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PARTIAL HYPERBOLICITY OF G-INDUCED FLOWS

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We prove that a topologically transitive G-induced flow on a homogeneous space G/K is a partially hyperbolic dynamical system (cf. [1], 9.1) provided that $\dim G/K > 1$ and at least one of the horospherical subgroups of G acts nontrivially on G/K.

Given a connected Lie group G, G-induced flow is the action of a one parameter subgroup $\{g_t\}$, $gK \to g_t gK$, on a homogeneous space G/K, where K is a closed subgroup of G. The manifold G/K is considered with a Riemannian metric induced by some right-invariant metric on G via the natural isomorphism between each tangent space $T_{hK}(G/K)$ and $(T_h(hK))^{\perp}$. If K is discrete and the spectrum of the operator $\operatorname{Ad} g_1$ is not contained in the unit circle then the corresponding flow is partially hyperbolic. For solvflows (that is, G-induced flows with solvable group G) Auslander in [2] obtained the reduction to the case of discerte K (the exact formulation is as follows: an ergodic solvflow on G/K, $\mu(G/K) < \infty$, admits a finite covering by a solvflow on $\tilde{G}/D$, where D is a discrete subgroup in $\tilde{G}$). For G-induced flows on homogeneous spaces of semisimple Lie groups the reduction to the case of a discrete subgroup is provided by the Borel density theorem ([3]).

It turns out that the representation of stable and unstable submanifolds as the orbits of the corresponding horospherical subgroups (cf. [4]) allows to obtain partial hyperbolicity of topologically transitive G-induced flows.

Let $\mathfrak{G}$ be the Lie algebra of G, X – an element of $\mathfrak{G}$. The subgroup G^+ (resp. G^-) $= \{g \in G \mid (\exp tX) g (\exp tX)^{-1} \to e,\ t \to -\infty\ (+\infty)\}$ is called the expanding (contracting) horospherical subgroup of G, corresponding to the element X (here e denotes the unit element of G). Certainly, G^+ and G^- act on any G/K by left translation $gK \to hgK$.

THEOREM. *If the flow on the homogeneous manifold G/K induced by the element $X \in \mathfrak{G}$ is topologically transitive and* $\dim G/K > 1$, *then the partition of G/K into the orbits of the group G^+ (or G^-) is a foliation.*

Proof. Let us split the linear space $\mathfrak{G}$ into the direct sum of the eigenspaces of the linear operator $\operatorname{ad} X$. We denote by $\mathfrak{G}^+$ the sum of all the eigenspaces such that the real parts of the corresponding eigenvalues are positive and by $\mathfrak{G}^-$ and $\mathfrak{G}^0$ – the sum of all the eigenspaces with $\operatorname{Re}\lambda < 0$ and $\operatorname{Re}\lambda = 0$ respectively. It is easy to check up that $\mathfrak{G}^+$ is the Lie algebra of G^+, $\mathfrak{G}^-$ of G^-, and $\mathfrak{G}^0$ is a subalgebra in $\mathfrak{G}$.

Consider the point $hK \in G/K$ and the projection

$$P_h\colon G \to G/K\colon g \to ghK.$$

Let $dP_h\colon \mathfrak{G} \to T_{hK}(G/K)$ be the tangential mapping for P_h at the unit point of G.

LEMMA. *There exists an everywhere dense set in G/K such that for each point of it we have*

$$(*) \qquad \operatorname{Ker} dP_h = (\operatorname{Ker} dP_h \cap \mathfrak{G}^+) \oplus (\operatorname{Ker} dP_h \cap \mathfrak{G}^-) \oplus (\operatorname{Ker} dP_h \cap \mathfrak{G}^0)$$

