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GALERKIN METHODS FOR NONLINEAR VOLTERRA TYPE EQUATIONS

1. Introduction. Our aim in this paper is to construct an approximate solution of the nonlinear Volterra equation of the second kind

$$x(t)-\int_0^t m(t, s)k(t-s)G(x)(s)ds = f(t), \qquad t \in [0, 1], \tag{1}$$

where $G(x)(s) = g(x(s), s)$ for a.e. s in $[0, 1]$, k is in a suitable Nikolskiĭ space, f is in a suitably chosen Hölder or Nikolskiĭ space, and the regularity of other functions in (1) will be specified later.

In the case of smooth integral kernels k the collocation method gives the best results in the approximate solution of equation (1) (see [2]).

Since we are interested in the case of nonsmooth functions k, we use the Galerkin method to approximate equation (1). We seek the approximate solutions in spline function spaces.

We write (1) in the standard operator notation as

$$(I-KG)x = f. \tag{2}$$

We apply the theory of topological degree, which is excellently presented in [6]. The estimates of the error of the approximation are obtained in the supremum norm.

Our method was influenced by the paper of Graham [3], where similar questions for linear integral equations of Fredholm type were considered.

2. Basic notation and assumptions. We put

$$\Delta_\delta f(t) = f(t+\delta)-f(t) \quad \text{and} \quad [0, 1]_\delta = \{t \in [0, 1]\colon t+\delta \in [0, 1]\}.$$

For any $n \in N$, let Π_n denote the partition of $[0, 1]$ given by

$$\Pi_n\colon 0 = t_0 < t_1 < \ldots < t_n = 1.$$

We use the spline function spaces $S_q^\nu(\Pi_n)$, where $-1 \leqslant \nu < q$ are integers, which are defined as follows:

$$S_q^\nu(\Pi_n) = \{v \in C^\nu[0, 1]: v|[t_{i-1}, t_i] \text{ is a polynomial of degree} \leqslant q\}$$

for $\nu \geqslant 0$ and

$$S_q^{-1}(\Pi_n) = \{v: v|(t_{i-1}, t_i) \text{ is a polynomial of degree} \leqslant q\}.$$

To ensure that $v \in S_q^{-1}(\Pi_n)$ is well defined we assume its left continuity at each knot and right continuity at 0. Throughout this paper ν and q are fixed, therefore we write $S(\Pi_n)$ without indices.

In the sequel, as n varies, we assume that partitions Π_n remain quasiuniform, i.e., there exists a constant C with the property

$$\frac{\max\limits_{i=1,\ldots,n} (t_i - t_{i-1})}{\min\limits_{j=1,\ldots,n} (t_j - t_{j-1})} \leqslant C \tag{3}$$

for each partition Π_n.

Note that condition (3) implies $h \to 0$ as $n \to \infty$, where

$$h = \max_i (t_i - t_{i-1}).$$

By P_n we denote the orthogonal projection of $L^2(0, 1)$ onto $S(\Pi_n)$. All derivatives are understood in the weak (distributional) sense.

Throughout the paper C denotes an inessential constant. We permit it to change its value from paragraph to paragraph.

3. Some auxiliary definitions and theorems. In the sequel the following function spaces are used:

1° The Hölder space $C^\beta[0, 1]$, $0 < \beta < 1$, endowed with the norm

$$\|u\|_\beta = \sup_{t \in [0,1]} |u(t)| + \sup_{\substack{t \in [0,1]_\sigma \\ \delta \neq 0}} |\delta|^{-\beta} |\Delta_\delta u(t)|.$$

2° The Nikolskiĭ space $N_p^\gamma[0, 1]$ with noninteger $\gamma > 0$, $1 \leqslant p \leqslant \infty$, the space of all functions $u \in L^p(0, 1)$ satisfying the condition

$$|u|_{\gamma,p} = \sup_{\delta \neq 0} |\delta|^{-\gamma_0} \|\Delta_\delta D^{[\gamma]} u\|_{L^p(0,1)_\delta} < \infty,$$

where $[\gamma]$ is the integer part of γ and $\gamma_0 = \gamma - [\gamma]$.

It is known (see [5]) that $N_p^\gamma[0, 1]$ equipped with the norm

$$\|u\|_{\gamma,p} = \|u\|_p + |u|_{\gamma,p}$$

is a Banach space. Moreover, the following well-known results hold (see [5]):

THEOREM A. *We have*

$$N_p^{1+\alpha}[0, 1] \subseteq N_p^{1+\beta}[0, 1] \subseteq W_1^1(0, 1)$$

for $0 < \alpha < \beta$ *and* $1 \leqslant p \leqslant \infty$.

