speed after firing all his bullets.

Suppose that Player I has fired all his bullets and he evades. In this case Player II will do the best (if he survives) if he fires all his bullets immediately after the last shot of Player I. If, on the other hand, Player II has fired his bullet and Player I survives and has yet bullets, the best what he can do is to reach the opponent and to achieve the success surely. We shall keep these assumptions throughout the paper.

2. Duel (1, 1). Let us consider, at the beginning, the case where Players I and II have one bullet each.

Let $K(\hat{\xi}, \hat{\eta})$ be the *payoff function* (the expected gain for Player I) for strategies $\hat{\xi}$ and $\hat{\eta}$ of Players I and II and let a' mean (strategy) that Player I (II) fires in the distance $1-a'$ if his opponent did not fire before. The moment when Player I has been in a' will be denoted by $\langle\langle a'\rangle\rangle$. Since there can be many moments of time satisfying this condition, by $\langle a'\rangle$ we shall denote the earliest one.

Let the point $a_{m1} \in [0, 1]$. Denote by a_{mn}^{ε} the random moment,

$$\langle a_{m1}\rangle \leqslant a_{m1}^{\varepsilon} \leqslant \langle a_{m1}\rangle + \alpha(\varepsilon),$$

distributed according to an absolute continuous probability distribution (ACPD) in the above interval, $\alpha(\varepsilon) \to 0$ when $\varepsilon \to 0$.

Assume that at the moment 0 Player I is at the point 0. Consider the following strategies ξ and η of Players I and II:

S t r a t e g y o f P l a y e r I. Reach the point a_{11}, and if Player I did not fire before, fire a shot at a_{11}^{ε}.

S t r a t e g y o f P l a y e r II. Fire at the earliest moment when Player I reaches the point a_{11} (i.e., at the moment $\langle a_{11}\rangle$). If he does not reach this point, do not fire.

The number a_{11} satisfies the condition

$$P^2(a_{11}) = 1 - 2P(a_{11}),$$

i.e.,

$$P(a_{11}) = \sqrt{2} - 1. \tag{1}$$

We shall prove that ξ *is an ε-maximin strategy of Player* I, η *is a minimax strategy of Player* II *and the value of the game is*

$$v_{11} = 1 - 2P(a_{11}) = 3 - 2\sqrt{2}. \tag{2}$$

Suppose that Player II fired in $a' < a_{11}$. Under the assumptions made about the behaviour of the players after the shots, we obtain

$$K(\xi; a') = -P(a') + 1 - P(a') \geqslant -P(a_{11}) + 1 - P(a_{11}) = v_{11} \tag{a}$$

if v_{11} is given by (2).

Suppose that Player II does not fire before the moment $\langle a_{11}\rangle + \alpha(\varepsilon)$ if Player I does the same. Then if $\langle a'\rangle \geqslant \langle a_{11}\rangle + \alpha(\varepsilon)$, we have

$$K(\xi; a') \geqslant P(a_{11}) - (1 - P(a_{11}))P(a_{11}) - \varepsilon/2 = P^2(a_{11}) - \varepsilon/2 = v_{11} - \varepsilon/2 \tag{b}$$

for appropriate $\alpha(\varepsilon)$ in the definition of the strategy.

From inequalities (a) and (b) it follows that for appropriate $\alpha(\varepsilon)$

$$K(\xi; a') \geqslant v_{11} - \varepsilon \tag{3}$$

for any a'

On the other hand, suppose that Player I does not reach the point a_{11} in any time and that he does not fire a shot. In this case, for such a strategy ξ

$$K(\xi;\eta) = 0 < v_{11}.$$

If Player I fires at the point $a' < a_{11}$, then

$$K(a';\ \eta) = P(a') - (1-P(a'))P(a') = P^2(a') < P^2(a_{11}) = v_{11}.$$

If Player I fires at the moment $\langle a_{11}\rangle$, i.e., at the same moment when Player II does, then

$$K(a';\eta) = 0 < v_{11}.$$

If Player II reaches the point a_{11} but does not fire before or at a_{11}, then for such a strategy ξ

$$K(\xi;\eta) = 1-2P(a_{11}) = v_{11}.$$

Then Player II applying the strategy η assures that he does not lose (in mean) more than v_{11}.

From the above and inequality (3) it follows that the strategy ξ is ε-maximin, the strategy η is minimax, and the value of the game is v_{11} given by (2).

In classical duels considered in the literature it is assumed that the players do not evade after firing all their bullets and they move all the time with constant velocity. In the case "one bullet each" the optimal strategies in a classical noisy duel are similar to the above considered but the number a_{11} satisfies the equation $P(a_{11}) = 1/2$ and the value of the game is zero. Then in the duel (1, 1) considered in the paper it is necessary to fire sooner than in the corresponding classical duel and the fact which player is moving (or has greater speed) has substantial influence on the value of the game.

