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# INTERPOLATION PROJECTORS AND CLOSED IDEALS

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## 0. Introduction

It is the objective of this paper to derive a functional analytic approach to interpolation in the algebra $C(E)$ of continuous real valued functions defined on a compact metric space $E$. We will study a class of interpolation projectors on $C(E)$ whose null spaces are determined by their precision sets. The basic tool is the Stone characterization of closed ideals in the algebra $C(E)$. We will establish algebraic properties of interpolation projectors by showing that products and Boolean sums of commutative interpolation projectors are again interpolation projectors determined by their precision sets. Furthermore, we will indicate the use of the functional analytic method of parametric extension [2] to the construction of interpolation projectors on product spaces.

## 1. Closed ideals

Let $E$ be a compact metric space. A linear subspace $J$ of $C(E)$ is an *ideal* if for any $f \in J$ and $g \in C(E)$ the product $fg$ is again an element of $J$. To describe ideals we introduce for any closed subset $T$ of $E$ the function set

$$C(E\|T) = \{f \in C(E): f|_T = 0\}.$$

It is easily seen that $C(E\|T)$ is an ideal of $C(E)$ which is in addition closed with respect to the norm topology of $C(E)$. Moreover, the following relations are true:

$$C(E\|\emptyset) = C(E), \tag{1}$$

$$C(E\|E) = \{0\}, \tag{2}$$

$$T_1 \subseteq T_2 \Leftrightarrow C(E\|T_2) \subseteq C(E\|T_1). \tag{3}$$

It is the content of the basic theorem of M. H. Stone that every closed ideal $J$ of $C(E)$ is of the form $C(E\|T)$ for some closed subset $T$ of $E$. In the following we present the main ideas of the proof of this result (see also Semadeni [6] and Stone [8]).

Let $V$ be a closed linear subspace of $C(E)$. Then $Z(V)$ denotes the set of common zeros of all $f \in V$. Obviously, $Z(V)$ is closed and the following relations are easily established:

$$V_1 \subseteq V_2 \Rightarrow Z(V_2) \subseteq Z(V_1), \tag{4}$$

$$Z(\{0\}) = E, \tag{5}$$

$$Z(C(E)) = \emptyset. \tag{6}$$

In particular, we have for any closed subset $T$ of $E$:

$$Z(C(E\|T)) = T. \tag{7}$$

If $V$ is a closed subspace of $C(E)$ and $Z(V)$ is its zero set we have the inclusion

$$V \subseteq C(E\|Z(V)). \tag{8}$$

We will show that for closed ideals $J$ the sharper relation

$$J = C(E\|Z(J))$$

holds.

PROPOSITION 1. *Let $J$ be a closed ideal in $C(E)$. Then we have*

$$|f| \in J \quad \text{for all } f \in J.$$

For a proof of Proposition 1 which is based on the binomial series we refer to Stone [8], p. 54. Note that

$$\sup\{f, g\} = (f+g+|f-g|)/2,$$

$$\inf\{f, g\} = (f+g-|f-g|)/2.$$

It follows that a closed ideal $J$ in $C(E)$ is also a vector lattice, i.e.,

$$\sup\{f, g\},\ \inf\{f, g\} \in J \quad (f, g \in J).$$

PROPOSITION 2. *Let $V$ be a nonvoid subset of $C(E)$ such that*

$$\sup\{f, g\},\ \inf\{f, g\} \in V \quad \text{for all } f, g \in V.$$

*Then a sufficient condition for a function $f \in C(E)$ to be in the closure $V^-$ of $V$ is that, whatever the points $x$, $y$ in $E$ and whatever the positive number $\varepsilon$, there exists a function $f_{xy}$ in $V$ such that*

$$|f(x) - f_{xy}(x)| < \varepsilon, \quad |f(y) - f_{xy}(y)| < \varepsilon.$$

For a proof of Proposition 2 we refer to Stone [8], p. 35, and Semadeni [6], p. 115.

Proposition 1 is due to Stone [8]. It is used in combination with Proposition 2 to characterize closed ideals in $C(E)$. For the case of compact metric spaces we include a simplified proof of this result (see Stone [8] and Semadeni [6], p. 119).

