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ONE-STEP METHODS FOR NEUTRAL DELAY-DIFFERENTIAL EQUATIONS WITH STATE DEPENDENT DELAYS

Abstract. We discuss one-step methods for the numerical solution of Volterra neutral delay-differential equations with state dependent delays

$$\begin{aligned} y'(t) &= f\big(t, y(t), y(\alpha(t, y(t))), y'(\beta(t, y(t)))\big), && t \in [a, b], \\ y(t) &= g(t) && t \in [\gamma, a], \end{aligned}$$

$\gamma \leqslant a < b$. A convergence theorem is given and the asymptotic behaviour of the global discretization error is discussed. These methods are implemented in variable-step mode with local discretization error estimated by local extrapolation. The results of the paper* are illustrated by numerical examples.

1. Introduction. This paper is concerned with the numerical solution of the initial-value problem for neutral delay-differential equations (NDDEs) with state dependent delays

$$\begin{aligned} y'(t) &= F(t, y, y'), && t \in [a, b], \\ y(t) &= g(t), && t \in [\gamma, a], \end{aligned} \tag{1}$$

$\gamma \leqslant a < b$, where for any $t \in [a, b]$ and functions $y, z \in C[a, b]$, F is defined by

$$F(t, y, z) := f\big(t, y(t), y(\alpha(t, y(t))), z(\beta(t, y(t)))\big),$$

$\gamma \leqslant \alpha(t, y) \leqslant t$, $\gamma \leqslant \beta(t, y) \leqslant t$. Here $C[a, b]$ denotes the set of real-valued continuous functions defined on $[a, b]$. The function f, the initial function g, and the delay functions α, β are assumed to satisfy certain conditions which will be given later.

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Equations of type (1) form a model for a two-body problem of classical electrodynamics and have been studied by Driver [4]–[6]. He proved, among other things, existence and uniqueness for the case where $\gamma \leqslant \beta(t, y) < t$. These equations have been also studied by Hale and Cruz [8], Grimm [12], Kamont and Kwapisz [19], and the author [18].

It is assumed throughout this paper that (1) has a unique solution $Y \in C^1[\gamma, b]$, where $C^1[\gamma, b]$ denotes the space of real-valued functions with continuous derivative on $[\gamma, b]$.

Let a stepsize $h > 0$ be given and put

$$t_i = a + ih, \quad i = 0, 1, \ldots, N, \; Nh = b - a.$$

We will consider the class of fully implicit one-step methods for (1) defined by

$$\begin{aligned} y_h(t_i + rh) &= y_h(t_i) + h \sum_{j=1}^{p} a_j^p(r) z_h(t_i + b_j^p h), \\ z_h(t_i + rh) &= \sum_{j=1}^{p+1} c_j^{p+1}(r) z_h(t_i + b_j^{p+1} h), \qquad (2) \\ z_h(t_i + b_j^{p+1} h) &= F(t_i + b_j^{p+1} h, y_h, z_h), \end{aligned}$$

$i = 0, 1, \ldots, N-1$, $r \in (0, 1]$, where y_h and z_h are continuous approximations to Y and Y', respectively, and Y is the solution of (1). It is assumed that y_h and z_h are given on the initial interval $[\gamma, a]$. The coefficients b_j^p, $b_j^{p+1} \in [0, 1]$ and the functions a_j^p, $c_j^{p+1} \in C[0, 1]$ will be chosen in such a way that method (2) would have the highest possible order. Usually, the b_j^q are distinct, $b_1^{p+1} = 0$, $b_{p+1}^{p+1} = 1$, and

$$a_j^q(r) = \int_0^r c_j^q(s) ds, \tag{3}$$

where c_j^q are Lagrange's fundamental polynomials

$$c_j^q(r) = \prod_{\substack{i=1 \\ i \neq j}}^{q} \frac{r - b_i^q}{b_j^q - b_i^q}. \tag{4}$$

Methods of type (2) have been examined in [16] for the equations

$$\begin{aligned} y'(t) &= F(t, y(\cdot), y'(\cdot)), & t &\in [a, b], \\ y(t) &= g(t), & t &\in [\gamma, a], \end{aligned} \tag{5}$$

where F: $[a, b] \times C[a, b] \times C[a, b] \to R$ is a Volterra operator which is not state dependent (see also [14], [17]).

In recent years the numerical solution of functional differential equations (FDEs), i.e., equations of type (5) with F independent of y', has received wide attention in the literature (Cryer [3] gives an extensive survey of the work done in this area prior to 1972; for more recent developments see [11], [13], [20]–[23]). On the other hand, less is known about equations with delays dependent on the solution. Neves [20], [21] described an algorithm based on Runge–Kutta–Merson formulas of the fourth order for a system of FDEs which is also applicable in the case of solution-dependent delays. An adaptation of algorithms for ordinary differential equations to delay-differential equations (DDEs) has also been considered by Arndt [1]. Tavernini [25] discussed construction of one-step methods of any order for FDEs where the delays can depend on the solution. High order methods for state-dependent DDEs with non-smooth solutions were also developed by Feldstein and Neves in a recent paper [7]. All these papers are primarily concerned with equations of type (1) with F independent of y', while there is an almost complete lack of results for equations of neutral type. The only paper the author is aware of concerning the numerical solution of (1) is the paper by Castleton and Grimm [2] in which two first order methods were constructed. In [15] the results of numerical experiments were presented which suggest that the construction of higher order methods for (1) is possible. However, the theoretical analysis of numerical methods considered in [15] was restricted to the equations of type (5). This paper partially fills this gap providing the analysis of one-step methods (2) for NDDEs (1). The analysis of linear multistep methods for these equations is currently under investigation.

The organization of this paper is as follows. In Section 2 we investigate if method (2) is well defined and also consider the iteration scheme for solving (2). A convergence theorem is given in Section 3 and the asymptotic behaviour of the global discretization error is examined in Section 4. Local error estimation and step changing strategy are discussed in Section 5. Finally, in Section 6, some results of numerical experiments are presented.

2. Existence and uniqueness of solutions of system (2). We have the following existence result:

THEOREM 1. *Assume that f, α, β are continuous; $g \in C^1[\gamma, a]$; b_j^p, b_j^{p+1} are distinct; $a_j^p(0) = 0$; $c_j^{p+1}(b_k^{p+1}) = \delta_{j,k}$; and that (1) has a unique solution $Y \in C^1[\gamma, b]$. Then for any $h > 0$ system (2) with $y_h(t) = g(t)$, $z_h(t) = g'(t)$ for $t \in [\gamma, a]$ has a continuous solution (y_h, z_h) (not necessarily unique).*

Proof. Observe first that the condition that (1) has a unique solution Y allows us to assume, without loss of generality, that the function f is bounded. Indeed, put

$$\bar{f} = f \,|\, [a, b] \times [-M, M] \times [-M_\alpha, M_\alpha] \times [-M_\beta, M_\beta],$$

where M, M_α, and M_β are constants such that

$$M > \sup\{|Y(t)|:\ t \in [\gamma, b]\},$$
$$M_\alpha > \sup\{|Y(\alpha(t, Y(t)))|:\ t \in [\gamma, b]\},$$
$$M_\beta > \sup\{|Y(\beta(t, Y(t)))|:\ t \in [\gamma, b]\},$$

and denote by $\bar{f}$ a continuous, bounded extension of $\bar{f}$ to $[a, b] \times R^3$. Then problems (1) with f and $\bar{f}$, respectively, have the same solution Y.

