ZASTOSOWANIA MATEMATYKI
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Z. POROSIŃSKI (Wrocław)

# OPTIMAL STOPPING OF A RANDOM LENGTH SEQUENCE OF MAXIMA OVER A RANDOM BARRIER

**0. Introduction.** An optimal stopping problem is considered. Let $X_1, X_2, \ldots, X_N, Y_1, Y_2, \ldots, Y_N$ be independent copies of a known continuous distributed random variable (r.v.). Suppose we observe the realization of the sequence $\xi_n = \max(X_1, \ldots, X_n)$ only. We want to stop the observation at the moment when $\xi$ exceeds the maximum of the unobservable sequence with maximal probability. A related problem has been studied by Szajowski [3]. Moreover, he considered some cases where the length of the observation is infinite.

In this paper we allow the number of observations $N$ to be an r.v. independent of observations with a known distribution. We can look at this generalization of the above problem as taking into consideration various exterior factors having an influence on the length of the observation.

Some continuous time version of this problem, where $X_n$, $Y_n$ appear according to the Poisson process and the decision about stopping must be made before a random moment, has been considered by Porosiński [1].

The organization of the present paper is as follows. In Section 1 the precise formulation of the problem is given. It can be stated as the classical optimal stopping problem for some Markov chain (see, e.g., [2]). In Section 1 this reduction is also presented. Sections 2 and 3 contain the solutions (under additional assumptions) of this problem when $N$ is bounded or unbounded, respectively. The optimal gain is found and the optimal stopping time is obtained. In Section 4 we discuss the assumptions of the theorems obtained. In Section 5 the cases where $N$ has the geometric distribution, the one-point distribution and the uniform distribution are considered in detail.

**1. Model and its reduction.** Assume that

(1) $X_1, X_2, \ldots, Y_1, Y_2, \ldots$ are independent identically distributed r.v.'s with a continuous distribution function $F$, defined on the probability space $(\Omega, \mathscr{F}, P)$.

Let

$$\xi_n = \max(X_1, \ldots, X_n), \quad \eta_n = \max(Y_1, \ldots, Y_n).$$

We observe the sequence $\xi_n$ only and our object is to stop the observation at the moment $n$ when the value of $\xi_n$ exceeds the unobservable value of $\eta_n$. It is assumed that the horizon $N$ of the observation is an r.v. independent of $X_n$, $Y_n$ $(n = 1, 2, \ldots)$ with a known distribution

$$P(N = n) = p_n, \quad n = 0, 1, \ldots, \sum_{n=0}^{\infty} p_n = 1. \tag{2}$$

At the moment $n$, if we observe the value of $\xi_n$, we also know that $N \geqslant n$, i.e., we know events from a $\sigma$-field

$$\mathscr{F}_n = \sigma(\xi_1, \ldots, \xi_n, I_{\{0\}}(N), \ldots, I_{\{n-1\}}(N)),$$

where $I_A$ denotes the indicator function of the event $A$. Let $\mathscr{T}$ be the set of all Markov moments with respect to the family $(\mathscr{F}_n)_{n=1}^{\infty}$. Consider the following problem:

(P) Find a stopping time $\tau^* \in \mathscr{T}$ such that

$$P(\tau^* \leqslant N, \xi_{\tau^*} > \eta_{\tau^*}) = \sup_{\tau \in \mathscr{T}} P(\tau \leqslant N, \xi_\tau > \eta_\tau).$$

Let

$$Z_n = P(n \leqslant N, \xi_n > \eta_n \mid \mathscr{F}_n), \quad n = 1, 2, \ldots \tag{3}$$

Thus

$$\mathrm{E}(Z_\tau) = P(\tau \leqslant N, \xi_\tau > \eta_\tau).$$

Since $F$ is a continuous function, $F(X_n)$ is uniformly distributed on $[0, 1]$ and the inequalities $X_n > Y_n$ and $F(X_n) > F(Y_n)$ are equivalent almost surely (a.s.). Therefore, without loss of generality we may additionally assume that

(4) $X_n$, $Y_n$ $(n = 1, 2, \ldots)$ are uniformly distributed on $[0, 1]$.

Using (4) we can write (3) in the form

$$Z_n = \begin{cases} \xi_n^n & \text{for } n \leqslant N, \\ 0 & \text{for } n > N. \end{cases}$$

For $B$ being a Borel subset of $[0, 1]$ we have

$$P(n+1 \leqslant N, \xi_{n+1} \in B \mid \mathscr{F}_n) = \begin{cases} \pi_n(\xi_n + |B \cap (\xi_n, 1]|) & \text{if } \xi_n \in B, \\ \pi_n |B \cap (\xi_n, 1]| & \text{if } \xi_n \notin B, \end{cases}$$

where

$$\pi_n = P(N \geqslant n+1 \mid N \geqslant n) = g_{n+1}/g_n, \quad g_n = P(N \geqslant n) = \sum_{i=n}^{\infty} p_i,$$

and $|\cdot|$ stands for the Lebesgue measure. Therefore, the sequence $\beta = (\beta_n)_{n=0}^{\infty}$, where

$$\beta_n = \begin{cases} (n, \xi_n) & \text{if } n \leqslant N, \\ \partial & \text{if } n > N, \end{cases}$$

and $\partial$ is a label for the final state, is a homogeneous Markov chain with respect to $(\mathscr{F}_n)_{n=0}^{\infty}$ with the state space $\{0, 1, \ldots\} \times [0,1] \cup \{\partial\}$ and the transition function

(5) $$p(n, x; n+1, B) = \begin{cases} \pi_n(x + |B \cap (x, 1]|) & \text{if } x \in B, \\ \pi_n |B \cap (x, 1]| & \text{if } x \notin B, \end{cases}$$

$$p(n, x; \partial) = P(N = n \mid N \geqslant n) = p_n/g_n = 1 - \pi_n,$$

and $\partial$ is an absorbing state (we assume that $\xi_0 = 0$ a.s. and $\mathscr{F}_0 = \{\emptyset, \Omega\}$).