Remark 1. It is known (see [1], Lemma 5.8) that $W_1^1(0, 1)$ consists of absolutely continuous functions.

THEOREM B. *We have*

$$N_p^{\alpha}[0, 1] \subseteq N_q^{\beta}[0, 1]$$

for $\alpha > 0$, $1 \leqslant p \leqslant \infty$ *and* $\beta = \alpha-(1/p-1/q) > 0$.

We need the following well-known facts from the Lebesgue theory:

THEOREM C. *Let* $f \in L^1(0, 1)$. *Then*

$$\frac{d}{dt}\Big(\int_0^t f(s)ds\Big) = f(t) \quad \textit{a.e. in } [0, 1].$$

THEOREM D. *Let* $f \in C[0, 1]$ *be absolutely continuous. Then*

$$\lim_{\delta\to 0} \int_{[0,1]_\delta} \left|\frac{f(s+\delta)-f(s)}{\delta} - f'(s)\right| ds = 0.$$

For convenience of the reader we present a simple proof of Theorem D.

Proof. For an absolutely continuous function f we have

$$f(x+\delta)-f(x) = \int_0^\delta f'(x+s)ds.$$

Reversing the order of integration we get

$$\int_{[0,1]_\delta} \left|\frac{f(x+\delta)-f(x)}{\delta} - f'(x)\right| dx \leqslant \frac{1}{\delta}\int_0^\delta \int_{[0,1]_\delta} |f'(x+s)-f'(x)|dxds \tag{4}$$
$$\leqslant \sup_{s\leqslant|\delta|} \int_{[0,1]_\delta} |f'(x+s)-f'(x)|dx.$$

Since translation is a continuous operation in L^1, the right-hand side of (4) tends to 0 as $\delta\to 0$. This completes the proof.

We make use of the following facts:

PROPOSITION 1. *Let* $k \in N_1^\alpha[0, 1]$ *for some* $0 < \alpha < 1$ *and let*

$$I(t) = \int_0^t k(s)ds \quad \textit{for } t \in [0, 1].$$

Then $I \in C^\alpha[0, 1]$.

Proof. To estimate $|I(t+\delta)-I(t)|$ we observe that

$$\int_t^{t+\delta} k(s)ds = \int_{t_1}^{t+\delta} [k(s)-k(s-\delta)]ds + \int_{t_1-\delta}^{t_1} k(s)ds \tag{5}$$

when all the integrals make sense. Recalling that $k \in N_1^\alpha[0, 1]$, we can estimate the first term on the right-hand side of (5) by $\text{const}|\delta|^\alpha$.

We turn to the latter one. By Theorem C, the expression

$$\delta^{-1} \int_{t_1-\delta}^{t_1} k(s)ds$$

tends to $k(t)$ for a.e. t in $[0, 1]$ as $\delta \to 0$. Hence selecting t_1 so that this limit is finite, we get the estimate

$$|I(t+\delta)-I(t)| \leqslant C|\delta|^\alpha \tag{6}$$

for $t \in [0, 1]_\delta$ and $|\delta| < \delta_0$, where δ_0 is a sufficiently small number and the constant C does not depend on t and δ. Now our assertion follows from (6).

Theorem 4 in [3] on approximation can be adapted to our case, which gives

PROPOSITION 2. *Let $f \in N_\infty^\eta[0, 1] \cap C[0, 1]$ for some $0 < \eta < 1$. Then for any $n \in N$ there exists a function $\xi_n \in S(\Pi_n)$ such that*

$$\|f-\xi_n\|_\infty \leqslant Ch^\eta,$$

where C is independent of n.

PROPOSITION 3. *Let the nonnegative functions $x, f, k \in L^1(0, d)$, $0 < d < \infty$, satisfy the inequality*

$$x(t) \leqslant \int_0^t k(t-s)x(s)ds+f(t) \quad \textit{a.e. in } [0, 1]. \tag{7}$$

Then $\|x\|_1 \leqslant C\|f\|_1$, where C depends on k only.

Proof. Put

$$X(T) = \int_0^T x(s)ds, \quad I(T) = \int_0^T k(s)ds \quad \text{and} \quad F(T) = \int_0^T f(s)ds$$

for $T \in [0, d]$. After changing the order of integration we get

$$\int_0^T dt \int_0^T k(t-s)x(s)ds = \int_0^T x(s)I(T-s)ds \quad \text{for every } T \in [0, d].$$

Hence integrating both sides of (7) and making use of the monotonicity of the functions X and I we obtain

$$X(T) \leqslant I(T)X(T_1)+I(T-T_1)X(T)+F(T)$$

for every $0 \leqslant T_1 \leqslant T \leqslant d$. Therefore, selecting $\delta > 0$ so that $I(t) \leqslant 1/2$ for $0 \leqslant t \leqslant \delta$ and applying once more the monotonicity argument, we can write

$$X(T) \leqslant 2\big(I(d)X(T_1)+F(d)\big),$$

for every $0 \leqslant T_1 \leqslant T \leqslant d$ with $T-T_1 \leqslant \delta$.

