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KRIPKE MODELS WITH RELATIVE ACCESSIBILITY AND THEIR APPLICATIONS TO INFERENCES FROM INCOMPLETE INFORMATION

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1. Introduction

Management of uncertainty and incompleteness of information is the important issue in a number of fields of computer science such as expert systems, knowledge representation, pattern recognition, natural language processing. Techniques and theories about incomplete information have undergone rapid development in the last years, and several models has been proposed for representation of incomplete knowledge. In the present paper we deal with those application domains in which information about objects under consideration is provided by means of some of their properties, for example functions may be characterized by their values for arguments from some intervals, human beings by age and profession, programs by some preconditions and postconditions, relational structures by formulas which are true in these structures.

The natural consequence of the fact that we are given a restricted information about properties of objects is that in general we are not able to distinguish objects as individual entities, but rather we grasp them as classes. In each class there are objects which cannot be distinguished from some of the others by means of the given properties. As a consequence we are also not able to characterize precisely sets of objects, since membership of an object in a set is defined modulo the admitted properties of objects. It follows that we cannot define a sharp boundary between a set and its complement, and hence the set is determined with some tolerance. There are several formal approaches to incompleteness of this kind, for example Narin'yani [3], Pawlak [9], Dubois and Prade [1], Rasiowa and Skowron [11], Orłowska [5].

Our claim is that to reason about incompletely defined sets of objects we need a special logic in which approximations of the sets with respect to different sets of properties could be expressed. In the present paper we introduce a modification of the ordinary Kripke structures (Kripke [2]). We consider structures in which a universe consists of objects, and accessibility relations are determined by sets of parameters interpreted as properties of objects. Next, we introduce a logic with modal operators defined with respect to relative accessibility relations. The logic is intended to provide a means to reason about objects which are defined modulo some properties.

The classical Kripke structures are systems of the form $K = (S, R)$, where S is a nonempty set whose elements are called *states*, and R is a binary relation in set S. Kripke structures serve as semantical structures for propositional modal languages with operations $\langle\ \rangle$ and $[\ ]$ interpreted as possibility and necessity, respectively. Given a structure K, we define meaning function m which assigns sets of states to propositional variables used in the language. Set $m(p) \subseteq S$ is interpreted as the set of those states in which variable p represents a true sentence. Any system of the form $M = (S, R, m)$ is called a *model*. Next, we define satisfiability of the formulas in a model. Satisfiability of atomic formulas, that is propositional variables, and satisfiability of formulas with modal operators of possibility and necessity is defined as follows. We say that a formula F is satisfied by state s in a model M ($M, s \operatorname{sat} F$) whenever the following conditions are satisfied:

$M, s \operatorname{sat} p$ iff $s \in m(p)$,

$M, s \operatorname{sat} \langle\ \rangle F$ iff there is $t \in S$ such that $(s, t) \in R$ and $M, t \operatorname{sat} F$,

$M, s \operatorname{sat} [\ ] F$ iff for all $t \in S$ if $(s, t) \in R$ then $M, t \operatorname{sat} F$.

This means that according to information provided by the model formula F is possible in such a state for which there is a successor state (with respect to accessibility given in M) in which F is satisfied. Similarly, formula F is necessary in such a state whose all the successor states satisfy F. We say that a formula F is true in a model M whenever for all $s \in S$ we have $M, s \operatorname{sat} F$. A formula F is valid whenever F is true in all models.

In the next section we show that to reason about incomplete knowledge it is reasonable to consider accessibility relations to be relative, or in other words, to be dependent on a set of parameters. These parameters are a formal counterpart of some properties of states which are associated by means of the accessibility relation.

2. Relative accessibility relations

In many recent aplications of modal logics states and accessibility relations from a semantical structure and modal operators of the language receive special interpretation. In logic S5 accessibility is assumed to be an equivalence relation. It provides a classification of states into a family of pairwise disjoint subsets (equivalence classes) of set ST such that in one class there are those states which cannot be distinguished one from the others. In this connection the natural question is with respect to what properties or what characteristic features some states are considered to be equivalent and some other not equivalent. They are not absolutely equivalent, but rather relatively equivalent, that is equivalent according to some criteria. Thus it seems to be reasonable to consider an accessibility relation to be determined by a set of parameters which are a formal counterpart of properties with respect to which equivalence was established.