Thus we reduce Problem (P) to optimal stopping of the Markov chain $\beta$ with the reward function $f$, where

$$f(0, 0) = 0,$$
$$f(n, x) = x^n \quad \text{for } n = 1, 2, \ldots, \ x \in [0, 1],$$
$$f(\partial) = 0.$$

The problem of optimal stopping of the Markov chain $\beta$ with the reward function $f$ consists in calculating

$$v(x) = \sup_{\tau \in \mathscr{T}} \mathrm{E}_{0,x} f(\tau, \xi_\tau)$$

($\mathrm{E}_{0,x}$ denotes the expectation with respect to the distribution $P_{0,x}$ given by (5), $x$ is the initial state of $\xi$) and in finding a stopping time $\tau^* \in \mathscr{T}$ such that

$$\mathrm{E}_{0,x} f(\tau^*, \xi_{\tau^*}) = v(x)$$

(see [2]).

**2. Bounded** $N$. Let $\xi_n = x$. Put

(6) $$\begin{aligned} Tf(n, x) &= \mathrm{E}_{n,x} f(n+1, \xi_{n+1}), \\ Qf(n, x) &= \max\{f(n, x),\ Tf(n, x)\}, \\ v_K(n, x) &= \sup_{\tau \in \mathscr{T}_n^K} \mathrm{E}_{n,x} f(\tau, \xi_\tau), \end{aligned}$$

where $\mathscr{T}_n^K = \{\tau \in \mathscr{T}\colon\ n \leqslant \tau \leqslant K\}$.

In this section we assume that

(7) there exists $K$ such that $P(N \leqslant K) = 1$, $p_K > 0$.

To solve Problem (P) in this case, we use the following lemma (see [2], Theorem 15, p. 108):

LEMMA 1. *Let $((n, \xi_n))_{n=0}^{K}$ be a Markov chain with state space $E$ and let*

$$f\colon \{0, 1, \ldots, K\} \times E \to R$$

*be a bounded non-negative function. Then the function $v_K(n, x)$ satisfies the equations*

$$v_K(n, x) = \max\{f(n, x),\ Tv_K(n, x)\} = Q^{K-n} f(n, x)$$

*and the moment*

$$\tau^*_{K,n} = \min\{n \leqslant k \leqslant K\colon v_K(k, \xi_k)\} = f(k, \xi_k)$$

*is optimal in* $\mathcal{T}^K_n$. *The optimal gain is* $v(x) = v_K(0, x)$ *and the optimal stopping time is* $\tau^*_K = \tau^*_{K,0}$.

For the solution of (P), first of all we show the following lemma:

LEMMA 2. *If* $\pi_n > 0$, *then the equation* $Tf(n, x) = f(n, x)$ *has in the interval* $(0, 1)$ *the unique solution* $x^0_n$ *and*

$$Qf(n, x) = \begin{cases} f(n, x) & \text{for } x \geqslant x^0_n, \\ Tf(n, x) & \text{for } x < x^0_n \end{cases}$$

(*if* $\pi_n = 0$, *then* $Qf(n, x) = f(n, x)$ *for each* $x \in [0, 1]$).

Proof. Let $x \in [0, 1]$ and let $n$ be fixed. Taking into account (5) we have

$$Tf(n, x) = \mathrm{E}_{n,x} f(n+1, \xi_{n+1}) = x^{n+1}\pi_n x + \int_x^1 y^{n+1}\pi_n dy$$
$$= \pi_n\left(\frac{n+1}{n+2}x^{n+2} + \frac{1}{n+2}\right).$$

The function $g(n, x) = Tf(n, x) - f(n, x)$ is continuous on $[0, 1]$, has at most one extremum and $g(n, 0) > 0$, $g(n, 1) < 0$ if $0 < \pi_n < 1$; if $\pi_n = 1$, then $g(n, 1) < 0$ and the unique extremum of $g(n, x)$ is the minimum; if $\pi_n = 0$, then $g(n, x) = 0$ for $x = 0$ only. These properties imply that the unique root $x^0_n$ of the equation $g(n, x) = 0$ exists on $[0, 1)$. For $x \geqslant x^0_n$ the function $g(n, x)$ is non-positive. The lemma is proved.