Hence beginning with $T_1 = 0$ and $T = \delta$ we get the recurrent estimates of $X(\delta)$, $X(2\delta)$, and so on. After a finite number of steps we obtain the required estimate of $X(d) = \|x\|_1$.

Now we return to the operators P_n. It is known (see [4]) that the norms of P_n considered as operators on $L^\infty(0, 1)$ are uniformly bounded. Namely, there exists a constant C independent of n such that

$$\|P_n\| \leqslant C \quad \text{for all } n \in N. \tag{8}$$

For any $\xi \in S(\Pi_n)$ we have

$$\|f - P_n f\|_\infty = \|f - \xi - P_n(f - \xi)\|_\infty \leqslant (1 + C)\|f - \xi\|_\infty.$$

Therefore, in view of Proposition 2, for any $f \in C^\gamma[0, 1]$ we get

$$\|f - P_n f\|_\infty \to 0 \quad \text{as } n \to \infty. \tag{9}$$

Since $C^\gamma[0, 1]$ is a dense subset of $C[0, 1]$, (9) is valid for every $f \in C[0, 1]$.

4. Exact problem. In the sequel, $g\colon R \times [0, 1] \to R$ denotes a function satisfying the Lipschitz condition with constant M:

(i) $|g(x_1, s) - g(x_2, s)| \leqslant M|x_1 - x_2|$ for a.e. $s \in [0, 1]$;

(ii) $|g(0, s)| \leqslant M$ a.e. in $[0, 1]$.

Moreover, we assume that m and k appearing in (1) satisfy

(iii) $k \in N_1^\alpha[0, 1]$;

(iv) $m \in C([0, 1] \times [0, 1])$ and satisfies the Hölder condition

$$|m(t_1, s) - m(t_2, s)| \leqslant M|t_1 - t_2|^\alpha \quad \text{for } t_1, t_2 \in [0, 1],$$

uniformly with respect to s; the exponent $\alpha \in (0, 1)$ is the same in both cases.

Let us define the operators

$$Kx(t) = \int_0^t m(t, s)k(t - s)x(s)ds, \qquad G(x)(t) = g(x(t), t) \qquad t \in [0, 1].$$

So we have

$$\begin{aligned} KG(x)(t) &= \int_0^t m(t, s)k(t - s)G(x)(s)ds \\ &= \int_0^t m(t, t - s)k(s)G(x)(t - s)ds \quad \text{for } t \in [0, 1]. \end{aligned} \tag{10}$$

The essential properties of the operator KG are collected in the following

LEMMA 1. *Let* $0 < d \leqslant 1$. *Then*

(a) *for any* x_1, x_2 *in* $L^1(0, d)$,

$$\|KG(x_1) - KG(x_2)\|_1 \leqslant M \int_0^d |k(s)|ds \, \|x_1 - x_2\|_1 \tag{11}$$

(*if* $x_1, x_2 \in L^\infty(0, d)$, *then* (11) *is valid also with the* $\| \; \|_\infty$*-norm*);

(b) $KG(x) \in N_1^\alpha[0, d]$ *for every* $x \in L^1(0, d)$;
(c) $KG(x) \in C^\alpha[0, d]$ *for every* $x \in L^\infty(0, d)$.

Proof. First let us note that for $x \in L^1(0, d)$, by the Lipschitz condition, we have

$$(12) \qquad |G(x)(s)| \leqslant M|x(s)| + |G(0)(s)| \quad \text{a.e. in } [0, d].$$

Hence the operator KG is well defined both in $L^1(0, d)$ and in $L^\infty(0, d)$.

To prove (a) note that for every $x_1, x_2 \in L^1(0, d)$ we have

$$|G(x_1)(s) - G(x_2)(s)| \leqslant M|x_1(s) - x_2(s)| \quad \text{a.e. in } [0, d],$$

which obviously implies our assertion.

Now we turn to (b). Assume that $\delta > 0$ (the case $\delta < 0$ is similar). Making the substitution $\tau = t - s$ in (10) we get

$$(13) \qquad |KG(x)(t+\delta) - KG(x)(t)| \leqslant |I_1| + |I_2| + |I_3|,$$

where

$$I_1 = \int_0^t m(t+\delta, t-\tau)[k(\tau+\delta) - k(\tau)]G(x)(t-\tau)d\tau,$$

$$I_2 = \int_{-\delta}^0 m(t+\delta, t-\tau)k(\tau+\delta)G(x)(t-\tau)d\tau$$

$$= \int_0^\delta m(t+\delta, t+\delta-\tau)k(\tau)G(x)(t+\delta-\tau)d\tau,$$

$$I_3 = \int_0^t [m(t+\delta, t-\tau) - m(t, t-\tau)]k(\tau)G(x)(t-\tau)d\tau$$

for every $0 \leqslant t < t+\delta \leqslant d$.