For noisy duels see [4], [9], [13], [18].

3. Duel $(m, 1)$, $m \geq 2$. Now, let us consider the case where Player I has m bullets and Player II has one bullet. Denote now by ξ and η strategies of Players I and II defined as follows.

Strategy of Player I. Reach the point a_{m1} and if Player II did not fire before, fire a shot at $\langle a_{m1}\rangle$ and play ε-optimally the obtained duel $(m-1, 1)$.

Strategy of Player II. If Player I reached the point a_{m1}, fire a shot at a_{m1}^{ε}. If he does not reach this point, do not fire.

"Play ε-optimally" means "apply an ε-optimal strategy".

The number a_{m1} is determined from the equations

$$v_{m1} = P(a_{m1}) + (1-P(a_{m1}))v_{m-1,1} = 1-2P(a_{m1}), \tag{4}$$

where v_{11} is given by (2).

Solving this system of equations we obtain

(5) $$v_{m1} = \frac{1+v_{m-1,1}}{3-v_{m-1,1}},$$

(6) $$P(a_{m1}) = \frac{1-v_{m1}}{2} = \frac{1-v_{m-1,1}}{3-v_{m-1,1}}.$$

From (5) we get

(7) $$v_{m1} = \frac{1+(m-3)P(a_{11})}{1+(m-1)P(a_{11})}.$$

Then

(8) $$P(a_{m1}) = \frac{P(a_{11})}{1+(m-1)P(a_{11})},$$

where $P(a_{11})$ is given by (1).

Let us notice that from (8) it follows that $a_{m+1,1} < a_{m1}$, as required. Then the strategies ξ and η are well defined.

We shall prove that *the strategy ξ is ε-maximin, the strategy η is ε-minimax and the value of the game is given by* (7).

Suppose that Player II fires at the point a', where $a' < a_{m1}$. We have

$$K(\xi;\, a') = 1-2P(a') \geqslant 1-2P(a_{m1}) = v_{m1}.$$

Suppose that Player II fires at $\langle a_{m1}\rangle$. For such a strategy a_0' we have

$$K(\xi;\, a_0') \geqslant (1-P(a_{m1}))^2-\varepsilon > 1-2P(a_{m1})-\varepsilon = v_{m1}-\varepsilon.$$

Suppose that Player II playing against ξ has not intended to fire before $\langle a_{m1}\rangle$ or at $\langle a_{m1}\rangle$ if Player I did not fire before. For such a strategy $\hat{\eta}$ we have

$$K(\xi;\, \hat{\eta}) \geqslant P(a_{m1})+(1-P(a_{m1}))(v_{m-1,1}-\varepsilon) \geqslant v_{m1}-\varepsilon$$

by (4). Then Player I applying the strategy ξ assures for himself (in mean) the payoff $v_{m1}-\varepsilon$.

We shall prove that Player II applying η assures that he does not lose (in mean) more than $v_{m1}+\varepsilon$.

Assume that Player I applying the strategy ξ against η never reaches the point a_{m1} and never fires. Then

(a) $$K(\xi;\, \eta) = 0 < v_{m1}.$$

Assume that Player I playing against η fires his first shot in $a' < a_{m1}$ and later plays according to a strategy ξ_0 (maybe dependent on a') in the subsequent duel. For such a strategy (a', ξ_0) we have

(b) $$\begin{aligned} K(a', \xi_0;\, \eta) &\leqslant P(a')+(1-P(a'))(v_{m-1,1}+\varepsilon/2) \\ &\leqslant P(a_{m1})+(1-P(a_{m1}))v_{m-1,1}+\varepsilon/2 = v_{m1}+\varepsilon/2. \end{aligned}$$

Assume that Player I playing against η has the intention to fire his first shot at a moment later than $\langle a_{m1}\rangle+\alpha(\varepsilon)$ or not to fire at all. For such a strategy ξ we obtain

(c) $$K(\xi;\eta) = 1-2P(a_{m1}) = v_{m1}.$$

From (a)–(c) it follows that

(9) $$K(\xi;\eta) \leqslant v_{m1}+\varepsilon$$

for any strategy ξ of Player I. Then Player II applying η assures that he does not lose (in mean) this value. The proposition is proved.

Let us notice that the strategies ξ and η defined for $m = 1$ are of different kind from those defined for $m \geqslant 2$.

The duel (m, n), whose special case $(m, 1)$ is solved in this paper, is considered also in papers [15] of the author.

For games of timing see also [1]–[3], [7], [8], [10], [14], [17].

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