SINGULARITIES
BANACH CENTER PUBLICATIONS, VOLUME 20
PWN POLISH SCIENTIFIC PUBLISHERS
WARSAW 1988

A NATURAL EQUIVALENCE RELATION ON SINGULARITIES

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1. Motivation

The classification problem is always fundamental in every branch of mathematics. For singularities, one would like to classify germs of real and complex analytic functions in n variables.

Let $\mathscr{A}_n$ denote the set of all real analytic germs g: $(\boldsymbol{R}^n, 0) \to (\boldsymbol{R}, 0)$. Each g can be represented by its Taylor expansion, which has no constant term.

When should two elements of $\mathscr{A}_n$ be declared equivalent? Each equivalence class should be as large as possible, making the classification simpler; each class should also be as small as possible, so that equivalent germs are "very much similar". Therefore the task is to search for a nice and natural, God-given, equivalence relation in $\mathscr{A}_n$, an ideal compromise between these contradictory demands. This is called the *Equisingularity Problem.*

Consider, as an illustrative example, the Whitney family

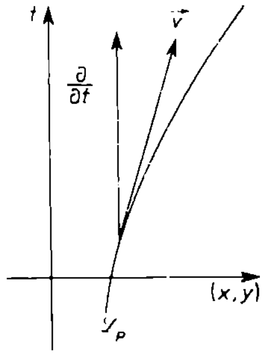
$$W_t(x, y) = xy(x-y)(x-ty), \qquad (x, y, t) \in \boldsymbol{R}^3.$$

Let us restrict the parameter t to the interval $(1, \infty)$, so that W_t is a non-degenerate form for each t; in particular, $W_t = 0$ consists of four distinct lines.

Intuitively, W_t and $W_{t'}$, $t \neq t'$, are very much similar; yet, there does not exist a local C^1-diffeomorphism h such that $W_{t'} \circ h = W_t$. (This can be proved using a simple Linear Algebra argument on dh.)

This phenomenon had cast serious doubt on the existence of an ideal equivalence relation on $\mathscr{A}_n$.

Let us not be discouraged. There is, at least, a God-given way to construct a vector field, $\vec{v}$, which generates a one-parameter family of homeomorphisms trivializing the Whitney family. Consider any point $P(x, y, t)$ off the t-axis. Let $\mathscr{L}_P$ denote the level surface of W_t through P, and



$\frac{\partial}{\partial t}$ denote the unit vector in the t-direction. Take the orthogonal projection of $\frac{\partial}{\partial t}$ to the tangent plane of $\mathscr{L}_P$ at P, and then adjust its length so that the t-component equals 1. The resulting vector is $\vec{v}(P)$. An easy calculation leads to

$$\vec{v}(x, y, t) = -\frac{\frac{\partial W}{\partial t}\frac{\partial W}{\partial x}}{\left(\frac{\partial W}{\partial x}\right)^2+\left(\frac{\partial W}{\partial y}\right)^2}\frac{\partial}{\partial x}-\frac{\frac{\partial W}{\partial t}\frac{\partial W}{\partial y}}{\left(\frac{\partial W}{\partial x}\right)^2+\left(\frac{\partial W}{\partial y}\right)^2}\frac{\partial}{\partial y}+\frac{\partial}{\partial t}$$

where W stands for $W_t(x, y)$. Along the t-axis, define

$$\vec{v}(0, 0, t) = \frac{\partial}{\partial t}.$$

The flow of $\vec{v}$ trivializes $W_t(x, y)$ topologically (cf. [K$_1$], [K$_2$], [K$_3$]), but, of course, not diffeomorphically.

However, one should not be satisfied with a more topological trivialization. Since $\vec{v}$ is God-given, it must offer something stronger.