We need only to estimate each term on the right-hand side of (13).

Reversing the order of integration we have

$$(14) \qquad \int_0^{d-\delta} |I_1|dt \leqslant \int_0^{d-\delta} |k(\tau+\delta) - k(\tau)| \int_\tau^{d-\delta} |m(t+\delta, t-\tau)||G(x)(t-\tau)|dtd\tau,$$

$$(15) \qquad \int_0^{d-\delta} |I_2|dt \leqslant \int_0^\delta |k(\tau)| \int_0^{d-\delta} |m(t+\delta, t+\delta-\tau)||G(x)(t+\delta-\tau)|dtd\tau.$$

First let us note that the integrals with respect to t in (14) and (15) do not exceed $M\|G(x)\|_1$. Hence, by $k \in N_1^\alpha[0, 1]$, the right-hand side of (14) can be estimated by $C\|G(x)\|_1\delta^\alpha$, where the constant C is independent of x and d.

Now, noting that $|k| \in N_1^\alpha[0, 1]$, we can apply inequality (6) with $t = 0$ to get a similar estimate of the right-hand side of (15).

Since m satisfies the Hölder condition and

$$\int_0^{d-\delta} \int_0^t |k(s)G(x)(t-s)|dsdt \leqslant \|k\|_1 \|G(x)\|_1,$$

we get

$$\int_0^{d-\delta} |I_3|\,dt \leqslant M\,\|k\|_1\,\|G(x)\|_1\,\delta^\alpha.$$

From the above considerations we obtain the estimate of the left-hand side of (13) in the form $C\,\|G(x)\|_1\,\delta^\alpha$, where C is independent of d. This completes the proof.

(c) It follows from (12) that $G(x) \in L^\infty(0, d)$. Therefore the integrals I_1, I_2 and I_3 can be estimated by $\operatorname{const}\|G(x)\|_\infty\,\delta^\alpha$ even in a simpler way than in the proof of (b).

COROLLARY 1. *The operator $KG\colon L^\infty(0, 1) \to C[0, 1]$ is continuous and compact.*

Proof. The continuity of KG follows from part (a) of Lemma 1. Inequality (12) and the estimate

$$|KG(x)(t+\delta) - KG(x)(t)| \leqslant \operatorname{const}\|G(x)\|_\infty\,\delta^\alpha \quad \text{for } t \in [0, 1]_\delta$$

obtained in the proof of part (c) of Lemma 1 allow us to apply the Ascoli–Arzela theorem to verify the compactness of KG.

We are now ready to consider equation (1). The essential results concerning the existence and the uniqueness of solutions of that equation are given in the following theorem:

THEOREM 1. *Let $f \in L^1(0, 1)$. Then equation* (1) *has a unique solution x_0 which belongs to $L^1(0, 1)$. If $f \in C[0, 1]$, then the solution x_0 is in $C[0, 1]$.*

Proof. Choose a sufficiently small $d > 0$ so that

$$M\int_0^d |k(s)|\,ds < 1.$$

By part (a) of Lemma 1 the operator KG, considered either on $L^1(0, d)$ or on $C[0, d]$, is a contraction. Hence there exists a unique solution x_0 of (1) defined on the interval $[0, d]$ in both cases.

Now we consider the complete metric space

$$X_1 = \{x \in L^1(0, 2d)\colon x = x_0 \text{ on } [0, d]\}$$

in the first case and the complete metric space

$$X_2 = \{x \in C[0, 2d]\colon x = x_0 \text{ on } [0, d]\}$$

in the second one. For any x_1 and x_2 either in X_1 or in X_2 we have

$$|KG(x_1)(t) - KG(x_2)(t)| = 0 \quad \text{for } t \in [0, d],$$

$$|KG(x_1)(t) - KG(x_2)(t)| \leqslant M\int_d^t |k(t-s)|\,|x_1(s) - x_2(s)|\,ds \quad \text{for } d \leqslant t \leqslant 2d.$$

Hence we get

$$\|KG(x_1)-KG(x_2)\|_1 \leqslant M\int_0^d |k(s)|ds\,\|x_1-x_2\|_1$$

in the first case and

$$\|KG(x_1)-KG(x_2)\|_\infty \leqslant M\int_0^d |k(s)|ds\,\|x_1-x_2\|_\infty$$

in the second one. Therefore KG is a contraction on X_1 and X_2. As a consequence of this result there exists a unique extension of x_0 to the solution of (1) on the interval $[0, 2d]$. Repeating this procedure we are led to a unique solution of (1) defined on the whole interval $[0, 1]$.