Now, Lemmas 1 and 2 give us

THEOREM 1. *Under the assumptions* (1), (2), (4) *and* (7), *if there exists* $M < K$ *such that*

$$x^0_1 \leqslant x^0_2 \leqslant \ldots \leqslant x^0_M, \qquad x^0_M \geqslant x^0_{M+1} \geqslant \ldots \geqslant x^0_K = 0$$

($x^0_n$ *defined in Lemma* 2), *then a solution of Problem* (P) *exists and is of the form*

(8) $$\tau^*_K = \inf\{n \leqslant K\colon \xi_n \geqslant x_n\},$$

*where* $x_{K-n} = x^0_{K-n}$ *for* $n = 0, 1, \ldots, K-M$, *and* $x_{M-n}$ *for* $n = 1, 2, \ldots, M-1$ *is the unique root of the equation*

(9) $$\psi_{i_{M-n}}(M-n, x) = x^{M-n},$$

*where*

$$\psi_i(M-n, x) = \frac{g_{M+i}}{g_{M-n}}\,\frac{M+i}{M+n+2i}\left(x^{M+n+2i} - x^{M+n+2i}_{M-n+1}\right)$$
$$+ \frac{g_{M-n+1}}{g_{M-n}}\left(\frac{M-n+1}{M-n+2}x^{M-n+2}_{M-n+1} - \frac{1}{M-n+2}\right) \quad \text{if } i = i_{M-n+1},$$

$$\psi_i(M-n, x) = \frac{g_{M+i}}{g_{M-n}} \frac{M+i}{M+n+2i}(x^{M+n+2i} - x_{M+i-1}^{M+n+2i})$$
$$+ \frac{g_{M+i_{M-n+1}}}{g_{M-n}} \frac{M+i_{M-n+1}}{M+n+2i_{M-n+1}}(x_{M+i_{M-n+1}}^{M+n+2i_{M-n+1}} - x_{M-n+1}^{M+n+2i_{M-n+1}})$$
$$+ \sum_{j=i_{M-n+1}+1}^{i-1} \frac{g_{M+j}}{g_{M-n}} \frac{M+j}{M+n+2j}(x_{M+j}^{M+n+2j} - x_{M+j-1}^{M+n+2j})$$
$$+ \frac{g_{M-n+1}}{g_{M-n}}\left(\frac{M-n+1}{M-n+2} x_{M-n+1}^{M-n+2} + \frac{1}{M-n+2}\right) \quad \text{if } i > i_{M-n+1}, \tag{10}$$

*where* $i_M = 1$ *and*

$$i_{M-n} = \min\{i_{M-n+1} \leqslant i \leqslant K-M\colon \psi_i(M-n, x_{M+i}) \geqslant x_{M+i}^{M-n}\}. \tag{11}$$

*If* $\xi_0 = 0$ *a.s., then the optimal gain is*

$$P(\tau^* \leqslant N, \xi_{\tau^*} > \eta_{\tau^*}) = \frac{1}{2}\Big[g_1(x_1^2+1) + g_{M+i_1}(x_{M+i_1}^{2M+2i_1} - x_1^{2M+2i_1})$$
$$+ \sum_{j=i_1+1}^{K-M} g_{M+j}(x_{M+j}^{2M+2j} - x_{M+j-1}^{2M+2j})\Big]. \tag{12}$$

Proof. Lemmas 1 and 2 together with (5) and (7) give $x_K = x_K^0 = 0$, and omitting simple transformations we obtain

$$v_K(K-1, x) = \begin{cases} x^{K-1} & \text{for } x \geqslant x_{K-1}, \\ \dfrac{g_K}{g_{K-1}} \dfrac{K}{K+1}(x^{K+1} - x_K^{K+1}) + \dfrac{g_K}{g_{K-1}}\left(\dfrac{K}{K+1} x_K^{K+1} + \dfrac{1}{K+1}\right) & \\ & \text{for } x < x_{K-1}, \end{cases}$$

where $x_{K-1} = x_{K-1}^0$. Assume by induction that we have obtained

$$x_K = x_K^0 \leqslant x_{K-1} = x_{K-1}^0 \leqslant \ldots \leqslant x_{K-m+1} = x_{K-m+1}^0 \quad \text{for } m \leqslant K-M$$

and

$$v_K(K-n,x) = \begin{cases} x^{K-n} & \text{for } x \in [x_{K-n}, 1], \\ \varphi_i(K-n,x) & \text{for } x \in [x_{K-i}, x_{K-i-1}),\ i = 0, \ldots, n-1, \end{cases} \tag{13}$$

where

$$\varphi_i(K-n, x) = \frac{g_{K-i}}{g_{K-n}} \frac{K-i}{K+n-2i}(x^{K+n-2i} - x_{K-i-1}^{K+n-2i})$$
$$+ \frac{g_{K-n+1}}{g_{K-n}}\left(\frac{K-n+1}{K-n+2} x_{K-n+1}^{K-n+2} + \frac{1}{K-n+2}\right)$$
$$+ \sum_{j=i}^{n-2} \frac{g_{K-j}}{g_{K-n}} \frac{K-j}{K+n-2j}(x_{K-j}^{K+n-2j} - x_{K-j-1}^{K+n-2j}) \tag{14}$$

for $n = 1, 2, \ldots, m-1$ (here and in the sequel we adopt the convention that $\sum_{j=a}^{b} = 0$ if $b < a$).

Now we calculate $Tv_K(K-m, x)$ and then we obtain the form of $v_K(K-m, x)$. If $x \geqslant x_{K-m+1}$, then using Lemma 2 we have

$$(15)\qquad Tv_K(K-m, x) = x^{K-m+1}\frac{g_{K-m+1}}{g_{K-m}}x + \int_x^1 y^{K-m+1}\frac{g_{K-m+1}}{g_{K-m}}dy$$

$$= \frac{g_{K-m+1}}{g_{K-m}}\frac{K-m+1}{K-m+2}(x^{K-m+2} - x_{K-m}^{K-m+2}) + x_{K-m}^{K-m} = \varphi_{m-1}(K-m, x).$$

If $x \in [x_{K-i}, x_{K-i-1})$ for $i = 0, 1, \ldots, m-2$, then

$$(16)\qquad Tv_K(K-m,x) = \varphi_i(K-m+1, x)\pi_{K-m}x + \int_x^{x_{K-i-1}} \varphi_i(K-m+1, y)\pi_{K-m}dy$$

$$+ \sum_{l=i+1}^{m-2} \int_{x_{K-l}}^{x_{K-l-1}} \varphi_l(K-m+1, y)\pi_{K-m}dy + \int_{x_{K-m+1}}^{1} y^{K-m+1}\pi_{K-m}dy = \varphi_i(K-m, x).$$

We obtain the above equality calculating the integrals and making tedious but simple transformations.