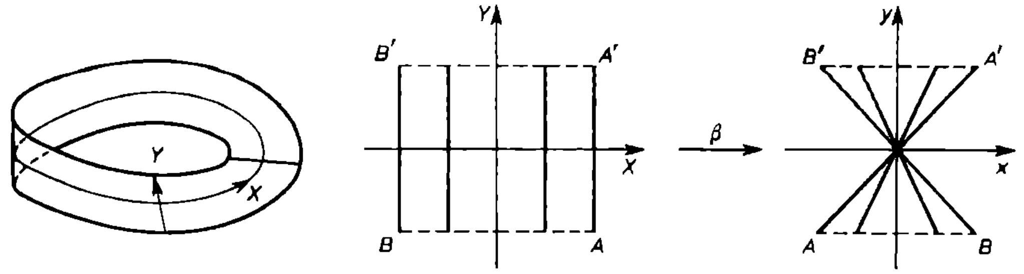
A closer examination of the components of $\vec{v}$ reveals their resemblance to the familiar example in Calculus:

$$f(x, y) = \frac{P_7(x, y)}{x^6+y^6}, \quad f(0, 0) = 0, \quad O(P_7) \geqslant 7,$$

which is continuous but not C^1.

Now, let β: $(\mathscr{M}, C) \to (\boldsymbol{R}^2, 0)$ be the blowing-up of $\boldsymbol{R}^2$ at 0, where $\mathscr{M}$ is the Möbius band, C its centre circle. Two charts are needed to cover $\mathscr{M}$. In one chart, β is expressed as $\beta(X, Y) = (XY, Y)$.
Hence

$$(f \circ \beta)(X, Y) = \frac{YQ(X, Y)}{X^6+1}$$



is analytic. The situation is similar in the other chart. We have thus made an important observation: $f\circ\beta$ is analytic on $\mathscr{M}$!

Returning to the Whitney family, one finds that $d(\beta\times\mathrm{id})^{-1}(\vec{v})$ is an analytic vector field on $\mathscr{M}\times\boldsymbol{R}$, tangent to $C\times\boldsymbol{R}$. Hence the topological trivialization generated by $\vec{v}$ *lifts* to an analytic isomorphism, leaving $C\times\boldsymbol{R}$ invariant. A detailed calculation is carried out in [K$_3$]. This result leads naturally to the notion of blow-analytic equivalence of singularities defined in the following section.

2. Blow-analytic equisingularities

The notion of blowing-up can be slightly generalized. For instance, one ought to consider a succession of them. A proper, surjective holomorphic map $\sigma^*\colon \tilde{X}^*\to X^*$ of complex spaces is called a *modification* if σ^* is a biholomorphism outside $\sigma^{*-1}(N)$, N a thin subset of X^* ([W]). By a modification of real spaces we shall mean a (proper surjective) real analytic map $\sigma\colon \tilde{X}\to X$ whose complexification σ^* is a modification.

Given $g_1, g_2\in\mathscr{A}_n$, we say they are *blow-analytically equivalent* if

(i) there is a local homeomorphism ϕ, $g_2\circ\phi = g_1$;

(ii) there exist two (real) modifications μ_1, μ_2, and an analytic isomorphism Φ such that the following diagram is commutative:

$$\begin{tikzcd}
(\mathscr{M}_1, \mu_1^{-1}(0)) \arrow[r, "\mu_1"] \arrow[d, "\Phi"', "\cong"] & (\boldsymbol{R}^n, 0) \arrow[dr, "g_1"] \arrow[d, "\phi"', "\approx"] & \\
(\mathscr{M}_2, \mu_2^{-1}(0)) \arrow[r, "\mu_2"] & (\boldsymbol{R}^n, 0) \arrow[r, "g_2"'] & (\boldsymbol{R}, 0)
\end{tikzcd}$$

(Thus, ϕ is a "collapsed" isomorphism.)

A succession of blowing-ups is of course a modification. The converse is almost true: Chow's lemma asserts that if $\mu\colon \mathscr{M}\to X$ is a modification, then there exists a modification $\mu'\colon \mathscr{M}'\to\mathscr{M}$ such that $\mu\circ\mu'$ is equivalent to a succession of blowing-ups ([H]).