As a consequence of Lemma 1, Theorem 1 and the obvious imbeddings for the Nikolskiĭ and Hölder spaces we get

COROLLARY 2. *Let $f\in N_1^\beta[0, 1]$ for some $0<\beta<1$. Then the solution x_0 of the problem* (1) *belongs to $N_1^\gamma[0, 1]$, where $\gamma = \min(\alpha, \beta)$. If $f\in C^\beta[0, 1]$, then x_0 is in $C^\gamma[0, 1]$.*

Before considering the case of more regular solutions of (1), we prove a technical lemma essential for our considerations.

Write

$$(16)\qquad F(\delta, t) = \delta^{-1}\Big[\Delta_\delta KG(x)(t) - \int_0^t m(t, s)k(t-s)\big[g\big(x(s+\delta), s+\delta\big)-g\big(x(s), s+\delta\big)\big]ds\Big]$$

$$\text{for } \delta\neq 0 \text{ and } t\in[0, 1]_\delta.$$

LEMMA 2. *Let* (i) $m\in C^2([0, 1]\times[0, 1])$; (ii) $g\in C^1(R\times[0, 1])$; (iii) $x\in C[0, 1]$. *Then for every $d\in(0, 1)$*

$$F(\delta, t)\to F(t) \quad \text{in } L^1(0, d) \text{ as } \delta\to 0,$$

where

$$F(t) = m(t, 0)k(t)G(x)(0)+\int_0^t [m_t(t, s)+m_s(t, s)]k(t-s)G(x)(s)ds$$

$$+\int_0^t m(t, s)k(t-s)g_s\big(x(s), s\big)ds \quad \text{for } t\in[0, 1].$$

Moreover, $F\in N_1^\alpha[0, 1]$.

Proof. Let $d\in(0, 1)$. Making the substitution $\tau = t-s$ in the first integral, adding the second one and at last making the substitution $s = t+\delta-\tau$ in the obtained integral we notice that

$$\int_0^t m(t+\delta,\, s+\delta)k(t-s)g\big(x(s+\delta),\, s+\delta\big)ds$$

$$+\int_t^{t+\delta} m(t+\delta,\, t-s+\delta)k(s)g\big(x(t+\delta-s),\, t+\delta-s\big)ds$$

$$= \int_0^{t+\delta} m(t+\delta,\, s)k(t+\delta-s)g\big(x(s),\, s\big)ds.$$

Therefore we can write

$$F(\delta,\, t) = I_1(\delta,\, t)+I_2(\delta,\, t)+I_3(\delta,\, t),$$

where

$$I_1(\delta,\, t) = \delta^{-1}\int_t^{t+\delta} m(t+\delta,\, t-s+\delta)k(s)g\big(x(t+\delta-s),\, t+\delta-s\big)ds,$$

$$I_2(\delta,\, t) = \delta^{-1}\int_0^t [m(t+\delta,\, s+\delta)-m(t,\, s)]k(t-s)g\big(x(s+\delta),\, s+\delta\big)ds,$$

$$I_3(\delta,\, t) = \delta^{-1}\int_0^t m(t+s)k(t-s)\big[g\big(x(s),\, s+\delta\big)-g\big(x(s),\, s\big)\big]ds.$$

If we compare $I_1(\delta, t)$ with

$$m(t,\, 0)g\big(x(0),\, 0\big)\delta^{-1}\int_t^{t+\delta} k(s)ds,$$

in view of the continuity of m, g, and x and Theorem D, we notice that

$$I_1(\delta,\, t)\to I_1(t) \quad \text{in } L^1(0,\, d) \text{ as } \delta\to 0,$$

where

$$I_1(t) = m(t,\, 0)g\big(x(0),\, 0\big)k(t).$$

By the mean value theorem applied to $\delta^{-1}[m(t+\delta,\, s+\delta)-m(t,\, \delta)]$ we obtain

$$I_2(\delta,\, t)\to I_2(t) \quad \text{in } L^1(0,\, d) \text{ as } \delta\to 0,$$

where

$$I_2(t) = \int_0^t [m_t(t,\, s)+m_s(t,\, s)]k(t-s)g\big(x(s),\, s\big)ds.$$

Finally, applying the mean value theorem to $\delta^{-1}\big[g\big(x(s),\, s+\delta\big)-g\big(x(s),\, s\big)\big]$ we get