Proof of the lemma. We will show that any everywhere dense trajectory of the flow $\{\exp tX\}$ is such a set. Let us take $h \in G$ so that hK belongs to the trajectory. Suppose that the splitting $(*)$ does not take place for the point hK. Then there exists such a vector $Y \in \operatorname{Ker} dP_h \subset \mathfrak{G}$ that at least two of the projections Y^+, Y^-, Y^0 of Y to the subspaces $\mathfrak{G}^+$, $\mathfrak{G}^-$, $\mathfrak{G}^0$ do not belong to $\operatorname{Ker} dP_h$. Since hK lays in the dense trajectory of the flow $\{\exp tX\}$ and $G/K > 1$, the trajectory returns into every neighborhood of the point hK, and there exists such a sequence $t_n \to \infty$ that $(\exp t_n X)(hK) \to hK$. Assume first that $t_k \to +\infty$.

Now each vector v of the tangent space $T_{hK}(G/K)$ defines the function

$$f_v\colon \{t_k\}_{k\in N} \to R, \qquad f_v(t_k) = \|(d\exp t_k X)v\|.$$

If $v = dP_h(Y)$ for some $Y \in \mathfrak{G}$, then

$$(d\exp t_k X)v = dP_{(\exp t_k X)h}[(\operatorname{Ad}\exp t_k X)Y]$$

and so $f_v(t_k)$ is proportional to the distance in some Euclidean metric in $\mathfrak{G}$ between the vector $\operatorname{Ad}(\exp t_k X)Y$ and the subspace $\operatorname{Ker} dP_{(\exp t_k X)h} = \operatorname{Ad}(\exp t_k Xh)(\mathfrak{k})$, where $\mathfrak{k}$ is the Lie algebra of K.

PROPOSITION. *If all the eigenvalues of the linear operator A in the linear space V are imaginary then the distance between the point $(\exp tA)Y$ and the subspace $(\exp tA)(V_0)$ behaves like a rational function when $t \to \infty$ for any vector Y and subspace V_0 in V.*

Proof. Given a metric in V, one can define certain metric in the exterior powers $\bigwedge^i V$ of the space V, so that the distance between Y and V_0 is equal to the ratio of the norms of the polyvectors $Y \wedge v_1 \wedge \ldots \wedge v_i$ and $v_1 \wedge \ldots \wedge v_i$, where $v_1, \ldots, v_i$ is a basis of V_0. The eigenvalues of the exterior powers of the operator A will be also purely imaginary, so the norms of these polyvectors may grow as a polynomial or be constant.

Let us take now the vector $Y^0 \in \mathfrak{G}^0$, $Y^0 \notin \mathrm{Ad}\, h(\mathfrak{k})$. The distance between $\mathrm{Ad}(\exp t_k X) Y^0$ and $\mathrm{Ad}(\exp t_k X)(\mathrm{Ad}\, h(\mathfrak{k}) \cap \mathfrak{G}^0)$ decreases not faster than a rational function. Since $(\exp t_k X) hK \to hK$, then $\mathrm{Ad}(\exp t_k X)(\mathrm{Ad}\, h(\mathfrak{k})) \to \mathrm{Ad}\, h(\mathfrak{k})$, and for any $k \in N$ the angles between $\mathfrak{G}^0$ and $\mathrm{Ad}(\exp t_k X)(\mathrm{Ad}\, h(\mathfrak{k}))$ (which are measured in all the 2-dimensional planes, orthogonal to its intersection) are greater than some positive number. So the distance between $\mathrm{Ad}(\exp t_k X) Y^0$ and $\mathrm{Ad}(\exp t_k X)(\mathrm{Ad}\, h(\mathfrak{k}))$ also decreases not faster than some rational function.