The function $h(K-m, x) = Tv_K(K-m, x) - f(K-m, x)$ is continuous on $[0, 1]$ and has the following properties:

(a) if $x \geqslant x_{K-m+1}$, then $h(K-m, x) = g(K-m, x)$ defined in Lemma 2; thus

$$v_K(K-m, x) = \begin{cases} f(K-m, x) & \text{for } x \geqslant x_{K-m} = x_{K-m}^0, \\ \varphi_{m-1}(K-m, x) & \text{for } x \in [x_{K-m+1}, x_{K-m}); \end{cases}$$

(b) if $x < x_{K-m+1} < x_{K-m}$, then $h(K-m, x) > g(K-m, x) > 0$; thus

$$v_K(K-m, x) = Tv_K(K-m, x)$$

because $Tv_K(K-m, x) > Tf(K-m, x)$.

These properties imply that conditions (13) and (14) hold also for $m$. In this way we prove (13) for $n = 1, \ldots, K-M$.

Since $Tv_K(M-1, x) = \mathrm{E}_{M-1,x}v_K(M, \xi_M)$, $Tv_K(M-1, x)$ takes the form (15) and (16) for $K-m = M-1$. The function

$$h(M-1, x) = Tv_K(M-1, x) - f(M-1, x)$$

is continuous on $[0, 1]$ and

(a) if $x > x_M$, then $h(M-1, x) = g(M-1, x) < 0$ because $x_{M-1}^0 < x_M = x_M^0$;

(b) if $x < x_M$, then $h(M-1, x) > g(M-1,x)$ because $v_K(M,x) > f(M, x)$;

(c) $h(M-1, x)$ has for $x < x_M$ at most one extremum.

The properties (a), (b) and (c) yield that the equality $h(M-1, x) = 0$ has on $(0, 1)$ the unique solution $x_{M-1}$ and $x_{M-1} \in (x_{M-1}^0, x_M)$. Thus we can write $v_K(M-1, x)$ in the form

$$v_K(M-1, x) = x^{M-1} \quad \text{for } x \geq x_{M-1},$$

$$v_K(M-1, x) = \frac{g_{M+i}}{g_{M-1}} \frac{M+i}{M+1+2i}(x^{M+1+2i} - x_{M+i-1}^{M+1+2i}) + x_{M-1}^{M-1}$$
$$+ \frac{g_{M+i_{M-1}}}{g_{M-1}} \frac{M+i_{M-1}}{M+1-2i_{M-1}}(x_{M+i_{M-1}-1}^{M+1+2i_{M-1}} - x_{M-1}^{M+1+2i_{M-1}})$$
$$+ \sum_{j=i_{M-1}}^{i-1} \frac{g_{M+j}}{g_{M-1}} \frac{M+j}{M+1+2j}(x_{M+j}^{M+1+2j} - x_{M+j-1}^{M+1+2j})$$

for $x \in [x_{M+i_{M-1}}, x_{M-1})$ if $i = i_{M-1}$ and $x \in [x_{M+i}, x_{M+i-1})$ if $i = i_{M-1}+1, \ldots, K-M$, where $x_{M-1}$ and $i_{M-1}$ fulfil (9) and (11) for $n = 1$, respectively.

Now, suppose we have obtained

$$x_M > x_{M-1} > \ldots > x_{M-m+1} > x_{M-m+1}^0, \quad 1 = i_M \leq i_{M-1} \leq \ldots \leq i_{M-m+1}$$

for $m < M$, which fulfil (9) and (11), and for $n = 1, \ldots, m-1$

(17) $$v_K(M-n, x) = x^{M-n} \quad \text{for} \quad x \geq x_{M-n},$$

$$v_K(M-n, x) = \frac{g_{M+i}}{g_{M-n}} \frac{M+i}{M+n+2i}(x^{M+n+2i} - x_{M+i-1}^{M+n+2i}) + x_{M-n}^{M-n}$$
$$+ \frac{g_{M+i_{M-n}}}{g_{M-n}} \frac{M+i_{M-n}}{M+n+2i_{M-n}}(x_{M+i_{M-n}-1}^{M+n+2i_{M-n}} - x_{M-n}^{M+n+2i_{M-n}})$$
$$+ \sum_{j=i_{M-n}}^{i-1} \frac{g_{M+j}}{g_{M-n}} \frac{M+j}{M+n+2j}(x_{M+j}^{M+n+2j} - x_{M+j-1}^{M+n+2j})$$

for $x \in [x_{M+i}, x_{M+i-1})$ if $i = i_{M-n}+1, \ldots, K-M$ and for $x \in [x_{M+i_{M-n}}, x_{M-n})$ if $i = i_{M-n}$.

If we calculate $Tv_K(M-n, x)$ analogously to (15) and (16), then making indispensable simplifications we obtain

(18) $$Tv_K(M-m, x) = \psi_i(M-m, x)$$

for $x \in [x_{M+i}, x_{M+i-1})$ if $i = i_{M-m+1}+1, \ldots, K-M$ and $x \in [x_{M+i_{M-m+1}}, x_{M-m+1})$ if $i = i_{M-m+1}$, where $\psi_i$ is given by (10).