$$I_3(\delta,\, t)\to I_3(t) \quad \text{in } L^1(0,\, d) \text{ as } \delta\to 0,$$

where

$$I_3(t) = \int_0^t m(t,\, s)k(t-s)g_s\big(x(s),\, s\big)ds.$$

Since $F(t) = I_1(t)+I_2(t)+I_3(t)$, we have finally

$$F(\delta, t) \to F(t) \quad \text{in } L^1(0, d) \text{ as } \delta \to 0.$$

Since m is regular, $I_1(t)$ is in $N_1^\alpha[0, 1]$. By Lemma 1 (c), the integrals I_2 and I_3 are also in $N_1^\alpha[0, 1]$. Hence $F \in N_1^\alpha[0, 1]$.

Assume that m and g appearing in equation (1) are more regular. Namely, assume that they satisfy assumption (i) and (ii) of Lemma 2, respectively. Then we get

THEOREM 2. *Let $f \in N_1^{1+\beta}[0, 1]$ for some β ($0 < \beta < 1$). Then there exists a unique solution $x_0 \in N_1^{1+\gamma}[0, 1]$ of equation* (1), *where $\gamma = \min(\alpha, \beta)$.*

Proof. Since $f \in N_1^{1+\beta}[0, 1] \subseteq C^\beta[0, 1]$, by Corollary 2 a unique solution x_0 of (1) is in $C^\gamma[0, 1]$.

Write $x_\delta = \delta^{-1}\Delta_\delta x_0(t)$ and $f_\delta(t) = \delta^{-1}\Delta_\delta f(t)$. In view of (i) and (16) we see that x_δ satisfies the equation

$$x_\delta(t) - \int_0^t m(t, s)k(t-s)\psi(s, \delta)x_\delta(s)ds = F(\delta, t)+f_\delta(t), \tag{17}$$

where

$$\psi(s, \delta) = \begin{cases} \dfrac{g(x_0(s+\delta), s+\delta)-g(x_0(s), s+\delta)}{x_0(s+\delta)-x_0(s)} & \text{if } x_0(s) \neq x_0(s+\delta), \\ g'_x(x_0(s), s+\delta) & \text{if } x_0(s) = x_0(s+\delta), \end{cases}$$

valid for a.e. $t \in [0, d]$.

Together with (17) we consider the equation

$$x(t) - \int_0^t m(t, s)k(t-s)g_x(x_0(s), s)x(s)ds = F(t)+f'(t), \qquad t \in [0, d]. \tag{18}$$

As $f' \in N_1^\beta[0, 1]$ and $F \in N_1^\alpha[0, 1]$, the right-hand side of (18) is in $N_1^\gamma[0, 1]$. Therefore, by Corollary 2, there exists a unique solution $x \in N_1^\gamma[0, 1]$ of (18).

Subtracting equation (17) from (18) we get

$$\begin{aligned} (x(t)-x_\delta(t)) - \int_0^t m(t, s)k(t-s)\psi(\delta, s)(x(s)-x_\delta(s))ds \\ = [F(\delta, t)-F(t)]+(f_\delta(t)-f'(t)) \\ + \int_0^t m(t, s)k(t-s)[\psi(s, \delta)-g'_x(x(s), s)]\,x(s)ds. \end{aligned} \tag{19}$$

Using the mean value theorem we get $\psi(s, \delta) \to g'_x(x(s), s)$ uniformly on $[0, d]$ as $\delta \to 0$.

By Theorem D, $f_\delta \to f'$ in $L^1(0, d)$. Therefore the right-hand side of (19) tends to 0 in $L^1(0, d)$ as $\delta \to 0$.

By definition, $\|\psi(\cdot, \delta)\|_\infty \leqslant M$. Therefore we can apply Proposition 3 to verify that $x_\delta(t) \to x(t)$ in $L^1(0, d)$ as $\delta \to 0$.

Since d is an arbitrary number in (0, 1), we derive

$$x_\delta \to x \quad \text{in } D'(0, 1),$$

and hence $x = x_0'$ on [0, 1]. Therefore $x_0' \in N_1^\gamma[0, 1]$. This completes the proof.

5. Approximate problem. Now we are going to construct some approximations to the solution of (1). They are defined as solutions to the following problem:

$$x_n - P_n KG(x_n) = P_n f. \tag{20}$$

To study this problem it is convenient to introduce the auxiliary function

$$y_n = f + KG(x_n). \tag{21}$$

It is easy to see that y_n satisfies the equations

$$x_n = P_n y_n, \tag{22}$$

$$y_n - KGP_n(y_n) = f. \tag{23}$$

Before proving the unique solvability of equations (20) and (23) we prove a lemma which is essential for our further considerations.