Assume that y_h and z_h are already defined on $[\gamma, t_i]$. We will show that they can be continuously extended on $[\gamma, t_{i+1}]$. To this end define the mapping

$$\mathscr{F}\colon R^{p+1} \to R^{p+1}$$

by

$$(\mathscr{F}(x))_j = f\big(t_i + b_j^{p+1} h, (y_x)_j, (y_{x,\alpha})_j, (z_{x,\beta})_j\big),$$

$j = 1, 2, \ldots, p+1$, where

$$(y_x)_j = y_h(t_i) + h \sum_{\mu=1}^{p} \sum_{\nu=1}^{p+1} a_\mu^p(b_j^{p+1}) c_\nu^{p+1}(b_\mu^p) x_\nu;$$

$$(y_{x,\alpha})_j = \begin{cases} y_h\big(\alpha(t_i + b_j^{p+1} h, (y_x)_j)\big), & \alpha(t_i + b_j^{p+1} h, (y_x)_j) \leqslant t_i, \\ y_h(t_i) + h \displaystyle\sum_{\mu=1}^{p} \sum_{\nu=1}^{p+1} a_\mu^p((r_{x,\alpha})_j) c_\nu^{p+1}(b_\mu^p) x_\nu, & \alpha(t_i + b_j^{p+1} h, (y_x)_j) > t_i; \end{cases}$$

$$(z_{x,\beta})_j = \begin{cases} z_h\big(\beta(t_i + b_j^{p+1} h, (y_x)_j)\big), & \beta(t_i + b_j^{p+1} h, (y_x)_j) \leqslant t_i, \\ \displaystyle\sum_{\nu=1}^{p+1} c_\nu^{p+1}((r_{x,\beta})_j) x_\nu, & \beta(t_i + b_j^{p+1} h, (y_x)_j) > t_i; \end{cases}$$

$$(r_{x,\alpha})_j = \big(\alpha(t_i + b_j^{p+1} h, (y_x)_j) - t_i\big)/h;$$

$$(r_{x,\beta})_j = \big(\beta(t_i + b_j^{p+1} h, (y_x)_j) - t_i\big)/h;$$

and for any $z \in R^{p+1}$, $(z)_j$ denotes the j-th component of z. Since we can assume f to be bounded, it follows from the Schauder fixed point theorem that the system $x = \mathscr{F}(x)$ has at least one solution x. Putting

$$z_h(t_i + rh) = \sum_{\nu=1}^{p+1} c_\nu^{p+1}(r) x_\nu,$$

$$y_h(t_i + rh) = y_h(t_i) + h \sum_{\mu=1}^{p} a_\mu^p(r) z_h(t_i + b_\mu^p h),$$

$r \in (0, 1]$, we can extend y_h and z_h on the interval $[\gamma, t_{i+1}]$. It is also clear that the conditions imposed on the coefficients of method (2) guarantee that these extensions are continuous. This completes the proof.

To discuss the uniqueness of solutions of (2) we require the following:

(i) g and g' are Lipschitz-continuous with constants L_g and $L_{g'}$, respectively;

(ii) f is bounded by a constant $M_f \geqslant 0$ and

$$|f(t_1, y_1, u_1, z_1) - f(t_2, y_2, u_2, z_2)| \leqslant L_1(|t_1 - t_2| + |y_1 - y_2| + |u_1 - u_2|) + L_2 |z_1 - z_2|,$$

$L_1, L_2 \geqslant 0$, $t_1, t_2 \in [a, b]$, $y_1, y_2, u_1, u_2, z_1, z_2 \in R$;

(iii) $$|\alpha(t_1, y_1)-\alpha(t_2, y_2)| \leqslant A_1|t_1-t_2|+A_2|y_1-y_2|,$$

$A_1, A_2 \geqslant 0,\ t_1, t_2 \in [a, b],\ y_1, y_2 \in R;$

(iv) $$|\beta(t_1, y_1)-\beta(t_2, y_2)| \leqslant B_1|t_1-t_2|+B_2|y_1-y_2|,$$

$B_1, B_2 \geqslant 0,\ t_1, t_2 \in [a, b],\ y_1, y_2 \in R.$

Additional conditions on some Lipschitz constants appearing above will be given in the formulation of the uniqueness theorem.

Using the Newton interpolation formula we can write the approximation z_h to Y' in the form

$$z_h(t_i+rh) = \sum_{j=1}^{p+1} h^{j-1}q_j(r)z_h[t_i+b_1^{p+1}h, \ldots, t_i+b_j^{p+1}h],$$

where

(6) $$q_j(r) = \prod_{\nu=1}^{j-1}(r-b_\nu^{p+1}),$$

$j = 1, 2, \ldots, p+1$, $\prod_{\nu=1}^{0} = 1$ and $z_h[t_i+b_1^{p+1}h, \ldots, t_i+b_j^{p+1}h]$ are Newton divided differences. Denote by Q_j, $j = 2, 3, \ldots, p+1$, Lipschitz constants of q_j on $[0, 1]$ and by E_j the Lipschitz constants of the coefficients a_j^p defined by (3). Define

$$\delta = \min\{|b_\mu^{p+1}-b_\nu^{p+1}|:\ \mu \neq \nu\},$$

$$Q = \sum_{j=2}^{p+1}(2^{j-2}Q_j/\delta^{j-2}), \qquad E = \sum_{j=1}^{p} E_j,$$

$$L_y = \max\{L_g, EM_f\}, \qquad C = \sup\{\sum_{j=1}^{p+1}|c_j^{p+1}(r)|:\ r \in (0, 1]\}.$$

We have the following uniqueness result:

THEOREM 2. *Assume that* (i)–(iv) *hold;* $L_2C < 1$; $QL_2(B_1+B_2L_y) < 1$; b_j^q, $q = p, p+1$, *are distinct numbers from* $[0, 1]$; $b_1^{p+1} = 0$; a_j^p *and* c_j^{p+1} *are defined by* (3) *and* (4), *respectively, and that problem* (1) *has a unique solution* Y. *Then for sufficiently small* h *system* (2) *has a unique solution* (y_h, z_h).

