APPROXIMATION AND FUNCTION SPACES
BANACH CENTER PUBLICATIONS, VOLUME 22
PWN–POLISH SCIENTIFIC PUBLISHERS
WARSAW 1989

SHORT PROOFS FOR THE INEQUALITIES OF SZEGÖ, MARKOV AND ZYGMUND

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1. Inequalities of Szegö and Markov

For real trigonometric polynomials van der Corput and Schaake [2] were the first to state explicitly the following extension of Bernstein's inequality [1] which was implicit in an earlier inequality due to Szegö [7].

THEOREM 1. *If T is a real trigonometric polynomial of degree n and $|T(t)| \leqslant 1$ for real t then*

$$n^2 T(t)^2 + T'(t)^2 \leqslant n^2, \qquad t \text{ real}. \tag{1}$$

Proof. We may suppose that $t = 0$, $T'(0) \geqslant 0$, and

$$\|T\| := \max_s |T(s)| < 1.$$

Let β, $-\pi < 2n\beta < \pi$, be defined by

$$T(0) = \sin n\beta.$$

At the points $t_j := (2j-1)\pi/(2n) - \beta$ the trigonometric polynomial $v := \sin n(\cdot + \beta) - T$ of degree n has the property sign $v(t_j) = (-1)^{j+1}$. Hence v has exactly one zero in each interval (t_j, t_{j+1}), and the inequalities

$$t_0 < 0 < t_1, \qquad v(t_0) < 0 = v(0) < v(t_1)$$

imply $v'(0) > 0$. Because of $T'(0) \geqslant 0$,

$$0 \leqslant T'(0) = n\cos n\beta - v'(0) \leqslant n\cos n\beta = n\left(1 - T(0)^2\right)^{1/2}. \tag{2}$$

Since $\|T\| < 1$ was arbitrary, (2) is still valid if $\|T\| = 1$. That concludes the proof of Theorem 1. ∎

In 1889 A. Markov [5] published his famous inequality concerning the derivatives of algebraic polynomials.

THEOREM 2. *If P is an algebraic polynomial of degree n and $|P(x)| \leqslant 1$ in $-1 \leqslant x \leqslant 1$, then in the same interval*

$$|P'(x)| \leqslant n^2. \tag{3}$$

Proof. We first consider real polynomials P. By the substitution $x = \cos t$, $0 \leqslant t \leqslant \pi$, we associate with P the real trigonometric polynomial $T(t) = P(\cos t)$ where

$$\begin{aligned} T'(t) &= -\sin t \; P'(\cos t), \\ T''(t) &= -\cos t \; P'(\cos t) + \sin^2 t \, P''(\cos t). \end{aligned} \tag{4}$$

We know by Theorem 1 that $\|T'\| \leqslant n$, $\|T''\| \leqslant n^2$. If $x = \pm 1$ and thus $t = 0$ or $t = \pi$, (4) leads to (3).

Let P' attain its maximal or minimal value at an interior point x of the interval $[-1, 1]$. Hence $P''(x) = 0$. The application of Theorem 1 for T' and the relations (4) then imply

$$n^4 \geqslant n^2 \|T'\|^2 \geqslant T''(t)^2 + n^2 T'(t)^2 = x^2 P'(x)^2 + n^2(1-x^2) P'(x)^2 \geqslant P'(x)^2$$

which establishes (3).

If P has complex coefficients we proceed in the following well-known way: Let x be a fixed number in $[-1, 1]$. We may assume that $P'(x)$ is real, otherwise we multiply P with a proper complex number of absolute value 1. Then, by (3) applied to $\operatorname{Re} P$, we get (3) because of

$$|P'(x)| = |(\operatorname{Re} P)'(x)|. \quad \blacksquare$$

2. The inequality of Zygmund for weighted L_p-norms

Recently G. G. Lorentz ([4], Theorem 2) established a new form of Bernstein's inequality which is expressed by means of the Hardy–Littlewood–Pólya order relation "$\prec$" on $[0, 2\pi]$. In particular, he gave a short and elegant proof of Zygmund's inequality

$$\int_0^{2\pi} |T'(t)|^p \, dt \leqslant n^p \int_0^{2\pi} |T(t)|^p \, dt, \qquad 1 \leqslant p < \infty \tag{5}$$

for trigonometric polynomials T of degree n. We will modify Lorentz's proof and earlier ideas of E. M. Stein [6] in order to prove a slightly more general version of Zygmund's and Lorentz's results.