THEOREM 1 (Stone). *Let $J$ be a closed ideal with zero set $Z(J)$. Then the relation*

$$C(E\|Z(J)) = J \tag{9}$$

*is true.*

*Proof.* It follows from the definition of $Z(J)$ and $C(E\|Z(J))$ that

$$J \subseteq C(E\|Z(J)). \tag{10}$$

In view of Proposition 1 we may apply Proposition 2 to $V = J$. Assume now $f \in C(E\|Z(J))$. Consider first the case $x, y \in Z(J)$. Then $f_{xy} := 0 \in J$ satisfies

$$f_{xy}(x) = f(x), \quad f_{xy}(y) = f(y).$$

Next suppose $x \in Z(J)$ and $y \in E - Z(J)$. Then there is a function $\Gamma_y \in J$ satisfying $\Gamma_y(y) = 1$. The function $f_{xy}$ defined by

$$f_{xy}(z) = \Gamma_y(z) f(y) \quad (z \in E)$$

is in $J$ and satisfies

$$f_{xy}(x) = f(x), \quad f_{xy}(y) = f(y).$$

Finally we consider the case $x, y \in E - Z(J)$. We introduce the functions $\varphi_1$ and $\varphi_2$ by setting

$$\varphi_1(z) = d(z, y)/(d(z, x) + d(z, y)),$$
$$\varphi_2(z) = d(z, x)/(d(z, x) + d(z, y)) \quad (z \in E)$$

where $d$ is the distance of $E$. $\varphi_1$ and $\varphi_2$ are in $C(E)$ and possess the interpolation properties

$$\varphi_1(x) = 1, \quad \varphi_1(y) = 0,$$
$$\varphi_2(x) = 0, \quad \varphi_2(y) = 1.$$

Then we define $f_{xy} \in J$ by

$$f_{xy}(z) = f(x)\varphi_1(z)\Gamma_x(z) + f(y)\varphi_2(z)\Gamma_y(z) \quad (z \in J).$$

The function $f_{xy} \in J$ again satisfies

$$f_{xy}(x) = f(x), \quad f_{xy}(y) = f(y).$$

Thus, Proposition 2 is applicable. It follows that

$$C(E\|Z(J)) \subseteq J \tag{11}$$

since $J = J^{-}$. This completes the proof of Theorem 1.

## 2. Interpolation projectors

Let $P$ be a bounded linear projector on $C(E)$. Then $\ker(P)$ is a closed linear subspace of $C(E)$ and we may consider the zero set $Z(\ker(P))$ of $\ker(P)$. For projectors this set has an interesting interpretation.

PROPOSITION 3. *Let $P$ be a bounded linear projector on $C(E)$ such that $Z(\ker(P)) \neq \emptyset$. Moreover, let $\operatorname{prec}(P)$ denote the set of all interpolation points of $P$, i.e.,*

$$\operatorname{prec}(P) = \{z \in E\colon\ P(f)(z) = f(z) \text{ for all } f \in C(E)\}. \tag{12}$$

*Then the relation*

$$\operatorname{prec}(P) = Z(\ker(P)) \tag{13}$$

*holds.*

*Proof.* Let $I$ designate the identity operator on $C(E)$. Since $P^2 = P$, we have

$$\ker(P) = \operatorname{ran}(I-P). \tag{14}$$

Thus, we can conclude

$$\begin{aligned} Z(\ker(P)) &= Z(\operatorname{ran}(I-P)) \\ &= \{z \in E\colon g(z) = 0 \text{ for all } g \in \operatorname{ran}(I-P)\} \\ &= \{z \in E\colon f(z) - P(f)(z) = 0 \text{ for all } f \in C(E)\} \end{aligned}$$

whence (13) follows.

The set $\operatorname{prec}(P)$ is called the *precision set* of $P$. It was introduced by Gordon and Wixom [5].

As a guiding example we consider the projector of polynomial Lagrange interpolation in $C[0, 1]$. We have

$$P_n(f)(x) = \sum_{j=1}^{n} f(x_j)\, l_{j,n}(x)$$

with $0 \leqslant x_1 < x_2 < \ldots < x_n \leqslant 1$ and

$$l_{j,n}(x) = w_n(x)/(w'(x_j)(x-x_j)) \qquad (j = 1, \ldots, n),$$

$$w_n(x) = (x-x_1)(x-x_2)\ldots(x-x_n).$$

It is evident that

$$\mathrm{prec}(P_n) = \{x_1, \ldots, x_n\}.$$

Moreover, $P_n(1) = 1$ and $\ker(P_n)$ is the closed ideal of functions in $C[0, 1]$ which vanish at $x_1, \ldots, x_n$. These properties lead to the abstract concept of interpolation projector.

A bounded projector $P$ on $C(E)$ is called *interpolation projector* iff

$$P(1) = 1, \tag{15}$$

$$\ker(P) \text{ is an ideal in } C(E). \tag{16}$$

Note that $\ker(P)$ is automatically closed. $I$ is a trivial interpolation projector while $0 = I - I$ is not an interpolation projector.

*Remark*. Besides polynomial Lagrange interpolation the method of broken-line interpolation yields another important example of interpolation projector. This interpolation method is used to construct the Schauder hat function system which is an interpolating basis for $C[0, 1]$. For a systematic discussion of interpolating bases in $C(E)$ and related interpolation projectors we refer to Semadeni [7].