Proof. The existence follows from Theorem 1. To show the uniqueness we will prove first that y_h and z_h defined by (2) are uniformly bounded and uniformly Lipschitz-continuous. For any function $x \in C[\gamma, b]$ and $[c, d] \subset [\gamma, b]$ put

$$\|x\|_{[c,d]} := \sup\{|x(t)|:\ t \in [c, d]\}.$$

In view of (ii) it is clear that

$$\|z_h\|_{[\gamma,b]} \leqslant M_z := CM_f.$$

We also have

$$\|y_h\|_{[\gamma,t_{i+1}]} \leqslant \|y_h\|_{[\gamma,t_i]} + hA\,\|z_h\|_{[\gamma,t_{i+1}]},$$

$i = 0, 1, \ldots, N-1$, where

$$A = \sup\Big\{\sum_{j=1}^{p} |a_j^p(r)|:\ r\in(0,\ 1]\Big\}.$$

Consequently,

$$\|y_h\|_{[\gamma,t_{i+1}]} \leqslant \|y_h\|_{[\gamma,t_i]} + hACM_f$$

and

$$\|y_h\|_{[\gamma,b]} \leqslant M_y := \|g\|_{[\gamma,a]} + (b-a)ACM_f.$$

Now we will show that y_h and z_h are uniformly Lipschitz-continuous. By (i) this is true on $[\gamma,\ a]$. Denote by $L_z \geqslant L_{g'}$ a constant such that

$$QL_1\big(1+L_y(1+A_1+A_2L_y)\big) + QL_2L_z(B_1+B_2L_y) < L_z,$$

the existence of which follows from the condition

$$QL_2(B_1+B_2L_y) < 1.$$

Assume that y_h and z_h are Lipschitz-continuous on $[\gamma,\ t_i]$ with constants L_y and L_z, respectively. We will show that there exist Lipschitz-continuous extensions of y_h and z_h on $[\gamma,\ t_{i+1}]$ and that the Lipschitz constants are preserved. Put

$$y_h^{[n]}(t) = y_h(t), \qquad z_h^{[n]}(t) = z_h(t), \qquad n = 0, 1, \ldots,$$

for $t\in[\gamma,\ t_i]$ and define the following iteration scheme on $[t_i,\ t_{i+1}]$:

$$z_h^{[n+1]}(t_i+b_j^{p+1}h) = F(t_i+b_j^{p+1}h,\ y_h^{[n]},\ z_h^{[n]}),$$

$$z_h^{[n+1]}(t_i+rh) = \sum_{j=1}^{p+1} c_j^{p+1}(r)z_h^{[n+1]}(t_i+b_j^{p+1}h),$$

$$z_h^{[n+1]}(t_i+rh) = y_h(t_i) + h\sum_{j=1}^{p} a_j^p(r)z_h^{[n+1]}(t_i+b_j^p h),$$

$n = 0, 1, \ldots,\ r\in(0,\ 1]$, with

$$z_h^{[0]}(t_i+rh) = z_h(t_i),$$

$$y_h^{[0]}(t_i+rh) = y_h(t_i) + hrz_h(t_i),$$

$r\in(0,\ 1]$. It is clear that $z_h^{[0]}$ and $y_h^{[0]}$ are Lipschitz-continuous on $[\gamma,\ t_{i+1}]$ with constants L_z and L_y, respectively. Assume that this is true for $z_h^{[n]}$ and $y_h^{[n]}$. Then in view of (ii)–(iv) we obtain

$$\begin{aligned}
&|z_h^{[n+1]}(t_i+b_\mu^{p+1}h)-z_h^{[n+1]}(t_i+b_\nu^{p+1}h)| \leqslant L_1(1+L_y)|b_\mu^{p+1}-b_\nu^{p+1}|h \\
&+L_1\left|y_h^{[n]}\left(\alpha\left(t_i+b_\mu^{p+1}h,\ y_h^{[n]}(t_i+b_\mu^{p+1}h)\right)\right)-y_h^{[n]}\left(\alpha\left(t_i+b_\nu^{p+1}h,\ y_h^{[n]}(t_i+b_\mu^{p+1}h)\right)\right)\right| \\
&+L_1\left|y_h^{[n]}\left(\alpha\left(t_i+b_\nu^{p+1}h,\ y_h^{[n]}(t_i+b_\mu^{p+1}h)\right)\right)-y_h^{[n]}\left(\alpha\left(t_i+b_\nu^{p+1}h,\ y_h^{[n]}(t_i+b_\nu^{p+1}h)\right)\right)\right| \\
&+L_2\left|z_h^{[n]}\left(\beta\left(t_i+b_\mu^{p+1}h,\ y_h^{[n]}(t_i+b_\mu^{p+1}h)\right)\right)-z_h^{[n]}\left(\beta\left(t_i+b_\nu^{p+1}h,\ y_h^{[n]}(t_i+b_\mu^{p+1}h)\right)\right)\right| \\
&+L_2\left|z_h^{[n]}\left(\beta\left(t_i+b_\nu^{p+1}h,\ y_h^{[n]}(t_i+b_\mu^{p+1}h)\right)\right)-z_h^{[n]}\left(\beta\left(t_i+b_\nu^{p+1}h,\ y_h^{[n]}(t_i+b_\nu^{p+1}h)\right)\right)\right| \\
&\qquad \leqslant \left[L_1(1+L_y(1+A_1+A_2L_y))+L_2L_z(B_1+B_2L_y)\right]|b_\mu^{p+1}-b_\nu^{p+1}|h \\
&\qquad \leqslant (L_z/Q)|b_\mu^{p+1}-b_\nu^{p+1}|h.
\end{aligned}$$

Using the Newton representation of $z_h^{[n+1]}$ we obtain

$$\begin{aligned}
&|z_h^{[n+1]}(t_i+r_1h)-z_h^{[n+1]}(t_i+r_2h)| \\
&\qquad \leqslant \sum_{j=2}^{p+1}|q_j(r_1)-q_j(r_2)|\ |h^{j-1}z_h^{[n+1]}[t_i+b_1^{p+1}h,\ \ldots,\ t_i+b_j^{p+1}h]|,
\end{aligned}$$

where the q_j are defined by (6). It can easily be proved by induction that

$$|h^{j-2}z_h^{[n+1]}[t_i+b_1^{p+1}h,\ \ldots,\ t_i+b_j^{p+1}h]| \leqslant \frac{2^{j-2}}{\delta^{j-2}}\frac{L_z}{Q},$$

$j = 2, 3, \ldots, p+1$. Hence

$$|z_h^{[n+1]}(t_i+r_1h)-z_h^{[n+1]}(t_i+r_2h)| \leqslant \sum_{j=2}^{p+1}\frac{2^{j-2}}{\delta^{j-2}}Q_j\frac{L_z}{Q}|r_1-r_2|h = L_z|r_1-r_2|h.$$

It follows from the Arzelà–Ascoli theorem that $\{y_h^{[n]}\}_{n\in N}$ and $\{z_h^{[n]}\}_{n\in N}$ are relatively compact in $C[\gamma, t_{i+1}]$. Let $\{y_h^{[n]}\}_{n\in N'}$ and $\{z_h^{[n]}\}_{n\in N'}$, $N' \subset N$, $\sup\{N'\} = \infty$, be any convergent subsequences of $\{y_h^{[n]}\}_{n\in N}$ and $\{z_h^{[n]}\}_{n\in N}$ and denote their limits by y_h and z_h. It is clear that y_h and z_h satisfy (2) on $[\gamma, t_{i+1}]$ and are Lipschitz-continuous with the same constants L_y and L_z, respectively.