The function $h(M-m, x) = Tv_K(M-m, x) - f(M-m, x)$ is continuous on $[0, 1]$ and

(a) if $x \geq x_{M-m+1}$, then

$$h(M-m, x) = g(M-m, x) < 0$$

because $x_{M-m}^0 \leq x_{M-m+1}^0 < x_{M-m+1}$; thus $v_K(M-m, x) = f(M-m, x)$ for $x \geq x_{M-m+1}$;

(b) if $x < x_{M-m+1}$, then

$$h(M-m, x) > g(M-m, x)$$

because $v_K(M-m+1, x) > f(M-m+1, x)$ for these $x$'s;

(c) the function $h(M-m, x)$ for $x_{M-m}^0 < x < x_{M-m+1}$ has at most one extremum (if $x \leqslant x_{M-m}^0$, then $v_K(M-m, x) = Tv_K(M-m, x)$ because $h(M-m, x) > g(M-m, x) \geqslant 0$).

From (a), (b) and (c) it follows that the equation $h(M-m, x) = 0$ has in (0, 1) exactly one solution $x_{M-m}$, $x_{M-m} \in (x_{M-m}^0, x_{M-m+1})$. If $i_{M-m}$ fulfils (11), then $x_{M-m}$ is the root of the equation (9). Taking into account that $x_{M-m}$ fulfils the appropriate equation we can transform $v_K(M-m, x)$ to the form (17).

In this manner we prove (17) for $n = 1, \ldots, M-1$, so to complete proof of Theorem 1 it suffices to show (12). Since we assume that $\xi_0 = 0$ a.s., the optimal gain is

$$\begin{aligned} v_K(0, 0) &= \mathrm{E}_{0,0} v_K(1, \xi_1) \\ &= \frac{1}{2}\Big[g_1(x_1^2+1) + g_{M+i_1}(x_{M+i_1}^{2M+2i_1} - x_1^{2M+2i_1}) + \sum_{j=i_1+1}^{K-M} g_{M+j}(x_{M+j}^{2M+2j} - x_{M+j-1}^{2M+2j})\Big] \end{aligned}$$

because $v_K(0, 0) = Tv_K(0, x)|_{x=0}$ and $Tv_K(0, x)$ has the same form as (18) in view of the form of $v_K(1, x)$ given by (17). The theorem is proved.

**3. Unbounded** $N$. Now we assume that (7) is not fulfilled, i.e., the number of observations $N$ is an unbounded r.v. In this case we use the following lemma corresponding to Lemma 1 (cf. [2], p. 108):

LEMMA 3. *Let* $((n, \xi_n))_{n=0}^{\infty}$ *be a homogeneous Markov chain with state space* $E$ *and let* $f\colon \{0, 1, 2, \ldots\} \times E \to R$ *be a non-negative bounded function. Then the function*

$$v(n, x) = \sup_{\tau \in \mathscr{T}_n} \mathrm{E}_{n,x} f(\tau, \xi_\tau) \quad \text{if } \xi_n = x,$$

*where* $\mathscr{T}_n = \{\tau \in \mathscr{T}\colon \tau \geqslant n\}$, *satisfies the equations*

$$\begin{aligned} v(n, x) &= \max\{f(n, x), Tv(n, x)\}, \\ v(n, x) &= \lim_{m \to \infty} Q^m f(n, x) \end{aligned}$$

(*the operators* $T$ *and* $Q$ *are given by* (6)) *and the stopping time*

$$\tau_{n,\varepsilon}^* = \inf\{i \geqslant n\colon v(i, \xi_i) \leqslant f(i, \xi_i) + \varepsilon\}, \quad \varepsilon > 0,$$

*is* $\varepsilon$*-optimal in* $\mathscr{T}_n$, *i.e.*,

$$v(n, x) \leqslant \mathrm{E}_{n,x} f(\tau_{n,\varepsilon}^*, \xi_{\tau_{n,\varepsilon}^*}) + \varepsilon.$$

*If* $\tau_{0,0}^* < \infty$ *a.s., then* $\tau_{0,0}^*$ *is the optimal stopping time and* $v(x) = v(0, x)$.

Here we can also assume the additional condition as stated in Theorem 1 but for some class of distributions (2) (see Section 5) the solution of Problem (P) can be formulated in a more simple form.

THEOREM 2. *Under the assumptions* (1), (2) *and* (4), *if* $N$ *is an unbounded r.v.* (*i.e.*, $\pi_n \neq 0$ *for each* $n$) *and the sequence* $(x_n^0)_{n=1}^{\infty}$ *given by Lemma* 2 *is non-decreasing, then there exists a solution of Problem* (P) *which takes the form*

$$\tau^* = \inf\{n\colon \xi_n \geqslant y_n\}, \tag{19}$$

*where the sequence* $(y_n)_{n=1}^{\infty}$ *is non-decreasing,*

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} (\pi_n)^{1/n},$$

*and it satisfies the recurrence relation*

(20)
$$y_n^n = \pi_n\left(\frac{n+1}{n+2}y_{n+1}^{n+2}+\frac{1}{n+2}\right).$$

*Moreover,* $y_n$ *can be obtained as the limit of the sequence of unique solutions* $z_{n,k}$, $k \geqslant 1$, *of the equation*