For an arbitrary function $\psi \in L^\infty(0, 1)$ with $\|\psi\|_\infty \leqslant M$ and for any $n \in N$ we define the linear operators $L(n, \psi)$ on $L^\infty(0, 1)$ by the formula

$$L(n, \psi)x = \psi P_n x \quad \text{for every } x \in L^\infty(0, 1).$$

The family of all the operators of this form is denoted by $\mathscr{L}$.

LEMMA 3. *There exists $N_0 > 0$ such that for any $n > N_0$, $\psi \in L^\infty(0, 1)$ with $\|\psi\|_\infty \leqslant M$, and $x \in C[0, 1]$.*

$$C_1 \|x\|_\infty \leqslant \|(I - KL(n, \psi))x\| \leqslant C_2 \|x\|_\infty, \tag{24}$$

where the positive constants C_1 and C_2 depend on M only.

Proof. Since the operators KL, $L \in \mathscr{L}$, are linear, it suffices to consider the case $\|x\|_\infty = 1$ only. By our assumptions the operator norms of $L \in \mathscr{L}$ are uniformly bounded. In view of Corollary 1 (the case $G = \mathrm{Id}$) the operator

$$K\colon L^\infty(0, 1) \to C[0,1]$$

is bounded and compact. So the operators KL, $L \in \mathscr{L}$, considered as operators on $C[0, 1]$ form a collectively compact family, i.e., the set $\bigcup_{L \in \mathscr{L}} KL(B)$, where B is the unit ball in $C[0, 1]$, is conditionally compact.

As all the operators KL, $L \in \mathscr{L}$, are uniformly bounded, the right-hand side of (24) is obvious. To prove the left-hand side we assume, to get a contradiction, that there exist a sequence of continuous functions x_l, $\|x_l\|_\infty = 1$, and operators $L_l = L(n_1, \psi_l)$ with $n_l \to \infty$ as $l \to \infty$ such that

$$\|(I-KL_l)x_l\|_\infty \to 0. \tag{25}$$

From the collective compactness of the operators $\{KL\}$, $L \in \mathscr{L}$, it follows that there exists a subsequence of $\{KL_l x_l\}$, $l \in N$, which is convergent in $C[0, 1]$. Choosing such a subsequence (for simplicity we do not change the notation) we have

$$KL_l x_l \to x_0 \tag{26}$$

in $C[0, 1]$ as $l \to \infty$ for some x_0.

Now from (25) it follows that $x_l \to x_0$ in $C[0, 1]$ as $l \to \infty$ and $\|x_0\|_\infty = 1$.

Note that $\psi_l x_0 - L_l x_l = \psi_l(x_0 - P_{n_l} x_0) + L_l(x_0 - x_l)$, and hence

$$\|\psi_l x_0 - L_l x_l\|_\infty \leqslant M\|x_0 - P_{n_l} x_0\|_\infty + \|L_l\| \, \|x_l - x_0\|_\infty, \tag{27}$$

where $\|L_l\|$ denotes the operator norm of L_l.

Since L_l are uniformly bounded and $x_l \to x_0$, by (9) and (27) it follows that $\psi_l x_0 - L_l x_l \to 0$ in $L^\infty(0, 1)$ as $l \to \infty$.

Now, as the operator K: $L^\infty(0,1) \to C[0, 1]$ is bounded, we deduce easily that

$$K(\psi_l x_0 - L_l x_l) \to 0 \quad \text{in } C[0, 1].$$

Finally, by (26) we get $K\psi_l x_0 \to x_0$ in $C[0, 1]$ as $l \to \infty$.

Consequently, the functions $f_l = x_0 - K\psi_l x_0$ converge to 0 in $C[0, 1]$ as $l \to \infty$. On the other hand, we have

$$|x_0(t)| \leqslant \text{const} \int_0^t |k(t-s)| \, |x_0(s)| \, ds + |f_l(t)|.$$

Hence, in view of Proposition 3, $\|x_0\|_1 \leqslant C\|f_l\|_1$ for each $l \in N$. So x_0 must be equal to 0, which contradicts $\|x_0\| = 1$, and (24) is proved.

The first step in obtaining the required estimates of the approximations is assured by the following

THEOREM 3. *Let $f \in C[0, 1]$. Then for sufficiently large n*

(i) *the problem* (20) *has a unique solution $x_n \in S(\Pi_n)$ for which*

$$C_1\|x_0 - P_n x_0\|_\infty \leqslant \|x_0 - x_n\|_\infty \leqslant C_2\|x_0 - P_n x_0\|_\infty; \tag{28}$$

(ii) *the problem* (23) *has a unique solution $y_n \in C[0, 1]$ for which*

$$\|x_0 - y_n\|_\infty \leqslant C_3\|x_0 - P_n x_0\|_\infty. \tag{29}$$

where C_1, C_2 and C_3 are independent of n and x_0 is the solution of (1).