To show that (y_h, z_h) is the unique solution let us assume that there is another solution $(\bar{y}_h, \bar{z}_h)$ of (2). Then it follows that

$$\begin{aligned}
&|z_h(t_i+b_\mu^{p+1}h)-\bar{z}_h(t_i+b_\mu^{p+1}h)| \leqslant L_1|y_h(t_i+b_\mu^{p+1}h)-\bar{y}_h(t_i+b_\mu^{p+1}h)| \\
&+L_1\left|y_h\left(\alpha\left(t_i+b_\mu^{p+1}h,\ y_h(t_i+b_\mu^{p+1}h)\right)\right)-y_h\left(\alpha\left(t_i+b_\mu^{p+1}h,\ \bar{y}_h(t_i+b_\mu^{p+1}h)\right)\right)\right| \\
&+L_1\left|y_h\left(\alpha\left(t_i+b_\mu^{p+1}h,\ \bar{y}_h(t_i+b_\mu^{p+1}h)\right)\right)-\bar{y}_h\left(\alpha\left(t_i+b_\mu^{p+1}h,\ \bar{y}_h(t_i+b_\mu^{p+1}h)\right)\right)\right| \\
&+L_2\left|z_h\left(\beta\left(t_i+b_\mu^{p+1}h,\ y_h(t_i+b_\mu^{p+1}h)\right)\right)-z_h\left(\beta\left(t_i+b_\mu^{p+1}h,\ \bar{y}_h(t_i+b_\mu^{p+1}h)\right)\right)\right| \\
&+L_2\left|z_h\left(\beta\left(t_i+b_\mu^{p+1}h,\ \bar{y}_h(t_i+b_\mu^{p+1}h)\right)\right)-\bar{z}_h\left(\beta\left(t_i+b_\mu^{p+1}h,\ \bar{y}_h(t_i+b_\mu^{p+1}h)\right)\right)\right| \\
&\qquad \leqslant (L_1(2+L_yA_2)+L_2L_zB_2)\|y_h-\bar{y}_h\|_{[\gamma,t_{i+1}]}+L_2\|z_h-\bar{z}_h\|_{[\gamma,t_{i+1}]},
\end{aligned}$$

and

$$\|z_h-\bar{z}_h\|_{[\gamma,t_{i+1}]} \leqslant C\big(L_1(2+L_yA_2)+L_2L_zB_2\big)\|y_h-\bar{y}_h\|_{[\gamma,t_{i+1}]}+CL_2\|z_h-\bar{z}_h\|_{[\gamma,t_{i+1}]}.$$

In view of the condition $L_2C<1$ we get

$$\|z_h-\bar{z}_h\|_{[\gamma,t_{i+1}]} \leqslant D\|y_h-\bar{y}_h\|_{[\gamma,t_{i+1}]}, \tag{7}$$

where

$$D:=\frac{C\big(L_1(2+L_yA_2)+L_2L_zB_2\big)}{1-L_2C}.$$

Subtracting the equations for y_h and $\bar{y}_h$ and taking (7) into account we obtain

$$\|y_h-\bar{y}_h\|_{[\gamma,t_{i+1}]} \leqslant \|y_h-\bar{y}_h\|_{[\gamma,t_i]}+hAD\|y_h-\bar{y}_h\|_{[\gamma,t_{i+1}]}.$$

Let $h_0>0$ be such that $1-h_0AD>0$. Then for $h<h_0$ we get

$$\|y_h-\bar{y}_h\|_{[\gamma,t_{i+1}]} \leqslant (1+hG)\|y_h-\bar{y}_h\|_{[\gamma,t_i]},$$

where $G:=AD/(1-h_0AD)$. But $\|y_h-\bar{y}_h\|_{[\gamma,a]}=0$, therefore $y_h\equiv\bar{y}_h$ and $z_h\equiv\bar{z}_h$ on $[\gamma,b]$, which is our claim.

3. A convergence theorem. Define the local errors η and ν of method (2) by

$$\begin{aligned}\eta(t,r,h)&:=Y(t+rh)-Y(t)-h\sum_{j=1}^{p}a_j^p(r)\,Y'(t+b_j^ph),\\ \nu(t,r,h)&:=Y'(t+rh)-\sum_{j=1}^{p+1}c_j^{p+1}(r)\,Y'(t+b_j^{p+1}h),\end{aligned} \tag{8}$$

where Y is the solution of (1). Method (2) is said to be *consistent* if

$$\eta(t,r,h)=o(1),\quad \eta(t,1,h)=o(h),\quad \text{and}\quad \nu(t,r,h)=o(1)$$

uniformly in t and r as $h\to 0$. Method (2) is said to be *of order* p if

$$\eta(t,r,h)=O(h^p),\quad \eta(t,1,h)=O(h^{p+1}),\quad \text{and}\quad \nu(t,r,h)=O(h^p)$$

uniformly in t and r as $h\to 0$ (cf. [14]–[17]). Method (2) is said to be *convergent* if

$$\|y_h-Y\|_{[\gamma,b]}\to 0\quad \text{as } h\to 0.$$

The *order of convergence* is p if

$$\|y_h-Y\|_{[\gamma,b]}=O(h^p)\quad \text{as } h\to 0.$$

We have the following convergence result:

THEOREM 3. *Assume that conditions* (i)–(iv) *hold;* $L_2C<1$; *method* (2) *is consistent and*

$$\|y_h-g\|_{[\gamma,a]}=o(1),\quad \|z_h-g'\|_{[\gamma,a]}=o(1)\quad \textit{as } h\to 0.$$

Moreover, assume that problem (1) *has a unique solution Y whose derivative is Lipschitz-continuous. Then every solution of* (2) *is convergent to Y. If, in addition, method* (2) *is of order p and starting errors are of order p, then the order of convergence is also p.*

Proof. The proof of this theorem is similar to that in [16] and [17] with some modifications to take into account state-dependent delays. Put

$$\varepsilon_h := y_h - Y, \quad e_h := z_h - Y',$$

and

$$\eta(h) := \sup\{|\eta(t, r, h)|:\ t \in [a, b-h],\ r \in (0, 1]\},$$

$$\mu(h) := \sup\{|\eta(t, 1, h)|:\ t \in [a, b-h]\},$$

$$v(h) := \sup\{|v(t, r, h)|:\ t \in [a, b-h],\ r \in (0, 1]\}.$$

Subtracting (2) and (8) we obtain

$$\varepsilon_h(t_i + rh) = \varepsilon_h(t_i) + h \sum_{j=1}^{p} a_j^p(r) e_h(t_i + b_j^p h) - \eta(t_i, r, h),$$

$$e_h(t_i + rh) = \sum_{j=1}^{p+1} c_j^{p+1}(r) e_h(t_i + b_j^{p+1} h) - v(t_i, r, h),$$

$$e_h(t_i + b_j^{p+1} h) = F(t_i + b_j^{p+1} h, y_h, z_h) - F(t_i + b_j^{p+1} h, Y, Y'),$$