When $Y^+ \in \mathfrak{G}^+$, let us consider the linear space $\mathfrak{G}^x \subset \mathfrak{G}$ which is the sum of all the root subspaces $\mathfrak{G}^\lambda$ of the operator $\mathrm{ad}\, X$ with $\mathrm{Re}\, \lambda = x$. Let x be maximal among those that the projection Y^x of Y^+ onto $\mathfrak{G}^x$ is not in $\mathfrak{G}^x \cap \mathrm{Ad}\, h(\mathfrak{k})$. Then $\mathrm{ad}\, X|\, \mathfrak{G}^x$ is the sum of the scalar operator with positive eigenvalue x and an operator with imaginary eigenvalues. Obviously, the distance between $\mathrm{Ad}(\exp t_k X) Y^x$ and $\mathrm{Ad}(\exp t_k X)(\mathcal{G}^x \cap \mathrm{Ad}\, h(\mathfrak{k}))$ and between $\mathrm{Ad}(\exp t_k X) Y^+$ and $\mathrm{Ad}(\exp t_k X)(\mathrm{Ad}\, h(\mathfrak{k}))$ grows exponentially as e^{tx} when $t_k \to +\infty$. (Observe that, as before, the angle between $\mathfrak{G}^x$ and $\mathrm{Ad}((\exp t_k X) h)(\mathfrak{k})$ is greater than some positive constant.)

We have shown, that if there exists $Y^+ \in \mathfrak{G}^+$ such that $dP_h(Y^+) = v$, then f_v grows exponentially in t, and if $v = dP_h(Y^0)$, then f_v grows or decreases not faster than a rational function. Obviously, if $v = dP_h(Y^-)$, f_v decreases exponentially to zero.

Suppose now, that the decomposition $(*)$ does not take place. Then there exist vectors $Y^i \in \mathfrak{G}^i$ such that $dP_h(Y^+ + Y^0 + Y^-) = 0$ and at least two of the vectors $dP_h(Y^i)$ are nonzero. Set at first $dP_h(Y^+) \neq 0$. Consider $v = dP_h(Y^+) = dP_h(-Y^- - Y^0)$. Then f_v on one hand, must grow exponentially, but on the other hand it grows not faster than a polynomial, and we come to a contradiction. The cases $dP_h(Y^+) = 0$ and $t_k \to -\infty$ can be considered in the same way.

The subspace $\mathrm{Ker}\, dP_h$ is exactly the subalgebra $\mathrm{Ad}\, h(\mathfrak{k})$ and $(*)$ is equivalent to the following condition on the dimensions

$$\dim \mathfrak{k} = \dim(\mathfrak{G}^+ \cap \mathrm{Ad}\, h(\mathfrak{k})) + \dim(\mathfrak{G}^- \cap \mathrm{Ad}\, h(\mathfrak{k})) + \dim(\mathfrak{G}^0 \cap \mathrm{Ad}\, h(\mathfrak{k})).$$

We notice now that for any arbitrary subspaces $V_1, V_2 \subset \mathfrak{G}$ $\dim(V_1 \cap \mathrm{Ad}\, g(V_2))$ as a function of g takes its minimal value $n(V_1, V_2)$ on some dense open subset in G. Indeed, the set of such points $g \in G$, that $\dim(V_1 \cap \mathrm{Ad}\, g(V_2)) > n(V_1, V_2)$ is a preimage in G of some analytic set in the Grassmann manifold $G(\dim G, \dim V_2)$, on which the group G acts analytically. Then it follows from the lemma that

$$\dim \mathfrak{k} = n(\mathfrak{G}^+, \mathfrak{k}) + n(\mathfrak{G}^-, \mathfrak{k}) + n(\mathfrak{G}^0, \mathfrak{k})$$

and the splitting $(*)$ takes place at all the points of G, where $n(\mathfrak{G}^i, \mathfrak{k})$

$= \dim(\mathfrak{G}^i, \operatorname{Ad} h(\mathfrak{k}))$. On the complement to the set of these points the dimension of the right part of (*) may only increase, so (*) is true at each point of G.

Suppose now that $\dim(\mathfrak{G}^+ \cap \operatorname{Ad} h(\mathfrak{k}))$ is not constant.