Proof. Consider first y_n. Since KG: $L^\infty(0, 1) \to C[0, 1]$, in view of Corollary 1, is continuous and compact and P_n, $n \in N$, are bounded on $L^\infty(0, 1)$, all the operators $tKGP_n$, $t \in [0, 1]$, are continuous and compact on $C[0, 1]$. Let us notice that $G(x)-G(0) = \psi_{(x)}x$ for some $\psi_{(x)} \in L^\infty(0, 1)$ satisfying $\|\psi_{(x)}\|_\infty \leqslant M$. So applying Lemma 2 we deduce easily the estimates

$$(30) \qquad C_1\|x\|_\infty - \|KG(0)\|_\infty \leqslant \|(I-tKGP_n)x\|_\infty \leqslant C_2\|x\|_\infty + \|KG(0)\|_\infty$$

valid for every $x \in C[0, 1]$, $t \in [0, 1]$ and $n > N_0$. Here C_1 and C_2 are independent of x and t.

Due to (30) the Leray–Schauder degree $\deg(f, B, I-tKGP_n)$ of the operators $I-tKGP_n$ may be defined in the ball $B \subseteq C[0, 1]$ centered at 0 with a sufficiently large radius r. In our case we can assume that

$$r = \frac{1+C_1}{C_1}(\|f\|_\infty + \|KG(0)\|_\infty).$$

From the inequality $\|f\|_\infty \leqslant r$ $\deg(f, B, I) = 1$ and from the properties of the topological degree we obtain

$$\deg(f, B, I-tKGP_n) = \deg(f, B, I) = 1 \qquad \text{for } t \in [0, 1].$$

So the problem (23) has a solution in B.

By (22) the problem (20) also has a solution.

Now let us notice, as above, that

$$GP_n(x_1) - GP_n(x_2) = \psi_{(x_1,x_2)}P_n(x_1-x_2)$$

for some $\psi_{(x_1,x_2)} \in L^\infty(0, 1)$ satisfying $\|\psi_{(x_1,x_2)}\|_\infty \leqslant M$. Therefore an application of Lemma 2 yields

$$(31) \qquad C\|x_1-x_2\|_\infty \leqslant \|(x_1-x_2)-(KGP_n(x_1)-KGP_n(x_2))\|_\infty.$$

From (31) we obtain immediately the uniqueness of the solution of the problem (23) and, by (21) and (22), of the problem (20).

To prove inequalities (28) and (29) we notice that from (2) and (23) we obtain

$$(32) \qquad (x_0-y_n)-(KGP_n(x_0)-KGP_n(y_n)) = KG(x_0)-KGP_n(x_0).$$

Estimating the left-hand side of (32) by using (31) and applying Lemma 1 (a) to the right-hand side we get (29).

To obtain (28) we observe that in view of (22) we have

$$x_0 - x_n = (x_0 - P_n x_0) + P_n(x_0 - y_n).$$

Now the uniform boundedness of the operator norms of P_n and (29) yield the right-hand inequality of (28).

To prove the left-hand inequality of (28) we observe, in view of (2) and (20), that

$$x_0 - P_n x_0 = x_0 - x_n - P_n(KG(x_0) - KG(x_n)).$$

Hence by (8) and Lemma 1 (a) we have

$$\|x_0 - P_n x_0\|_\infty \leqslant (1+C)\|x_0 - x_n\|_\infty.$$

This completes the proof of Theorem 3.

The regular solutions of equation (1), as in Corollary 1 or in Theorem 2, may be approximated with the following convergence rates:

THEOREM 4. *Let* $f \in C^{\beta}[0, 1]$ *for some* $0 < \beta < 1$. *Then*

$$\|x_0 - x_n\|_{\infty} \leqslant C_1 h^{\gamma} \quad \text{and} \quad \|x_0 - y_n\|_{\infty} \leqslant C_2 h^{\gamma},$$

where $\gamma = \min(\alpha, \beta)$ *and* C_1 *and* C_2 *are independent of* n.

Proof. From Corollary 2 it follows that

$$x_0 \in C^{\gamma}[0, 1] \subseteq N^{\gamma}_{\infty}[0, 1].$$

Now

$$\|x_0 - P_n x_0\|_{\infty} = \|(I - P_n)(x_0 - \xi_n)\|_{\infty} \leqslant Ch^{\gamma}$$

with ξ_n for given x_0 suitably chosen in $S(\Pi_n)$ in accordance with Proposition 2. Therefore the required result follows from Theorem 3.

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