$i = 0, 1, \ldots, N-1$, $r \in (0, 1]$. Denote by L_Y and $L_{Y'}$ the Lipschitz constants of Y and Y', respectively. Then proceeding similarly as in the proof of Theorem 2 we obtain

$$|e_h(t_i + b_j^{p+1} h)| \leqslant \big(L_1(2 + L_Y A_2) + L_2 L_{Y'} B_2\big) \|\varepsilon_h\|_{[\gamma, t_{i+1}]} + L_2 \|e_h\|_{[\gamma, t_{i+1}]},$$

and

$$\|e_h\|_{[\gamma, t_{i+1}]} \leqslant \|e_h\|_{[\gamma, a]} + \big(L_1(2 + L_Y A_2) + L_2 L_{Y'} B_2\big) C \|\varepsilon_h\|_{[\gamma, t_{i+1}]}$$
$$+ L_2 C \|e_h\|_{[\gamma, t_{i+1}]} + v(h).$$

Taking the condition $L_2 C < 1$ into account and putting

$$D = \max\left\{\frac{1}{1 - L_2 C}, \frac{\big(L_1(2 + L_Y A_2) + L_2 L_{Y'} B_2\big) C}{1 - L_2 C}\right\},$$

we get

$$\|e_h\|_{[\gamma, t_{i+1}]} \leqslant D\big(\|e_h\|_{[\gamma, a]} + \|\varepsilon_h\|_{[\gamma, t_{i+1}]} + v(h)\big), \tag{9}$$

$i = 0, 1, \ldots, N-1$. To estimate $\|\varepsilon_h\|_{[\gamma, t_{i+1}]}$ observe first that

$$|\varepsilon_h(t_{i+1})| \leqslant |\varepsilon_h(t_i)| + hA \|e_h\|_{[\gamma, t_{i+1}]} + \mu(h),$$

and by induction it follows that

$$|\varepsilon_h(t_i)| \leqslant |\varepsilon_h(t_0)| + hA \sum_{j=1}^{i} \|e_h\|_{[\gamma,t_j]} + i\mu(h).$$

We also have

$$\|\varepsilon_h\|_{[t_i,t_{i+1}]} \leqslant |\varepsilon_h(t_i)| + hA\,\|e_h\|_{[\gamma,t_{i+1}]} + \eta(h),$$

$$\|\varepsilon_h\|_{[\gamma,t_{i+1}]} \leqslant \|\varepsilon_h\|_{[\gamma,a]} + hA \sum_{j=1}^{i+1} \|e_h\|_{[\gamma,t_j]} + (b-a)\mu(h)/h + \eta(h). \tag{10}$$

Combining (9) and (10) we get

$$\begin{aligned}\|\varepsilon_h\|_{[\gamma,t_{i+1}]} \leqslant\ & \|\varepsilon_h\|_{[\gamma,a]} + (b-a)AD\,\|e_h\|_{[\gamma,a]} \\ & + hAD \sum_{j=1}^{i+1} \|e_h\|_{[\gamma,t_j]} + (b-a)AD\nu(h) + (b-a)\mu(h)/h + \eta(h),\end{aligned}$$

$i = 0, 1, \ldots, N-1$. Let $h_0 > 0$ be such that $1 - h_0 AD > 0$. Then for $h < h_0$, using standard arguments as in [16] and [17], we obtain

$$\|\varepsilon_h\|_{[\gamma,b]} \leqslant M[\|\varepsilon_h\|_{[\gamma,a]} + \|e_h\|_{[\gamma,a]} + \eta(h) + \mu(h)/h + \nu(h)]$$

for some non-negative constant M, which completes the proof.

4. Asymptotic behaviour of the global discretization error. Method (2) is said to be *of strong order p* if $\eta(t, r, h) = O(h^{p+1})$ and $\nu(t, r, h) = O(h^{p+1})$ as $h \to 0$ uniformly in t and r, where the local errors η and ν are defined by (8). It is easy to check that (2) has strong order p if b_j^q are distinct, a_j^p are defined by (3), and c_j^{p+1} are defined by (4). We have the following generalization of the result given in [17]:

THEOREM 4. *Assume that f and Y are sufficiently smooth; $\varepsilon_h(t) = O(h^{p+1})$ and $e_h(t) = O(h^{p+1})$ for $t \in [\gamma, a]$; method (2) is of strong order p; there exists a function $u \in C[a, b]$ such that*

$$\eta(t, 1, h) = h^{p+1}u(t) + O(h^{p+2}) \quad \textit{for } t \in [a, b] \textit{ as } h \to 0;$$

moreover, the system of equations

$$\begin{aligned} \varepsilon'(t) - e(t) &= -u(t), \quad t \in [a, b], \\ e(t) &= D_2 f(\omega)\varepsilon(t) + D_3 f(\omega)\big[\varepsilon\big(\alpha(t, Y(t))\big) + Y'\big(\alpha(t, Y(t))\big)D_2\alpha(t, Y(t))\varepsilon(t)\big] \\ &\quad + D_4 f(\omega)\big[e\big(\beta(t, Y(t))\big) + Y''\big(\beta(t, Y(t))\big)D_2\beta(t, Y(t))\varepsilon(t)\big], \quad t \in [a, b], \\ \varepsilon(t) &= e(t) = 0, \quad t \in [\gamma, a], \end{aligned} \tag{11}$$

where $\omega = \big(t, Y(t), Y\big(\alpha(t, Y(t))\big), Y'\big(\beta(t, Y(t))\big)\big)$, has a solution (ε, e) of class C^1. Then

$$\varepsilon_h(t) = y_h(t) - Y(t) = h^p\varepsilon(t) + O(h^{p+1}), \quad e_h(t) = z_h(t) - Y'(t) = h^p e(t) + O(h^{p+1})$$

for $t \in [\gamma, b]$ as $h \to 0$.