PROPOSITION 4. *Any interpolation projector $P$ has a nonvoid precision set* $\mathrm{prec}(P)$.

*Proof.* Assume $\mathrm{prec}(P) = \emptyset$. Taking into account (16), (13) and Theorem 1 we obtain

$$\ker(P) = C\big(E\|Z(\ker(P))\big) = C(E)$$

which implies $P = 0$. This contradicts $P(1) = 1$.

PROPOSITION 5. *Let $P$ be an interpolation projector. Then the remainder projector $I - P$ is not an interpolation projector.*

*Proof.* This follows from

$$(I - P)(1) = 0.$$

As an additional consequence we have

$$\mathrm{prec}(I - P) = \emptyset$$

for any interpolation projector.

It is well known that the product $PQ$ and the Boolean sum $P \oplus Q = P + Q - PQ$ of two commutative projectors are again projectors. Moreover, the relations

$$\ker(PQ) = \ker(P) + \ker(Q), \tag{17}$$

$$\ker(P \oplus Q) = \ker(P) \cap \ker(Q) \tag{18}$$

hold in this case.

THEOREM 2. *Let $P$ and $Q$ be commutative interpolation projectors. Then the product $PQ$ is again an interpolation projector satisfying*

$$\operatorname{prec}(PQ) = \operatorname{prec}(P) \cap \operatorname{prec}(Q). \tag{19}$$

*Proof.* It is obvious that

$$PQ(1) = 1.$$

Moreover, it follows from (17) that $\ker(PQ)$ is an ideal. By Theorem 1 and (13) we have

$$\ker(P) = C(E\|\operatorname{prec}(P)),$$
$$\ker(Q) = C(E\|\operatorname{prec}(Q)),$$
$$\ker(PQ) = C(E\|\operatorname{prec}(PQ)).$$

Taking into account (17) we obtain

$$C(E\|\operatorname{prec}(PQ)) = C(E\|\operatorname{prec}(P)) + C(E\|\operatorname{prec}(Q))$$

which implies

$$Z\big(C(E\|\operatorname{prec}(PQ))\big) = \operatorname{prec}(P) \cap \operatorname{prec}(Q).$$

Since by Proposition 3

$$\operatorname{prec}(PQ) = Z(\ker(PQ))$$

the proof is complete.

THEOREM 3. *The Boolean sum $P \oplus Q$ of the commutative interpolation projectors $P$ and $Q$ is again an interpolation projector with*

$$\operatorname{prec}(P \oplus Q) = \operatorname{prec}(P) \cup \operatorname{prec}(Q). \tag{20}$$

*Proof.* It is easily seen that

$$P \oplus Q(1) = 1.$$

Relation (18) implies that $\ker(P \oplus Q)$ is a closed ideal. Thus, $P \oplus Q$ is an interpolation projector. To establish (20) we apply Theorem 1 and Proposition 3. We have in view of (18), (13)

$$\begin{aligned} C(E\|\operatorname{prec}(P \oplus Q)) &= \ker(P \oplus Q) = \ker(P) \cap \ker(Q) \\ &= C(E\|\operatorname{prec}(P)) \cap C(E\|\operatorname{prec}(Q)), \end{aligned}$$

i.e.,

$$C(E\|\operatorname{prec}(P \oplus Q)) = C(E\|\operatorname{prec}(P)) \cap C(E\|\operatorname{prec}(Q)). \tag{21}$$

Using relation (7) we get

$$\operatorname{prec}(P \oplus Q) = \operatorname{prec}(P) \cup \operatorname{prec}(Q)$$

which completes the proof of Theorem 3.

PROPOSITION 6. *Let $P$ and $Q$ be commutative interpolation projectors satisfying* $\operatorname{prec}(P) = \operatorname{prec}(Q)$. *Then $P$ and $Q$ are identical.*

*Proof.* Note first that

$$\ker(P) = \ker(Q)$$

in view of

$$\ker(P) = C(E\|\operatorname{prec}(P)) = \ker(Q).$$

This implies

$$Qf = QP(f), \quad Pf = PQ(f)$$

for all $f \in C(E)$. Since $PQ = QP$ it follows $P = Q$.