In logic B accessibility is a similarity relation. An intuitive interpretation of a similarity class is that it consists of those states which are similar. However, in this case the clasification of states is not necessarily a partition, the classes may not be disjoint. And again in this case, if we say that some states are similar, we have in mind similarity with respect to some features. Thus we should consider accessibility to be relative, to be determined by some parameters corresponding to properties with respect to which similarity was defined.

In logic S4 accessibility is an ordering relation. In this case it is natural to ask what are criteria of ordering or with respect to what properties states are ordered. Thus the concept of relative accessibility seems to be natural also for this logic.

In epistemic logic modal operator of necessity is interpreted as knowledge operator. It seems to be reasonable to consider knowledge to be restricted to some aspects of reality and to define a family of relative accessibility relations reflecting these various aspects. Parameters used to define relative accessibility will be a formal counterpart of some background information.

In deontic logic modal operator of possibility is interpreted as permission operator. It seems natural to ask what is permitted with respect to some laws or rules, not just absolutely permitted. Thus we should introduce some parameters reflecting dependence of the act of permission on a set of laws. These parameters will enable us to discuss relationships between what is permitted with respect to different sets of laws.

In logics of information systems (Orłowska and Pawlak [6], [7], Orłowska [4]) states are interpreted as objects about which we would like to store information and accessibility relations are interpreted as indiscernibility relations determined by attributes admitted in the information system. Let OB be a set of objects and let AT be a set of attributes. For a subset $A \subseteq AT$ we define a binary relation $\mathrm{ind}(A)$ in set OB as follows:

$(s, t) \in \mathrm{ind}(A)$ iff s and t are the same with respect to attributes from A.

Relations $\mathrm{ind}(A)$ are equivalences or similarity relations. Given set $X \subseteq OB$, the lower (upper) aproximation $\underline{\mathrm{ind}}(A)X$ ($\overline{\mathrm{ind}}(A)X$) of set X with respect to indiscernibility relation $\mathrm{ind}(A)$ is defined as follows:

$\underline{\text{ind}}(A)X$ is the union of those equivalence (similarity) classes of $\text{ind}(A)$ which are included in X,

$\overline{\text{ind}}(A)X$ is the union of those equivalence (similarity) classes of $\text{ind}(A)$ which have an element in common with X.

Modal operators in logics of information systems are intended to correspond to operations of lower and upper approximation, respectively:

$\langle\text{ind}(A)\rangle F$ is satisfied by an object s iff s belongs to the upper approximation (with respect to $\text{ind}(A)$) of the set of objects satisfying formula F,

$[\text{ind}(A)]F$ is satisfied by an object s iff s belongs to the lower approximation (with respect to $\text{ind}(A)$) of the set of objects satisfying formula F.

In these logics accessibility relations are determined by sets of attributes, which play the role of parameters reflecting relativity of these relations.

Our proposal is to consider logics with semantics defined by means of Kripke structures with relative accessibility relations, that is relations which depend on sets of parameters. Let OB be a nonempty set of objects, PAR be a nonempty set whose elements are called *parameters*, and REL be a nonempty set whose elements are called *types* of relations (e.g. equivalence, similarity). We consider structures of the form:

$$K = (OB, \text{PAR}, \text{REL}, \{R(P)\}_{R \in \text{REL}, P \subseteq \text{PAR}}).$$

Each $R \in \text{REL}$ is a function R: $P(\text{PAR}) \to P(OB \times OB)$. For $P \subseteq \text{PAR}$ relation $R(P)$ is a binary relation in set OB. We assume that these relations satisfy the following conditions:

(R1) $$R(\emptyset) = OB \times OB,$$

(R2) $$R(P \cup Q) = R(P) \cap R(Q).$$

The first condition says that empty set of parameters does not enable us to distinguish any objects. In other words, if we have no criteria for distinguishing objects, then every object is associated with all the others. The second axiom says that if we have more parameters, then less objects will be associated with respect to these parameters, the relation will be smaller. If $R(P)$ is an equivalence (similarity) relation, then if we take into account more parameters, then in general less objects will be equivalent (similar). If $R(P)$ is an ordering relation, then if an object is comparable with some other object with respect to some parameters, then they will be not necessarily comparable if we take into account more parameters.

The straightforward consequences of these axioms are the following.

PROPOSITION 2.1. (a) *$P \subseteq Q$ implies $R(Q) \subseteq R(P)$,*
(b) *$R(P) \cup R(Q) \subseteq R(P \cap Q)$.*

EXAMPLE 2.1. Let PAR be a family of subsets of a nonempty set OB, and for any set $P \subseteq \text{PAR}$ let $R(P)$ be defined as follows:

$(s, t) \in R(P)$ iff for all $p \in P$ we have $s \in p$ iff $t \in p$.