Proof. As in [17], substituting $Y(t)+h^p\varepsilon(t)$ and $Y'(t)+h^p e(t)$ into (2) in place of $y_h(t)$ and $z_h(t)$ and denoting the resulting residues by ξ and ϱ, we obtain

$$Y(t_i+rh)+h^p\varepsilon(t_i+rh) = Y(t_i)+h^p\varepsilon(t)$$
$$+h\sum_{j=1}^{p} a_j^p(r)\big(Y'(t_i+b_j^p h)+h^p e(t_i+b_j^p h)\big)+\xi(t_i, r, h),$$

and

$$Y'(t_i+rh)+h^p e(t_i+rh)$$
$$=\sum_{j=1}^{p+1} c_j^{p+1}(r)F(t_i+b_j^{p+1}h,\ Y+h^p\varepsilon,\ Y'+h^p e)+\varrho(t_i, r, h).$$

It follows that

$$\xi(t_i, r, h) = \eta(t_i, r, h)+h^p\Big[\varepsilon(t_i+rh)-\varepsilon(t_i)-h\sum_{j=1}^{p} a_j^p(r)e(t_i+b_j^p h)\Big].$$

Expanding ε and e around t_i and taking the relation

$$\sum_{j=1}^{p} a_j^p(r) = r$$

into account, we get

$$\xi(t_i, r, h) = \eta(t_i, r, h)+h^{p+1}[\varepsilon'(t_i)-re(t_i)]+O(h^{p+2}).$$

Hence $\xi(t_i, r, h) = O(h^{p+1})$ and $\xi(t_i, 1, h) = O(h^{p+2})$ as $h\to 0$. We also have

$$\varrho(t_i, r, h) = \nu(t_i, r, h)$$
$$+h^p\Big(e(t_i+rh)-\sum_{j=1}^{p+1} c_j^{p+1}(r)\big(D_2 f(\omega_{i,j})\varepsilon(t_i+b_j^{p+1}h)$$
$$+D_3 f(\omega_{i,j})\big[\varepsilon\big(\alpha(\theta_{i,j})\big)+Y'\big(\alpha(\theta_{i,j})\big)D_2\alpha(\theta_{i,j})\varepsilon(t_i+b_j^{p+1}h)\big]$$
$$+D_4 f(\omega_{i,j})\big[e\big(\beta(\theta_{i,j})\big)+Y''\big(\beta(\theta_{i,j})\big)D_2\beta(\theta_{i,j})\varepsilon(t_i+b_j^{p+1}h)\big]\big)\Big)$$
$$+O(h^{p+1}),$$

where

$$\theta_{i,j} = \big(t_i+b_j^{p+1}h,\ Y(t_i+b_j^{p+1}h)\big),$$
$$\omega_{i,j} = \big(\theta_{i,j},\ Y\big(\alpha(\theta_{i,j})\big),\ Y'\big(\beta(\theta_{i,j})\big)\big).$$

If we expand around t_i, since $\sum_{j=1}^{p+1} c_j^{p+1}(r) = 1$, it follows that

$$\varrho(t_i, r, h) = \nu(t_i, r, h)+h^p\big(e(t_i)-D_2 f(\omega_i)\varepsilon(t_i)$$
$$-D_3 f(\omega_i)\big[\varepsilon\big(\alpha(\theta_i)\big)+Y'\big(\alpha(\theta_i)\big)D_2\alpha(\theta_i)\varepsilon(t_i)\big]$$
$$-D_4 f(\omega_i)\big[e\big(\beta(\theta_i)\big)+Y''\big(\beta(\theta_i)\big)D_2\beta(\theta_i)\varepsilon(t_i)\big]\big)+O(h^{p+1}),$$

where $\theta_i = (t_i, Y(t_i))$, $\omega_i = (\theta_i, Y(\alpha(\theta_i)), Y'(\beta(\theta_i)))$. Consequently, in view of (11),

$$\varrho(t_i, r, h) = O(h^{p+1}) \quad \text{as } h \to 0.$$

From this point, putting

$$\varphi(t) = Y(t) + h^p \varepsilon(t) - y_h(t),$$
$$\psi(t) = Y'(t) + h^p e(t) - z_h(t),$$

and proceeding exactly as in the proof of Theorem 3, we obtain

$$\varphi(t) = O(h^{p+1}) \quad \text{and} \quad \psi(t) = O(h^{p+1}) \quad \text{as } h \to 0,$$

which is our claim.

5. Local error estimation and step changing strategy. We use local extrapolation (see [9] and [10]) to estimate the local discretization error of method (2). Performing one step of size h we obtain

$$y_h(t_{i+1}) = y_h(t_i) + h \sum_{j=1}^{p} a_j^p(1) z_h(t_i + b_j^p h),$$

and the corresponding local discretization error is

$$\eta(t_i, 1, h) = Y(t_{i+1}) - Y(t_i) - h \sum_{j=1}^{p} a_j^p(1) Y'(t_i + b_j^p h).$$

Performing two steps of size $h/2$ we get

$$y_h^*(t_{i+1}) = y_h(t_i) + (h/2) \sum_{j=1}^{p} a_j^p(1) [z_h^*(t_i + b_j^p h/2) + z_h^*(t_{i+1/2} + b_j^p h/2)],$$

and the corresponding local discretization error is

$$\begin{aligned} \eta^*(t_i, 1, h) &= Y(t_{i+1}) - Y(t_i) - (h/2) \sum_{j=1}^{p} a_j^p(1) [Y'(t_i + b_j^p h/2) + Y'(t_{i+1/2} + b_j^p h/2)] \\ &= \eta(t_i, 1, h/2) + \eta(t_{i+1/2}, 1, h/2). \end{aligned}$$

Assuming that $\eta(t, 1, h) = h^{p+1} u(t) + O(h^{p+2})$ as $h \to 0$ and that the function u is continuously differentiable, we obtain

$$\eta^*(t_i, 1, h) = (\tfrac{1}{2})^p h^{p+1} u(t_i) + O(h^{p+2}) \quad \text{as } h \to 0.$$

The function u is called the *principal error function.* To estimate the principal part of $\eta(t_i, 1, h)$ and $\eta^*(t_i, 1, h)$, i.e., $h^{p+1} u(t_i)$ and $(\frac{1}{2})^p h^{p+1} u(t_i)$, we assume that the conditions of Theorem 4 are satisfied. Then

$$y_h(t) = Y(t) + h^p \varepsilon(t) + O(h^{p+1}),$$
$$z_h(t) = Y'(t) + h^p e(t) + O(h^{p+1}),$$

and it follows that

$$y_h(t_{i+1}) = Y(t_i)+\varepsilon_h(t_i)+h\sum_{j=1}^{p} a_j^p(1)[Y'(t_i+b_j^p h)+h^p e(t_i+b_j^p h)]+O(h^{p+2})$$

$$= Y(t_{i+1})-\eta(t_i, 1, h)+\varepsilon_h(t_i)$$

$$+h^{p+1}\sum_{j=1}^{p} a_j^p(1)e(t_i+b_j^p h)+O(h^{p+2}).$$

Similarly,

$$y_h^*(t_{i+1}) = Y(t_i)+\varepsilon_h(t_i)+(h/2)\sum_{j=1}^{p} a_j^p(1)[Y'(t_i+b_j^p h/2)+h^p e(t_i+b_j^p h/2)$$

$$+Y'(t_{i+1/2}+b_j^p h/2)+h^p e(t_{i+1/2}+b_j^p h/2)]+O(h^{p+2})$$

$$= Y(t_{i+1})-\eta^*(t_i, 1, h)+\varepsilon_h(t_i)+(h^{p+1}/2)\sum_{j=1}^{p} a_j^p(1)[e(t_i+b_j^p h/2)$$

$$+e(t_{i+1/2}+b_j^p h/2)]+O(h^{p+2}).$$

Hence

$$y_h(t_{i+1}) = Y(t_{i+1})-\eta(t_i, 1, h)+\varepsilon_h(t_i)+h^{p+1}e(t_i)+O(h^{p+2})$$

and

$$y_h^*(t_{i+1}) = Y(t_{i+1})-\eta^*(t_i, 1, h)+\varepsilon_h(t_i)+h^{p+1}e(t_i)+O(h^{p+2}).$$