THEOREM 3. *Let w be an integrable nonnegative function on $[0, 2\pi]$. Let q be a positive integer and $1 \leqslant p < \infty$. Then, for trigonometric polynomials T of degree n with real or complex coefficients,*

$$\int_0^{2\pi} w(s)\,|T^{(q)}(s)|^p\,ds \leqslant n^{pq} \max_{y\in Y_{qn}} \int_0^{2\pi} w(s)\,|T(s+y)|^p\,ds \tag{6}$$

where

$$Y_{qn} := \{(4k-1+(-1)^q)\pi/4n \mid k = 1, \ldots, 2n\}.$$

Proof. For a complex number $\beta \neq 0$ we put $\operatorname{sign}\beta := |\beta|/\beta$. We choose a number h, $0 < h < \pi/2n$, and a positive number M and define the trigonometric polynomial Q of degree $\leqslant n$ by

$$Q(t) := \int_0^{2\pi} w(s)\operatorname{sign}(T'(s))\,T(s+t)\,ds + M \int_0^{2h} \sin n(s+t-h)\,ds.$$

Let $y = y(M, h) \in [0, 2\pi)$ be such that $|Q(y)| = \|Q\|$. Using Bernstein's inequality or (1) for Q at $t = 0$, $Q'(0) \leqslant n\|Q\| = n|Q(y)|$, we get

$$\begin{aligned}\int_0^{2\pi} w(s)\,|T'(s)|\,ds + Mn \int_0^{2h} \cos n(s-h)\,ds &= Q'(0) \leqslant n|Q(y)|\\ &\leqslant n \int_0^{2\pi} w(s)\,|T(s+y)|\,ds + Mn \int_0^{2h} |\sin n(s+y-h)|\,ds\end{aligned}$$

and thus

$$\int_0^{2\pi} w(s)\,|T'(s)|\,ds \leqslant n \int_0^{2\pi} w(s)\,|T(s+y)|\,ds. \tag{7}$$

If $M \to \infty$, $h \to 0$ such that $Mh \to \infty$ the cluster points of $y(M, h)$ are contained in Y_{1n}. Hence (7) is valid for some $y \in Y_{1n}$.

Let $p > 1$. We apply (7) for the weight function $W := w\,|T'|^{p-1}$ and use the Hölder inequality

$$\int_0^{2\pi} FG\,ds \leqslant \Big(\int_0^{2\pi} |F|^r\,ds\Big)^{1/r} \Big(\int_0^{2\pi} |G|^p\,ds\Big)^{1/p}$$

for $r := p/(p-1)$, $F := w^{1/r}|T'|^{p-1}$, $G := w^{1/p}|T(\cdot+y)|$ for a proper $y \in Y_{1n}$. This leads to the relations

$$\begin{aligned}\int_0^{2\pi} w(s)\,|T'(s)|^p\,ds &= \int_0^{2\pi} W(s)\,|T'(s)|\,ds\\ &\leqslant n \int_0^{2\pi} W(s)\,|T(s+y)|\,ds = n \int_0^{2\pi} F(s)\,G(s)\,ds\\ &\leqslant n\Big(\int_0^{2\pi} w(s)\,|T'(s)|^p\,ds\Big)^{1/r} \Big(\int_0^{2\pi} w(s)\,|T(s+y)|^p\,ds\Big)^{1/p}\end{aligned}$$

which implies (6) for $q = 1$.

For $q = 2, 3, \ldots$ one obtains (6) by induction. ∎

3. Special cases of Theorem 3

3.1. Lorentz's version of Bernstein's inequality

For two integrable functions f and g on $I := [0, 2\pi]$ the Hardy–Littlewood–Pólya order relation $f \prec g$ means that for each measurable set $A \subset I$ there is another set $B \subset I$ of equal measure with the property $\int_A |f|\, ds \leqslant \int_B |g|\, ds$. Recently Lorentz ([4], Theorem 2) proved that

$$T^{(q)} \prec n^q T, \qquad q = 1, 2, \ldots \tag{8}$$

for each trigonometric polynomial of degree n. He pointed out that Zygmund's inequality (5) is an immediate consequence of (8). His inequality (8) follows from our slightly more general Theorem 3 if we set $p = 1$ and $w(s) := 1$ for $s \in A$, $w(s) := 0$ elsewhere.

3.2. An inequality of Duffin and Schaeffer

If we apply Theorem 3 for $p = 1$ at a fixed point $x \in [0, 2\pi)$ for the weight function $w(s) := 1/h$ if $x \leqslant s \leqslant x+h$, $w(s) := 0$ elsewhere, and if $h > 0$ tends to zero, we get the following inequality which is an extension of [3], Lemma IV:

$$|T^{(q)}(x)| \leqslant n^q \max_{y \in Y_{qn}} |T(x+y)|, \qquad q = 1, 2, \ldots \tag{9}$$

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Presented to the Semester
Approximation and Function Spaces
February 27–May 27, 1986