EXAMPLE 2.2. Let OB and PAR be nonempty families of subsets of a nonempty set and let $R(P)$ for $P \subseteq$ PAR be defined as follows:

$(s, t) \in R(P)$ iff for all $p \in P$ we have $s \subseteq p$ iff $t \subseteq p$.

The relations from these examples are relativised versions of extensional identities considered in Weingartner [12]. Objects associated by means of these relations can be considered to be identical with respect to properties from set P.

EXAMPLE 2.3. Let OB be a set of objects for which attributes length and weigth are meaningful, and assume that for an object s its length is in an interval l and its weight is in an interval w. This means that we do not known exactly the values of length and weigth for the objects, we know them with some tolerance determined by the boundaries of the given intervals. Let $P = \{\text{length, weigth}\}$. We define relation $R(P)$ as follows:

$(s, s') \in R(P)$ iff $l \cap l' \neq \emptyset$ and $w \cap w' \neq \emptyset$.

Relation $R(P)$ is a similarity relation. The objects associated by means of this relation can be considered to be similar with respect to length and weight.

EXAMPLE 2.4. Let OB be a set of human beings, let P consists of an attribute age, and let $R(\text{age})$ be the following ordering relation

$(s, t) \in R(\text{age})$ iff s is younger than t.

It results from the given axioms that the following holds.

PROPOSITION 2.2. *For each $R \in$ REL the algebra $(\{R(P)\}_{P \in \mathrm{PAR}}, \cap)$ with the operation of intersection is a lower semilattice where $R(\mathrm{PAR})$ is the zero element.*

In the next section we discuss how properties of sets of parameters influence properties of accessibility relations.

3. Properties of sets of parameters

Given a structure $K = (\mathrm{ST}, \mathrm{PAR}, \mathrm{REL}, \{R(P)\}_{R \in \mathrm{REL}, P \subseteq \mathrm{PAR}})$, we say that a set $P \subseteq$ PAR is *R-reducible* iff there is its proper subset $Q \subsetneq P$ such that $R(P) = R(Q)$. This means that we can drop some parameters from an R-reducible set of parameters without violating the structure of associations provided by relation $R(P)$ in set ST. A set $P \subseteq$ PAR is R-irreducible if it is not R-reducible. By 2.1(a) we easily obtain the following characterization of irreducible sets.

PROPOSITION 3.1. *The following conditions are equivalent*:

(a) *A set P is R-irreducible,*

(b) *For any $Q \subsetneq P$ we have $R(P) \subsetneq R(Q)$.*

The following fact is the immediate consequence of the above definitions.

PROPOSITION 3.2. (a) *If a set P is R-reducible, then every its superset is R-reducible,*

(b) *If a set P is R-irreducible, then every its subset is R-irreducible.*

PROPOSITION 3.3. *If* PAR *is R-reducible, then for any $P, Q \subseteq$ PAR we have $R(P) \subseteq R(Q)$ implies $Q \subseteq P$.*

Proof. Let $R(P) \subseteq R(Q)$. Then $R(P \cup Q) = R(P) \cap R(Q) = R(P)$. If we have $P \cup Q \neq P$, then $P \cup Q$ would be R-reducible. But PAR is R-irreducible and so is every its subset, a contradiction. Thus we have $P \cup Q = P$, and hence $Q \subseteq P$.

PROPOSITION 3.4. *If* PAR *is R-irreducible, then $R(P \cap Q)$ is the least relation of type R including $R(P)$ and $R(Q)$.*

Proof. By 2.1(a) we have $R(P) \subseteq R(P \cap Q)$ and $R(Q) \subseteq R(P \cap Q)$. Let $S \subseteq$ PAR be a set of parameters such that $R(P) \subseteq R(S)$ and $R(Q) \subseteq R(S)$. By 3.3 we have $S \subseteq P$ and $S \subseteq Q$. Hence $S \subseteq P \cap Q$ and $R(P \cap Q) \subseteq R(S)$.

PROPOSITION 3.5. *The following conditions are equivalent:*

(a) *A set P is R-reducible,*

(b) *There is $Q \not\subseteq P$ such that $R(P-Q) = R(P)$.*

We say that a set P is reducible whenever there is $Q \not\subseteq P$ such that for all $R \in$ REL we have $R(P) = R(Q)$. A set P is irreducible if it is not reducible.