Subtracting these equations we get

$$y_h^*(t_{i+1})-y_h(t_{i+1}) = \eta(t_i, 1, h)-\eta^*(t_i, 1, h)+O(h^{p+2}).$$

Consequently, the principal parts of the local errors $\eta(t_i, 1, h)$ and $\eta^*(t_i, 1, h)$ are

(12) $$h^{p+1}u(t_i) = \frac{y_h^*(t_{i+1})-y_h(t_{i+1})}{1-2^{-p}}+O(h^{p+2})$$

and

(13) $$(\tfrac{1}{2})^p h^{p+1}u(t_i) = \frac{y_h^*(t_{i+1})-y_h(t_{i+1})}{2^p-1}+O(h^{p+2}).$$

We summarize the above discussion in the following

THEOREM 5. *Under the conditions of Theorem* 4, *the principal part of the local discretization error* $\eta(t_i, 1, h)$ *corresponding to one step of size h is given by* (12) *and the principal part of the local discretization error* $\eta^*(t_i, 1, h)$ *corresponding to two steps of size* $h/2$ *is given by* (13).

After completing one step we have a choice of starting the integration with $y_h(t_{i+1})$ or $y_h^*(t_{i+1})$. In our numerical examples in the next section we start the integration with $y_h^*(t_{i+1})$. Put

$$\mathrm{LE} := \frac{y_h^*(t_{i+1}) - y_h(t_{i+1})}{2^p - 1}.$$

If LE is less than or equal to the given tolerance TOL, the step is accepted and the new step size is computed from the formula

$$h_{\text{new}} = \min\{2h_{\text{old}},\ 0.9h_{\text{old}}(\mathrm{TOL}/\mathrm{LE})^{1/(p+1)}\}.$$

If LE > TOL, the step is rejected. In this case the step size is halved and the computations start again.

6. Numerical examples. To illustrate the results of this paper we have solved the following initial-value problems.

EXAMPLE 1 (Castleton and Grimm [2]).

$$y'(t) = \frac{-4ty^2(t)}{\log^2(\cos(t))+4} + \tan(2t) + \tfrac{1}{2}\arctan(z(t)), \qquad t \in [0, 0.75],$$

$$y(0) = y'(0) = 0,$$

where $z(t) = y'(ty^2(t)/(1+y^2(t)))$. The theoretical solution is

$$Y(t) = -\tfrac{1}{2}\log(\cos(2t)).$$

EXAMPLE 2 (Castleton and Grimm [2]).

$$y'(t) = \cos(t(1+u(t))) + y(t)z(t) - \sin(t(1+\sin^2(t))), \qquad t \in [0, 1],$$

$$y(0) = 0, \qquad y'(0) = 1,$$

where $u(t) = y(ty^2(t))$, $z(t) = y'(ty^2(t))$. The theoretical solution is

$$Y(t) = \sin(t).$$

EXAMPLE 3 (Driver [5]).

$$y'(t) = -y'(t - y^2(t)/4), \qquad t \in [0, 1],$$

$$y(t) = 1 - t, \qquad t \leqslant 0.$$

The theoretical solution is

$$Y(t) = 1 + t.$$

Numerical methods described in this paper are also applicable to DDEs which are not of neutral type. We have included two examples of this form.

EXAMPLE 4 (Feldstein and Neves [7]).

$$y'(t) = -y(y(t) - \sqrt{2} + 1)/(2\sqrt{t}), \qquad t \in [1, 3],$$

$$y(t) = 1, \qquad t \leqslant 1.$$

The unique solution to this problem is

$$Y(t) = \begin{cases} \sqrt{t}, & t \in [1, 2], \\ t/4 + 1/2 + (1 - \sqrt{2}/2)\sqrt{t}, & t \in [2, 3]. \end{cases}$$

EXAMPLE 5 (Neves [20]).

$$y'(t) = y(t)y(\ln(y(t)))/t, \quad t \in [1, e],$$

$$y(t) = 1, \quad t \leqslant 1.$$

The unique solution is

$$Y(t) = \begin{cases} t, & t \in [1, e], \\ \exp(t/e), & t \in [e, e^2]. \end{cases}$$

These examples have been solved by methods (2) of orders from 2 to 10 with

$$b_j^q = (j-1)/q, \quad j = 1, 2, \ldots, q+1,$$

the functions a_j^p defined by (3) and the functions c_j^{p+1} defined by (4). The method was implemented in variable-step mode with step changing strategy described in Section 5. The local error was estimated by local extrapolation. Following Shampine and Watts [24] the initial step size h_0 was computed from the formula

$$h_0 = \min\{(b-a)/2, (\mathrm{TOL}/|z_h(a)|)^{1/(p+1)}\},$$

where TOL is a given tolerance. When solving systems of non-linear equations, the iterations were terminated if two successive approximations differed by less than TOL. In integrating Examples 1, 2, 4, 5 the computations were stopped if we passed the end point b, and the approximate solution $y_h(b)$ was computed by interpolation. In integrating Example 3 the end point was forced to be in a mesh. The reason for doing this is that, in this example, $t = 1$ is a bifurcation point (for $t \in [1, 3]$ both $y(t) = 1 + t$ and $y(t) = 3 - t$ are solutions). The selection of numerical results for the tolerances TOL $= 10^{-4}$ and 10^{-8} is given in Tables 1–10.

The abbreviations in the tables have the following meaning:

NS – number of successful steps (we count two steps of length $h/2$ as one step);
NRS – number of rejected steps;
NFE – number of evaluations of the right-hand side;
HMIN – the minimum step size;
HMAX – the maximum step size;
ERR – the absolute error at the end point b;
TIME – computation time in seconds on the Amdahl 370/V-II computer.