(21)
$$\gamma_k(n, x) = x^n$$

*in* $(0, z_{n+1,k-1})$, *where* $z_{n,0} = 1$ *for* $n \geqslant 1$ *and*

(22)
$$\gamma_k(n, x) = \frac{g_{n+k}}{g_n}\frac{n+k}{n+2k}(x^{n+2k}-z_{n+1,k-1}^{n+2k})+\pi_n\left(\frac{n+1}{n+2}z_{n+1,k-1}^{n+2}+\frac{1}{n+2}\right)$$

*as* $k$ *tends to infinity.*

*If* $\xi_0 = 0$ *a.s., then the optimal gain is*

(23)
$$P(\tau^* \leqslant N,\ \xi_{\tau^*} > \eta_{\tau^*}) = g_1(1+y_1^2)/2.$$

Proof. By Lemma 2 we have

$$Qf(n, x) = \begin{cases} x^n & \text{for } x \geqslant z_{n,1}, \\ \pi_n\left(\dfrac{n+1}{n+2}x^{n+2}+\dfrac{1}{n+2}\right) & \text{for } x < z_{n,1}, \end{cases}$$

where $z_{n,1} = x_n^0$, so $z_{n,1}$ satisfies (21) for $k = 1$. Since the sequence $x_n^0$ does not decrease, $z_{n,1} \leqslant z_{n+1,1}$.

Assume by induction that there exist unique solutions $z_{n,k}$ of the equation (21) for $k = 1, 2, \ldots, m-1$, $z_{n,1} < z_{n,2} < \ldots < z_{n,m-1}$, $z_{n,k} \leqslant z_{n+1,k}$ for $k = 1, 2, \ldots, m-1$ and each $n$, and

(24)
$$Q^k f(n, x) = \begin{cases} x^n & \text{for } x \geqslant z_{n,k}, \\ \gamma_k(n, x) & \text{for } x < z_{n,k} \end{cases}$$

for $k = 1, 2, \ldots, m-1$, where $\gamma_k(n, x)$ is given by (22).

We calculate $Q^m f(n, x)$ taking into account the relation

(25)
$$\begin{aligned} Q^m f(n, x) &= \max\{Q^{m-1}f(n, x),\ TQ^{m-1}f(n, x)\} \\ &= \max\{f(n, x),\ TQ^{m-1}f(n, x)\}. \end{aligned}$$

For $x \leqslant z_{n+1,m-1}$ we obtain

$$TQ^{m-1}f(n, x) = \gamma_{m-1}(n+1, x)\pi_n x + \int_x^{z_{n+1,m-1}} \gamma_{m-1}(n+1, y)\pi_n dy + \int_{z_{n+1,m-1}}^{1} y^{n+1}\pi_n dy$$

and calculating these integrals, taking into consideration that

$$\gamma_{m-1}(n+1, z_{n+1,m-1}) = z_{n+1,m-1}^{n+1},$$

for $x \leqslant z_{n+1,m-1}$ we can write $TQ^{m-1}f(n, x) = \gamma_m(n, x)$. The function

$$h_m(n, x) = TQ^{m-1}f(n, x) - f(n, x)$$

is continuous on [0, 1] and has the following properties:

(a) if $x \geqslant z_{n+1,m-1} > z_{n,m-1}$, then

$$h_m(n, x) = Tf(n, x) - f(n, x) = g(n, x) < 0$$

(cf. Lemma 2);

(b) if $x \leqslant z_{n,m-1}$, then

$$h_m(n, x) > h_{m-1}(n, x) \geqslant 0$$

because of (25) and $Q^m f(n, x) \geqslant Q^{m-1}f(n, x) > f(n, x)$ for $x < z_{n,m-1}$;

(c) for $x < z_{n+1,m-1}$ the function $h_m(n, x)$ has at most one extremum.

From these properties the equation $h_m(n, x) = 0$ has the unique solution $z_{n,m}$ and $z_{n,m-1} < z_{n,m} < z_{n+1,m-1}$. Hence $z_{n,m} \leqslant z_{n+1,m}$ and the statement (24) holds for each $k$.

The sequence $(z_{n,m})_{m=1}^{\infty}$ is increasing and bounded, so the limit

$$\lim_{m\to\infty} z_{n,m} = y_n$$

exists. Hence

$$v(n, x) = \lim_{m\to\infty} Q^m f(n, x) = \begin{cases} x^n & \text{for } x \geqslant y_n, \\ \pi_n\left(\dfrac{n+1}{n+2} y_{n+1}^{n+2} + \dfrac{1}{n+2}\right) & \text{for } x < y_n. \end{cases}$$

Since the function $v(n, x)$ is continuous at the point $y_n$, (20) is fulfilled. The relation (20) gives us

$$\left(\frac{\pi_n}{n+2}\right)^{1/n} \leqslant y_n \leqslant (\pi_n)^{1/n}$$

and, consequently,

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} (\pi_n)^{1/n}.$$

From Lemma 3 we infer that the optimal Markov moment is of the form (19). It is a stopping time because the chain $\xi$ attains the state $\partial$ a.s. The optimal gain is

$$v = v(0, 0) = \int_0^{y_1} \pi_1\left(\frac{2}{3} y_2^3 + \frac{1}{3}\right)\pi_0 dy + \int_{y_1}^{1} y\pi_0 dy = \frac{\pi_0}{2}(1 + y_1^2).$$

The proof is completed.