PROPOSITION 3.6. *If a set P is reducible, then P is R-reducible for all $R \in$ REL.*

PROPOSITION 3.7. *If* PAR *is irreducible, then for all $R \in$REL the algebra $(\{R(P)\}_{P \subseteq \mathrm{PAR}}, \sqcup, \sqcap)$, where $R(P) \sqcup R(Q) = R(P \cap Q)$, $R(P) \sqcap R(Q) = R(P \cup Q)$, is a lattice.*

4. RAL-relative accessibility logic

In the present section we define a logic with modal operators which are determined by relative accessibility relations. For indiscernibility relations the operators are intended to correspond to operations of lower and upper approximation. However, they will be defined in a more general way, namely we will use the notion of neighbourhood instead of the notions of equivalence or similarity class. Let R be a binary relation in a set ST. We define the set $\mathrm{pre}_R(s)$ of predecessors of an entity s and the set $\mathrm{suc}_R(s)$ of its successors with respect to relation R:

$$\mathrm{pre}_R(s) = \{t \in \mathrm{ST}\colon (t, s) \in R\}, \quad \mathrm{suc}_R(s) = \{t \in \mathrm{ST}\colon (s, t) \in R\}.$$

The union of these sets is referred to as a neighbourhood of s:

$$n_R(s) = \mathrm{pre}_R(s) \cup \mathrm{suc}_R(s).$$

Let R and S be binary relations in set ST. In the following we show how the sets of predecessors and successors with respect to complement $-R$, converse R^{-1}, union $R \cup S$, intersection $R \cap S$, and composition RS depend on the sets of predecessors and successors with respect to R and S.

PROPOSITION 4.1 (a) $\mathrm{pre}_{-R}(s) = -\mathrm{pre}_R(s)$, (b) $\mathrm{pre}_{R \cup S}(s) = \mathrm{pre}_R(s) \cup \mathrm{pre}_S(s)$, (c) $\mathrm{pre}_{R \cap S}(s) = \mathrm{pre}_R(s) \cap \mathrm{pre}_S(s)$, (d) $\mathrm{pre}_{R^{-1}}(s) = \mathrm{suc}_R(s)$.

Similarly, sets of successors of Boolean compositions of relations and the converse relation can be defined by means of sets of successors of their components according to the laws obtained from the above equalities by switching successors and predecessors.

PROPOSITION 4.2 (a) $\mathrm{pre}_{RS}(s) = \bigcup \{\mathrm{pre}_R(t) \colon t \in \mathrm{pre}_S(s)\}$, (b) $\mathrm{suc}_{RS}(s) = \bigcup \{\mathrm{suc}_S(t) \colon t \in \mathrm{suc}_R(s)\}$.

PROPOSITION 4.3. *If R is an equivalence (similarity) relation, then for all $s \in \mathrm{ST}$ we have* $\mathrm{pre}_R(s) = \mathrm{suc}_R(s)$ *and $n_R(s)$ is an equivalence (similarity) class of R.*

In the case of ordering relations set $n_R(s)$ may be interpreted as the set of those elements which are close to s with respect to relation R.

The language of logic RAL will be defined in two steps. First, we define an auxiliary set of parameter expressions, and second, the set of formulas. The modal operators in the language will be defined in terms of neighbourhoods. Parameter expressions are built from the symbols taken from the following disjoint sets:

VARPAR a set of variables representing sets of parameters,

$\{-, \cup, \cap\}$ the set of set-theoretical operations.

Set EPAR of expressions representing sets of parameters is the least set satisfying the following conditions:

VARPAR $\subseteq$ EPAR,

If $P, Q \in$ EPAR, then $-P$, $P \cup Q$, $P \cap Q \in$ EPAR.

Formulas of logic RAL are built from the symbols taken from the following pairwise disjoint sets:

VARPROP a set of propositional variables,

VARREL a set of variables representing relational types,

$\{\neg, \vee, \wedge, \rightarrow\}$ the set of classical propositional operations,

$\{\langle\ \rangle, [\ ]\}$ the set of modal operations.

Set FOR of formulas is the least set satisfying the following conditions:

$$\mathrm{VARPROP} \subseteq \mathrm{FOR},$$

If $F, G \in$ FOR, then $\neg F$, $F \vee G$, $F \wedge G$, $F \rightarrow G \in$ FOR.

If $R \in$ VARREL, $P \in$ EPAR, $F \in$ FOR, then $\langle R(P)\rangle F$, $[R(P)]F \in$ FOR.

By a relational type we mean a characteristics of a binary relation, e.g. equivalence, similarity, ordering are relational types.