TABLE 1. Example 1. TOL $= 10^{-4}$

p	NS	NRS	NFE	HMIN	HMAX	ERR	TIME
2	77	4	976	.16E−2	.23E−1	4.1E−4	2.98
3	52	4	1011	.24E−2	.29E−1	7.2E−8	3.04
4	35	3	916	.35E−2	.47E−1	4.3E−7	3.24
5	30	6	1085	.42E−2	.53E−1	3.3E−8	4.05
6	22	6	1014	.42E−2	.94E−1	7.1E−8	4.27
7	17	6	973	.52E−2	.96E−1	1.1E−7	4.70
8	13	5	896	.97E−2	.19	4.2E−7	5.34
9	10	5	900	.11E−1	.19	2.1E−7	6.00
10	9	5	960	.13E−1	.20	2.9E−7	7.22

TABLE 2. Example 1. TOL $= 10^{-8}$

p	NS	NRS	NFE	HMIN	HMAX	ERR	TIME
5	2922	9	87960	.40E−4	.73E−3	3.0E−14	247.18
6	2158	9	78048	.55E−4	.78E−3	6.9E−14	237.54
7	1597	8	67452	.74E−4	.15E−2	1.8E−13	233.24
8	1186	8	57360	.10E−3	.15E−2	4.3E−13	231.85
9	877	7	47781	.13E−3	.29E−2	1.1E−12	229.14
10	650	7	39470	.18E−3	.29E−2	2.6E−12	222.66

TABLE 3. Example 2. TOL $= 10^{-4}$

p	NS	NRS	NFE	HMIN	HMAX	ERR	TIME
2	42	1	404	.22E−1	.28E−1	7.7E−5	2.22
3	28	1	444	.32E−1	.50E−1	1.2E−5	2.32
4	22	2	532	.40E−1	.52E−1	1.4E−5	2.61
5	16	2	520	.54E−1	.68E−1	1.9E−5	2.76
6	10	1	462	.88E−1	.13	3.7E−5	2.96
7	8	1	483	.12	.16	9.6E−6	3.28
8	6	1	448	.16	.18	2.0E−5	3.50
9	5	1	459	.20	.20	3.3E−5	3.87
10	3	0	300	.37	.43	1.1E−3	3.58

TABLE 4. Example 2. TOL $= 10^{-8}$

p	NS	NRS	NFE	HMIN	HMAX	ERR	TIME
2	4088	3	38320	.22E−3	.30E−3	7.7E−9	94.40
3	2821	5	43161	.31E−3	.43E−3	1.7E−9	84.63
4	2032	6	45972	.39E−3	.59E−3	1.0E−9	84.90
5	1490	6	43785	.61E−3	.81E−3	6.4E−10	89.39
6	1101	6	39408	.82E−2	.11E−2	4.2E−10	94.98
7	818	6	34461	.11E−2	.16E−2	3.6E−10	100.88
8	608	6	29472	.15E−2	.20E−2	4.6E−10	105.51
9	453	6	24885	.20E−2	.20E−2	7.0E−10	108.30
10	334	5	20550	.27E−2	.59E−2	1.6E−9	109.34

TABLE 5. Example 3. TOL = 10^{-4}

p	NS	NRS	NFE	HMIN	HMAX	ERR	TIME
2	240	160	2898	.18E−3	.19	1.9E−4	3.02
3	164	110	2988	.39E−3	.20	9.8E−5	3.38
4	112	65	2612	.49E−3	.16	8.6E−4	3.58
5	85	56	2580	.14E−2	.22	1.8E−4	4.28
6	58	34	2052	.26E−2	.13	4.9E−3	4.44
7	44	28	1869	.11E−2	.16	9.9E−5	5.09
8	26	16	1264	.94E−2	.18	1.8E−2	4.53
9	16	7	810	.22E−1	.20	1.4E−3	4.13
10	13	8	830	.40E−1	.22	7.7E−2	4.69

TABLE 6. Example 3. TOL = 10^{-8}

p	NS	NRS	NFE	HMIN	HMAX	ERR	TIME
7	4471	2440	176433	.55E−6	.20	3.4E−6	583.54
8	3233	1668	143592	.18E−6	.13	2.8E−6	473.48
9	2361	1162	116523	.44E−6	.16	1.7E−6	408.11
10	1790	878	98100	.12E−5	.19	2.0E−6	389.78

TABLE 7. Example 4. TOL = 10^{-4}

p	NS	NRS	NFE	HMIN	HMAX	ERR	TIME
2	23	0	254	.59E−1	.12	1.8E−4	1.98
3	17	1	306	.60E−1	.17	1.1E−5	2.18
4	13	1	324	.91E−1	.23	1.1E−8	2.14
5	9	0	270	.22	.30	2.3E−6	2.00
6	7	0	252	.29	.38	1.1E−5	2.17
7	6	0	252	.35	.48	5.4E−6	2.41
8	5	0	248	.39	.56	1.5E−5	2.36
9	4	0	234	.43	.60	4.0E−5	2.42
10	4	0	280	.46	.71	1.2E−5	2.83

TABLE 8. Example 4. TOL = 10^{-8}

p	NS	NRS	NFE	HMIN	HMAX	ERR	TIME
2	2171	2	24294	.59E−3	.13E−2	1.5E−8	24.34
3	1499	4	25209	.74E−3	.18E−2	9.1E−10	21.48
4	1081	5	26064	.90E−3	.25E−2	1.6E−9	24.60
5	793	5	23940	.16E−2	.34E−2	8.0E−10	25.79
6	587	5	21312	.22E−2	.46E−2	1.1E−9	27.60
7	437	5	18564	.30E−2	.62E−2	1.2E−9	29.36
8	326	5	15888	.41E−2	.84E−2	2.6E−9	30.56
9	244	5	13446	.53E−2	.11E−1	2.9E−9	31.27
10	180	4	11040	.79E−2	.15E−1	3.7E−8	32.06

TABLE 9. Example 5. TOL $= 10^{-4}$

p	NS	NRS	NFE	HMIN	HMAX	ERR	TIME
2	156	1	1480	.23E−1	.49E−1	2.2E−3	2.82
3	107	1	1638	.46E−1	.71E−1	4.4E−4	3.07
4	77	1	1636	.67E−1	.98E−1	5.4E−5	3.54
5	56	1	1710	.95E−1	.13	3.2E−4	4.13
6	42	1	1566	.13	.18	3.4E−4	4.70
7	32	1	1547	.16	.23	3.7E−4	5.49
8	22	0	1376	.25	.36	9.9E−4	6.07
9	17	0	1314	.35	.41	1.4E−3	6.90
10	14	0	1220	.43	.52	1.9E−3	7.58

TABLE 10. Example 5. TOL $= 10^{-8}$

p	NS	NRS	NFE	HMIN	HMAX	ERR	TIME
7	3092	6	117110	.15E−2	.24E−2	3.1E−8	460.16
8	2301	6	99928	.20E−2	.33E−2	9.3E−8	411.78
9	1714	6	83916	.25E−2	.44E−2	9.8E−9	375.35
10	1274	5	77840	.37E−2	.59E−2	2.4E−7	405.60

It can be seen from the tables that, in most cases, the number of steps and the number of function evaluations decrease as the order of the method increases. However, due to the increase in overhead related to the solution of linear systems of equations of higher dimension, as well as the fact that for all examples considered here the function evaluations are relatively inexpensive, this does not, in general, result in a decrease in computational time.

We were unable to integrate Examples 1, 3, and 5 for TOL $= 10^{-8}$ by lower order methods ($p \leqslant 4$ for Example 1, $p \leqslant 6$ for Examples 3 and 5) because of storage limitations. This illustrates the advantage of high order methods over low order methods for high tolerances.

In the future we would like to apply the numerical methods discussed in this paper to more complicated test problems. Unfortunately, it is difficult to find such examples with known solutions in the literature on the subject.

All computations were carried out in double precision on the Amdahl 370/V-II computer at the University of Arkansas at Fayetteville.

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