We are now ready to generalize what we have proved for the Whitney family. Consider a parametrized family of functions

$$F(x, t)\colon \boldsymbol{R}^n\times\boldsymbol{R}^k\to\boldsymbol{R}$$

where F is analytic in (x, t), $F(0, t) \equiv 0$. For fixed t, write $F_t(x) \equiv F(x, t)$: $(\boldsymbol{R}^n, 0) \to (\boldsymbol{R}, 0)$.

THEOREM. *Suppose for each t, F_t admits $0 \in \boldsymbol{R}^n$ as an isolated singularity. Then there exists a finite filtration of the parameter space $\boldsymbol{R}^k$*

$$\boldsymbol{R}^k = P^{(0)} \supset P^{(1)} \supset \ldots \supset P^{(l)} \supset P^{(l+1)} = \emptyset$$

by subanalytic subsets $P^{(i)}$ with the following properties:

(i) $\dim P^{(i)} > \dim P^{(i+1)}$, $P^{(i)} - P^{(i+1)}$ *are smooth*;

(ii) *for t, t' in a same connected component of $P^{(i)} - P^{(i+1)}$, F_t and $F_{t'}$ are blow-analytically equivalent.*

The proof is given in [K_4].

CONJECTURE. The hypothesis that 0 be an isolated singularity is superfluous.

3. Algebraic geometry

Consider a real variety $V_f = f^{-1}(0)$ defined by an analytic function f: $\boldsymbol{R}^n \to \boldsymbol{R}$. Given a point $a \in V_f$, let $T_a(f)$ denote the Taylor expansion of f centered at a. Thus $T_a(f) \in \mathscr{A}_n$.

The blow-analytic equivalence relation in $\mathscr{A}_n$ *induces* an equivalence relation $\sim_f$ on V_f as follows. Define $a \sim_f a'$ if and only if $T_a(f)$ and $T_{a'}(f)$ are blow-analytically equivalent.

WEAK CONJECTURE. V_f admits a (locally finite) stratification, of which each stratum is subanalytic and is contained in a single equivalence class of $\sim_f$.

This conjecture is closely related to the conjecture in Section 2.

STRONG CONJECTURE. Each equivalence class of $\sim_f$ is an analytic manifold; these manifolds form a stratification of V_f which satisfies the (W)-regularity condition ([V]).

More details can be found in [K_4].

In an attempt to prove the conjecture of the last section, we have come across a problem on desingularization of a holomorphic map, which is formulated as the following conjecture.

Let σ: $\mathscr{M}_1 \to \mathscr{M}_2$ be a given proper surjective holomorphic map of complex manifolds.

CONJECTURE. There exists blowing-ups β_i: $\tilde{\mathscr{M}}_i \to \mathscr{M}_i$, $i = 1, 2$, with possibly singular centers, whose exceptional divisors are smooth and forming normal crossing families, and a holomorphic map $\tilde{\sigma}$: $\tilde{\mathscr{M}}_1 \to \tilde{\mathscr{M}}_2$ such that $\beta_2 \circ \tilde{\sigma} = \sigma \circ \beta_1$, and $\tilde{\sigma}$ maps each canonical stratum of $\tilde{\mathscr{M}}_1$ *submersively* onto a canonical stratum of $\tilde{\mathcal{M}}_2$. (The canonical stratification of $\tilde{\mathcal{M}}_i$ is provided by the normal crossing family of exceptional divisors.)

Notice that when $\mathcal{M}_2 = C$, this reduces to Hironaka's desingularization theorem.

4. Complex singularities

We may call two complex germs $g_1, g_2 \in \mathcal{O}_n$ *blow-analytically equivalent* if

(i) there is a local homeomorphism ϕ of $(C^n, 0)$ such that $g_2 \circ \phi = g_1$;

(ii) there exist real modifications μ_1, μ_2 of C^n (as real spaces), and a real analytic isomorphism Φ such that $\phi \circ \mu_1 = \mu_2 \circ \Phi$.

Using this definition, the theorem of Section 2 remains true for complex singularities. The proof is the same.

However, if one requires Φ to be a biholomorphism, then the problem of moduli can not be avoided, there would be no locally finite classification in $\mathcal{O}_n$.

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Presented to the semester
Singularities
15 February–15 June, 1985