Semantics of the language is defined by means of notions of model and satisfiability of the formulas in a model. By model we mean any structure $M = (\mathrm{ST}, \mathrm{PAR}, \mathrm{REL}, \{R(P)\}_{P \subseteq \mathrm{PAR}, R \in \mathrm{REL}}, m)$, where ST is a nonempty set whose elements are called states; PAR is a nonempty set whose elements, called parameters, are interpreted as characteristics of states; REL is a nonempty set of relational types; For each $P \subseteq \mathrm{PAR}$ relation $R(P)$ is a binary relation in set ST determining associations between states; m is a meaning function assigning values to variables such that $m(p) \subseteq \mathrm{ST}$ for $p \in \mathrm{VARPROP}$, $m(R) \in \mathrm{REL}$ for $R \in \mathrm{VARREL}$, $m(P) \subseteq \mathrm{PAR}$ for $P \in \mathrm{VARPAR}$, and moreover, $m(-P) = -m(P)$, $m(P \cup Q) = m(P) \cup m(Q)$, $m(P \cap Q) = m(P) \cap m(Q)$.

Given a model M, we define inductively satisfiability of the formulas by states. We say that a state s satisifies a formula F in a model M ($M, s \text{ sat } F$) whenever the following conditions are satisfied:

$M, s \text{ sat } p$ iff $s \in m(p)$,

$M, s \text{ sat } \neg F$ iff not $M, s \text{ sat } F$,

$M, s \text{ sat } F \vee G$ iff $M, s \text{ sat } F$ or $M, s \text{ sat } G$,

$M, s \text{ sat } F \wedge G$ iff $M, s \text{ sat } F$ and $M, s \text{ sat } G$,

$M, s \text{ sat } F \rightarrow G$ iff $M, s \text{ sat } \neg F \vee G$,

$M, s \text{ sat } \langle R(P) \rangle F$ iff there is $s' \in \mathrm{ST}$ such that $s \in n_{m(R)(m(P))}(s')$ and there is $t \in \mathrm{ST}$ such that $t \in n_{m(R)(m(P))}(s')$ and $M, t \text{ sat } F$,

$M, s \text{ sat } [R(P)] F$ iff there is $s' \in \mathrm{ST}$ such that $s \in n_{m(R)(m(P))}(s')$ and for all $t \in \mathrm{ST}$ if $t \in n_{m(R)(m(P))}(s')$, then $M, t \text{ sat } F$.

A formula F is said to be true in a model M if for all $s \in OB$ we have $M, s \text{ sat } F$. A formula is valid if it is true in all models. By an extension of a formula F in model M we mean the set of those objects which satisfy F in M:

$$\mathrm{ext}_M F = \{s \in OB\colon\ M, s \text{ sat } F\}.$$

The definition of semantics of modal operators results in the following fact.

PROPOSITION 4.3. (a) $\mathrm{ext}_M \langle R(P) \rangle F$ *is the union of those neighbourhoods with respect to relation* $m(R)$ $(m(P))$ *which have an element in common with* $\mathrm{ext}_M F$.

(b) $\mathrm{ext}_M [R(P)] F$ *is the union of those neighbourhoods with respect to the relation* $m(R)$ $(m(P))$ *which are included in* $\mathrm{ext}_M F$.

As a consequence, if R is the indiscernibility type, that is either equivalence or similarity, then the operators correspond to upper and lower approximation, respectively. Moreover, we have the following facts.

PROPOSITION 4.4. *If R and S are equivalence relations, then the following conditions are satisfied*:

(a) $n_{-R}(s) = -n_R(s)$,

(b) $n_{R \cup S}(s) = n_R(s) \cup n_S(s)$,

(c) $n_{R \cap S}(s) = n_R(s) \cap n_S(s)$,

(d) $n_{RS}(s) = \bigcup \{n_R(z)\colon\ z \in n_S(s)\} \cup \bigcup \{n_S(z)\colon\ z \in n_R(s)\}$.

PROPOSITION 4.5. *If R is the equivalence type, then the following conditions are satisfied*:

(a) M, s sat $\langle R(P)\rangle F$ *iff there is* $t \in \mathrm{ST}$ *such that* $(s, t) \in m(R)(m(P))$ *and* M, t sat F,

(b) M, s sat $[R(P)]$ *iff for all* $t \in \mathrm{ST}$ *if* $(s, t) \in m(R)(m(P))$, *then* M, t sat F.

This means that for the equivalence accessibility relations the modal operators of logic RAL coincide with the ordinary modal operators.