We note that Theorem 2 and Theorem 3 can be used to construct new interpolation projectors. Let $L$ be a set of commutative interpolation projectors. It follows from Theorem 2 that

$$L' = \{PQ\colon\ P, Q \in L\}$$

is again a set of commutative interpolation projectors such that

$$L \subseteq L'. \tag{22}$$

Applying Theorem 3 it follows that

$$L'' = \{P \oplus Q\colon\ P, Q \in L'\}$$

is also a set of commutative interpolation projectors satisfying

$$L' \subseteq L''. \tag{23}$$

This process leads to the recursive construction [1]:

$$L_1 := L'', \quad L_{k+1} := L_k'' \quad (k \in N).$$

Note that

$$L \subseteq L_1 \subseteq L_2 \subseteq \ldots \subseteq L_k \subseteq \ldots \tag{24}$$

Thus

$$\tilde{L} := \bigcup_{k=1}^{\infty} L_k$$

is a set of commutative interpolation projectors generated from $L$. It follows from the relations (22), (23), and (24) that $\tilde{L}$ is closed with respect to product and Boolean sum operations, i.e.,

$$PQ,\ P \oplus Q \subseteq \tilde{L} \quad \text{for all } P, Q \in \tilde{L}.$$

In view of Proposition 5 $\tilde{L}$ can not be complemented, i.e., $P \in \tilde{L}$ does not imply $I - P \in \tilde{L}$. This is an important difference to Boolean algebras of commutative projectors ([1]). For a more complete discussion we refer to [3].

## 3. Parametric extensions of interpolation projectors

In this section we will show that the abstract method of parametric extension ([2], [4]) is appropriate for the construction of new interpolation projectors. Let $E = X \times Y$ be the cartesian product of the compact metric spaces $X$, $Y$. For $f \in C(X \times Y)$, $x \in X$, and $y \in Y$ we consider the functions

$$f^x = f(x, \cdot) \in C(Y), \qquad {}^y f = f(\cdot, y) \in C(X).$$

Let $A$ be a bounded linear operator on $C(X)$ and $B$ be a bounded linear operator on $C(Y)$. We define the *bivariate* functions

$$A'(f)(x, y) = A({}^y f)(x), \qquad B''(f)(x, y) = B(f^x)(y)$$

with $f \in C(X \times Y)$ and $x \in X$, $y \in Y$. It was shown in [2] that

$$A'(f),\ B''(f) \in C(X \times Y) \qquad (f \in C(X \times Y)).$$

Moreover, $A'$ and $B''$ are bounded linear operators on $C(X \times Y)$ which are called *parametric extensions* of $A$ respectively $B$. They possess the following properties

$$(A_1 + A_2)' = A_1' + A_2', \quad (A_1 A_2)' = A_1' A_2', \quad (cA)' = cA',$$

$$(B_1 + B_2)'' = B_1'' + B_2'', \quad (B_1 B_2)'' = B_1'' B_2'', \quad (cB)'' = cB''.$$

Moreover, the parametric extensions $A'$ and $B''$ commute, i.e.,

$$A' B'' = B'' A'.$$

It follows from these relations that the parametric extensions $P'$ and $Q''$ of the bounded linear projectors $P$ and $Q$ are also bounded linear projectors on $C(X \times Y)$ which commute. Thus, the product operator $P'Q''$ and the Boolean sum operator $P' \oplus Q''$ are bounded linear projectors.

THEOREM 4. *Let $P$ be an interpolation projector on $C(X)$ and $Q$ be an interpolation projector on $C(Y)$. Then the parametric extensions $P'$ and $Q''$ are* *interpolation projectors with precision sets*

$$\operatorname{prec}(P') = \operatorname{prec}(P) \times Y, \qquad \operatorname{prec}(Q'') = X \times \operatorname{prec}(Q).$$

We omit the easy proof which follows from the properties of the method of parametric extension.

It follows from Theorem 4 and the properties of the parametric extensions that Theorem 2 is applicable to $P'$ and $Q''$. Thus, $P'Q''$ is an interpolation projector with precision set

$$\operatorname{prec}(P'Q'') = \operatorname{prec}(P) \times \operatorname{prec}(Q).$$

For the special case of polynomial Lagrange interpolation the interpolation projector $P'Q''$ describes polynomial tensor product interpolation:

$$P'Q''(f)(x, y) = \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i, y_j)\, l_{i,m}(x)\, l_{j,n}(y).$$

In the same way Theorem 3 is applicable to $P'$ and $Q''$. Thus, $P' \oplus Q''$ is an interpolation projector with precision set

$$\operatorname{prec}(P' \oplus Q'') = \operatorname{prec}(P) \times Y \cup X \times \operatorname{prec}(Q).$$

For the case of polynomial Lagrange interpolation the interpolation projector $P' \oplus Q''$ describes polynomial blending interpolation ([4]):

$$P' \oplus Q''(f)(x, y) = \sum_{i=1}^{m} f(x_i, y)\, l_{i,m}(x) + \sum_{j=1}^{n} f(x, y_j)\, l_{j,n}(y) - \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i, y_j)\, l_{i,m}(x)\, l_{j,m}(y).$$

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