**4. Discussion of the assumption about $(x_n^0)_{n=1}^{\infty}$.** The examination of the monotonicity of the sequence $(x_n^0)_{n=1}^{\infty}$ is difficult in the general case on account of the form of the function $g(n, x)$ (see Lemma 2). Even if we consider a distribution as simple as the uniform one (i.e., $p_i = 1/K$ for $i = 1, \ldots, K$), we fail to obtain analytic argumentation. For this distribution the numerical solution $x_n^0$ of the equation $g(n, x) = 0$ gives

$$x_1^0 < x_2^0 < \ldots < x_M^0, \qquad x_M^0 > x_{M+1}^0 > \ldots > x_K^0 = 0$$

for a certain $M < K$. Thus this distribution fulfils the assumption of Theorem 1.

Nevertheless, for some distributions we may use the following lemma:

LEMMA 4. *For fixed $n$, if $\pi_{n+1} \geqslant \pi_n$, then $x_n^0 < x_{n+1}^0$.*

Proof. Notice that we can write the function $g(n+1, x)$ in the form

$$g(n+1, x) = xg(n, x) + p(n, x),$$

where

$$p(n, x) = \left(\pi_{n+1}\frac{n+2}{n+3} - \pi_n\frac{n+1}{n+2}\right)x^{n+3} - \pi_n\frac{1}{n+2}x + \pi_{n+1}\frac{1}{n+3}.$$

Since $p'(n, x)$ is monotone and $p'(n, 0) < 0$, $p'(n, 1) > 0$, the function $p(n, x)$ has a minimum on $[0, 1]$ at the point $a_n$ when $p'(n, a_n) = 0$. Since

$$p(n, a_n) = (\pi_{n+1} - a_n\pi_n)/(n+3) > 0,$$

we have $p(n, x) > 0$ for each $x \in [0, 1]$. Thus

$$g(n+1, x_n^0) = x_n^0 g(n, x_n^0) + p(n, x_n^0) = p(n, x_n^0) > 0.$$

This inequality and the properties of $g(n+1, x)$ yield $x_{n+1}^0 > x_n^0$. The lemma is proved.

The condition $\pi_{n+1} \geqslant \pi_n$ is not necessary for $x_{n+1}^0 < x_n^0$ to hold. Let $n$ be fixed and $x_n^0$ be calculated. Let $\pi_{n+1} = \pi_{n+1}(a)$ be the value of $\pi_{n+1}$ for which $x_{n+1}^0 = x_n^0$ if $\pi_n = a$. Fig. 1 shows for $n = 1, 2, 3, 5, 10, 20$ how $\pi_{n+1}(a)$ depends on $\pi_n = a$. For $\pi_{n+1}$ exceeding this diagram we have $x_{n+1}^0 > x_n^0$, so it is easy to choose $\pi_{n+1} < \pi_n$ such that $x_{n+1}^0 > x_n^0$ for small $n$.

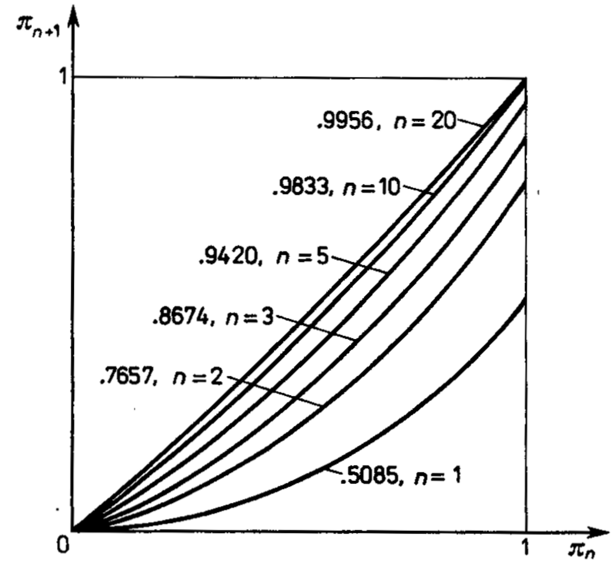


Fig. 1

Note that if $\pi_n$ is a fixed constant, then

$$\lim_{n\to\infty}\frac{\pi_{n+1}(\pi_n)}{\pi_n} = \lim_{n\to\infty}\left[\frac{n+1}{n+2}(x_n^0)^{n+3}+\frac{1}{n+2}\,x_n^0\right]\left[\frac{n+2}{n+3}(x_n^0)^{n+3}+\frac{1}{n+3}\right]^{-1} = 1,$$

so $\pi_{n+1}(\pi_n)\uparrow\pi_n$.

Unfortunately, the problem of analytic investigation of $(x_n)_{n=1}^{\infty}$ when $(\pi_n)_{n=1}^{\infty}$ decreases (this case contains distributions considered more often, e.g., the uniform distribution and the Poisson one) is still open. Nevertheless, Lemma 4 allows us to solve Problem (P) when the distribution given by (2) is a geometric one (cf. Example 1 below).