EXAMPLE 4.1. Assume that we are given set $X = \{A1, A2, A3, A4, A5, A6, A7\}$ of animals which are characterized by means of attributes size and animality according to the following table:

	size	animality
A1	small	bear
A2	medium	horse
A3	large	dog
A4	small	bear
A5	medium	horse
A6	large	horse
A7	large	horse

Indiscernibility relation ind(size, animality) determined by parameters size and animality provides the partition of set X into the following equivalence classes:

$$\{A1, A4\}\ \{A2, A5\}\ \{A3\}\ \{A6, A7\}.$$

Let us consider the structure $K = (X, PAR, \{\mathrm{ind}(P)\}_{P \subseteq \mathrm{PAR}})$, where $\mathrm{PAR} = \{\text{size, animality}\}$. Let us consider model M based on the structure K and the formula F such that $\mathrm{ext}_M F = \{A1, A2, A3\}$. We have $\mathrm{ext}_M \langle \mathrm{ind}(\text{size, animality})\rangle F = \{A1, A2, A3, A4, A5\}$, that is it coincides with the upper approximation of $\mathrm{ext}_M F$. We also have $\mathrm{ext}_M [\mathrm{ind}\ (\text{size, animality})] F = \{A3\}$, that is it coincides with the lower approximation of $\mathrm{ext}_M F$.

EXAMPLE 4.2. Assume that we are given the set $Y = \{P1, P2, P3, P4, P5\}$ consisting of five persons who are characterized by the attribute age, and assume that their age is given up to five years according to the following table:

	age
P1	20–24
P2	23–27
P3	27–31
P4	30–34
P5	31–35

We define the relation $R(\text{age})$ as follows:

$(s, t) \in R(\text{age})$ iff the age interval of s and the age interval of t have an element in common.

The relation generates the following similarity classes:

$$\{P1, P2\} \; \{P1, P2, P3\} \; \{P2, P3, P4, P5\} \; \{P3, P4, P5\}.$$

Let us consider the structure $K = (Y, \{\text{age}\}, \{R(\text{age})\})$, a model M based on this structure and the formula G such that $\text{ext}_M G = \{P4, P5\}$. We have $\text{ext}_M \langle R(\text{age}) \rangle G = \{P2, P3, P4, P5\}$ and $\text{ext}_M [R(\text{age})] G = \emptyset$.

5. Extensions of logic RAL

Some natural extensions of RAL might be of interest. Following the ideas of dynamic logic (Pratt [10]) consider RAL with operations on relations. That is assume that a family of accessibility relations forms an algebra with the usual relational operations: set-theoretical operations, composition, converse, reflexive and transitive closure etc. In this case the semantical structure for the logic is of the form:

$$K = (\text{ST}, \text{PAR}, \text{REL}, (\{R(P)\}_{P \subseteq \text{PAR}, R \in \text{REL}}, -, \cup, \cap, ^{-1}, \circ, *)).$$

In this case the problem arises of dependence of these operations on operations performed on sets of parameters and operations performed on relational types.

The other extension of RAL might be RAL with singleton constants, that is constants representing one-element sets of parameters and one-element sets of states. It was shown in Passy and Tinchev [8] that the operation of intersection of relations can be axiomatized in the language with state constants. And also in such a language it is possible to express several properties of a single model (Orłowska [4]). Thus we may add to the symbols of RAL sets CONPAR of parameter constants which are interpreted as one-element sets of parameters, and set CONPROP of propositional constants which are interpreted as one-element sets of states.

Our third proposal is to consider RAL for restricted classes of models, for example for the models in which set PAR is R-irreducible for all $R \in \text{REL}$. Due to Proposition 3.4 it might be possible in this case to find a suitable axioms characterizing modal operators.

6. Concluding remarks

We have presented an approach to modal logics via relative accessibility relations. Relative accessibility has been considered to provide information of two kinds. First, it tells us what states are associated, and second, with respect to which of their characteristics they are associated. This additional information about states in a semantical structure is useful for those applications of modal logics which deal with inferences from partial information. In such cases parameters determining accessibility relations reflect those characteristics of states to which our considerations are confined. And consequently, accessibility determined by these characteristics provides only a partial information about the structure of associations between states.

The modal logic RAL has been presented with operators defined by means of relative accessibility relations. Some extensions of the logic have been suggested.

It might be of interest to consider a relative accessibility formulation of the dynamic logic. However, in this case the standard definition of modal operators with respect to successors of accessibility relations would be sufficient.

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