Then for some $h \in G$ we have

$$\dim(\mathfrak{G}^+ \cap \operatorname{Ad} h(\mathfrak{k})) > n(\mathfrak{G}^+, \mathfrak{k}) \quad \text{and}$$

$$\dim(\mathfrak{G}^- \cap \operatorname{Ad} h(\mathfrak{k})) + \dim(\mathfrak{G}^0 \cap \operatorname{Ad} h(\mathfrak{k})) < n(\mathfrak{G}^-, \mathfrak{k}) + n(\mathfrak{G}^0, \mathfrak{k}).$$

This inequality contradicts the definition of $n(V_1, V_2)$, and so $\dim(\mathfrak{G}^+ \cap \operatorname{Ad} h(\mathfrak{k}))$ and the dimension of G^+-orbit, containing hK (which is equal to $\dim \mathfrak{G}^+ - \dim(\mathfrak{G}^+ \cap \operatorname{Ker} dP_h)$) do not depend on $h \in G$.

Now assume that at least one of the foliations in the Theorem has positive dimension. We will prove that our system is partially hyperbolic.

For each $hK \in G/K$ define

$$E^u_{hK} = T_{hK}(G^+ hK) = Dh(\mathfrak{G}^+),$$

$$E^s_{hK} = T_{hK}(G^- hK) = Dh(\mathfrak{G}^-), \qquad E^0_{hK} = Dh(\mathfrak{G}^0)$$

(Dh denotes the derivative of the map $x \to xhK$ at e).

Since $(\operatorname{ad} X)\, \mathfrak{G}^i = \mathfrak{G}^i$ for $i = u, s, 0,$ we have

$$E^i_{(\exp tX)hK} = (d \exp tX)(E^i_{hK}).$$

By the proof of the Theorem, the dimensions of E^i_{hK} do not depend on h. Since the foliations are not both zero-dimensional, $\dim E^u_{hK} \neq 0$ or $\dim E^s_{hK} \neq 0$. If $v = Dh(Y)$ where $Y \in \mathfrak{G}$, then

$$\|v\| = \operatorname{dist}(Y, \operatorname{Ad} h(\mathfrak{k})) \quad \text{and}$$

$$\|(d \exp tX)\, v\| = \operatorname{dist}\big(\operatorname{Ad}(\exp tX)\, Y, \operatorname{Ad}(\exp tX)(\operatorname{Ad} h(\mathfrak{k}))\big).$$

Using this fact and the Lemma (where $\operatorname{Ker} dP_h = \operatorname{Ad} h(\mathfrak{k})$) it is not hard to show that $\|(d \exp tX)\, v\| \leqslant Cr^t \|v\|$ with $C \in R^+$ and $r \in (0, 1)$ independent of h and $v \in E^s_{hK}$. Similarly, the spaces E^u_{hK} are unstable and E^0_{hK} neutral.

Let us consider now some examples when the partition into the orbits of the horospherical subgroup does not form a foliation. Let G be a semisimple Lie group, X – an element of the Cartan subalgebra of $\mathfrak{G}$ such that X is a linear combination of simple roots with real positive coefficients. Then $\mathfrak{G}^+$ is the maximal nilpotent subalgebra η^+, spanned by the root vectors of the positive roots, and $G^+ = N^+$. Let us take for K the horospherical subgroup N^+ itself. Then one of N^+-orbits consists only of one point, but since all the maximal unipotent subgroups are conjugated, there also exist the orbits which dimension is equal to $\dim N^+$. Now let B be the Borel subgroup of G. The space G/B is called the flag manifold (cf. [5]). It is compact and consists of a finite number of orbits of G^+ (or G^-), and among them there is one point and one open orbit. It should be pointed out that if $G = \mathrm{SL}(2, \boldsymbol{R})$ and K is the Borel subgroup of G, the the dimension of G/K is equal to one and the flow is topologically transitive, but the orbits of the horospherical group do not form a foliation. Thus the condition $\dim G/K > 1$ is essential.

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