**5. Examples.**

EXAMPLE 1. *The geometric distribution.*

Let the distribution (2) be a geometric one with the parameter $p$, i.e., $p_n = pq^n$, $p+q = 1$, $p > 0$, $n = 0, 1, \ldots$ Since $g_n = q^n$, we have $\pi_n = q = \mathrm{const}$ and the assumption of Lemma 4 is fulfilled for each $n$. We can write Theorem 2 for this important case in the following form:

THEOREM 3. *If the length of observation $N$ has the geometric distribution with parameter $p$, then the solution of Problem* (P) *exists and takes the form*

$$\tau^* = \inf\{n\colon \xi_n \geqslant y_n\},$$

*where the sequence $(y_n)_{n=1}^{\infty}$ is increasing, $y_n\uparrow 1$ and it satisfies the relation*

$$y_n^n = q\left(\frac{n+1}{n+2}\,y_{n+1}^{n+2}+\frac{1}{n+2}\right).$$

*The value of $y_n$ can be calculated as the limit*

$$y_n = \lim_{k\to\infty} z_{n,k},$$

*where $z_{n,k}$ is the unique root of the equation*

$$q^k\frac{n+k}{n+2k}(x^{n+2k}-z_{n+1,k-1}^{n+2k})+q\left(\frac{n+1}{n+2}\,z_{n+1,k-1}^{n+2}+\frac{1}{n+2}\right)-x^n = 0$$

($z_{n,0} = 1$, $n \geqslant 1$). *The optimal gain is equal to* $q(1+y_1^2)/2$.

EXAMPLE 2. *The one-point distribution.*

If $P(N = K) = 1$, then $\pi_n = 1$ for $n < K$ and $\pi_n = 0$ for $n \geqslant K$. From Lemma 4 we have $x_1^0 < x_2^0 < \ldots < x_{K-1}^0$, and naturally $x_K^0 = 0$. Thus in this case we can use Theorem 1. The solution has already been known (cf. [3]).

EXAMPLE 3. *The uniform distribution.*

Let $p_n = P(N = n) = 1/K$ for $n = 1, 2, \ldots, K$. We have

$$\pi_n = (K-n)/(K-n+1), \qquad n = 1, 2, \ldots, K.$$

In this case we present the numerical solution only. Table 1 gives $x_1$, $x_2, \ldots, x_{K-1}$ (written row-wise) and the optimal gain $v_K$ for some values of $K$.

TABLE 1. Solution of Problem (P) for the uniform distribution with parameter $K$ ([1])

| $K$ | $x_1, x_2, \ldots, x_{K-1}$ | | | | | | | $v_K$ |
|---|---|---|---|---|---|---|---|---|
| 2 | .1683 | | | | | | | .5140 |
| 3 | .2411 | .3626 | | | | | | .5290 |
| 4 | .2899 | .4349 | .4793 | | | | | .5420 |
| 5 | .3260 | .4828 | .5395 | .5561 | | | | .5531 |
| 6 | .3546 | .5181 | .5837 | .6120 | .6108 | | | .5629 |
| 7 | .3774 | .5440 | .6132 | .6464 | .6619 | .6519 | | .5712 |
| 8 | .3962 | .5643 | .6352 | .6709 | .6886 | .6989 | .6841 | .5785 |
| 9 | .4124 | .5811 | .6530 | .6905 | .7110 | .7208 | .7276 | |
| | .7101 | | | | | | | .5850 |
| 10 | .4263 | .5951 | .6673 | .7058 | .7278 | .7395 | .7457 | |
| | .7506 | .7316 | | | | | | .5909 |
| 11 | .4387 | .6074 | .6798 | .7191 | .7424 | .7564 | .7653 | |
| | .7714 | .7694 | .7498 | | | | | .5962 |
| 12 | .4496 | .6179 | .6901 | .7297 | .7536 | .7685 | .7783 | |
| | .7843 | .7890 | .7853 | .7653 | | | | .6011 |
| 13 | .4594 | .6271 | .6991 | .7388 | .7632 | .7788 | .7893 | |
| | .7960 | .8002 | .8037 | .7987 | .7787 | | | .6055 |
| 14 | .4683 | .6355 | .7072 | .7469 | .7716 | .7879 | .7985 | |
| | .8059 | .8104 | .8132 | .8162 | .8104 | .7905 | | .6097 |
| 15 | .4764 | .6430 | .7143 | .7541 | .7790 | .7957 | .8070 | |
| | .8150 | .8205 | .8242 | .8274 | .8270 | .8206 | .8009 | .6135 |
| 16 | .4838 | .6497 | .7206 | .7602 | .7852 | .8021 | .8137 | |
| | .8222 | .8280 | .8320 | .8349 | .8377 | .8364 | .8296 | |
| | .8102 | | | | | | | .6170 |
| 17 | .4906 | .6557 | .7263 | .7657 | .7908 | .8077 | .8196 | |
| | .8283 | .8346 | .8389 | .8419 | .8441 | .8466 | .8446 | |
| | .8376 | .8186 | | | | | | .6203 |
| 18 | .4970 | .6614 | .7316 | .7709 | .7959 | .8130 | .8251 | |
| | .8338 | .8403 | .8450 | .8484 | .8507 | .8525 | .8544 | |
| | .8520 | .8448 | .8261 | | | | | .6235 |
| 19 | .5029 | .6667 | .7365 | .7756 | .8005 | .8177 | .8300 | |
| | .8390 | .8459 | .8509 | .8546 | .8574 | .8595 | .8614 | |
| | .8614 | .8586 | .8512 | .8330 | | | | .6265 |
| 20 | .5084 | .6714 | .7408 | .7797 | .8046 | .8218 | .8342 | |
| | .8433 | .8503 | .8556 | .8595 | .8624 | .8646 | .8663 | |
| | .8681 | .8676 | .8645 | .8571 | .8393 | | | .6292 |

([1]) The optimal strategy: accept $\xi_n$ if $\xi_n \geqslant x_n$, otherwise reject $\xi_n$ and await $\xi_{n+1}$ (in all cases $x_K = 0$).

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INSTITUTE OF MATHEMATICS
TECHNICAL UNIVERSITY OF WROCŁAW
50-370 WROCŁAW

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