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SIGN CHANGING LYAPUNOV FUNCTIONS AND PERTURBATION OF INVARIANT TORI OF DYNAMICAL SYSTEMS

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We consider the system of differential equations

$$\frac{d\varphi}{dt} = a(\varphi), \quad \frac{dx}{dt} = A(\varphi)x, \tag{1}$$

where

$$\varphi = (\varphi_1, \ldots, \varphi_m), \quad x = (x_1, \ldots, x_n),$$

$a(\varphi)$ is a vector-valued function, $A(\varphi)$ – $n \times n$-matrix-valued function, continuous in variables $\varphi_1, \ldots, \varphi_m$ and 2π-periodic in φ_j, $j = \overline{1, m}$.

In addition we assume that $a(\varphi)$ is a function such that the Cauchy problem $d\varphi/dt = a(\varphi)$, $\varphi|_{t=0} = \varphi_0$ has a unique solution $\varphi_t(\varphi_0)$ for every fixed $\varphi_0 \in R^m$, continously depending on φ_0. We shall use the following notations.

1. $\Omega^t_\tau(\varphi_0; A)$ – the resolvent of the linear system $dx/dt = A(\varphi_t(\varphi_0))x$, $\Omega^t_\tau(\varphi, A)|_{t=\tau} = I_n$ – n-dimensional unit matrix

2. $\langle x, y\rangle = \sum_{i=1}^{n} x_i y_i$ – the usual scalar product in R^n, $\langle x, x\rangle = \|x\|^2$.

3. $C^0(\mathscr{T}_m)$ – the space of functions $F(\varphi)$, (vector-or matrix-valued), continuous in variables $\varphi_1, \ldots, \varphi_n$ and 2π-periodic in each of there;

4. $C^1(\mathscr{T}_m; a)$ – the subspace of $C^0(\mathscr{T}_m)$ consisting of functions $F(\varphi)$ such that the superposition $F(\varphi_t(\varphi_0))$ considered as a function of the variable t is continuously differentiable in t and $\frac{d}{dt}F(\varphi_t(\varphi))|_{t=0} \stackrel{\mathrm{df}}{=} \dot{F}(\varphi) \in C^0(\mathscr{T}_m)$.

DEFINITION 1. We say that the system of equations

$$\frac{d\varphi}{dt} = a(\varphi), \quad \frac{dx}{dt} = A(\varphi)x + f(\varphi), \quad f(\varphi) \in C^0(\mathscr{T}_m) \tag{2}$$

has an *invariant torus* $x = u(\varphi)$, if $u(\varphi) \in C^1(\mathcal{T}_m; a)$ and the following equality

$$\dot{u}(\varphi) \equiv A(\varphi)u(\varphi) + f(\varphi) \tag{3}$$

holds for all $\varphi \in R^m$.

DEFINITION 2. Suppose that $C(\varphi) \in C^0(\mathcal{T}_m)$ is an $n \times n$-matrix-valued function such that the function

$$G_0(\tau, \varphi) = \begin{cases} \Omega_\tau^0(\varphi; A)C)\varphi_\tau(\varphi)), & \tau \leqslant 0, \\ \Omega_\tau^0(\varphi; A)[C(\varphi_\tau(\varphi)) - I_n], & \tau > 0 \end{cases} \tag{4}$$

satisfies the estimate

$$\|G_0(\tau, \varphi)\| \leqslant K_0 \exp(-\gamma_0|\tau|), \quad K_0, \gamma_0 - \text{const} > 0, \quad \tau \in R, \tag{5}$$

then the function (4) is called the *Green function* of the problem of invariant torus for the system (1) (or simply – the Green function).

Obviously, the existence of a Green function $G_0(\varphi, \tau)$ for the system (1) guarantees the existence of an invariant torus of the system (3) with the vector-valued function

$$x = \int_{-\infty}^{\infty} G_0(\tau, \varphi) f(\varphi_\tau(\varphi))\, d\tau. \tag{6}$$

The converse is not true: it is possible that for any vector-valued function $f(\varphi)$ the system (3) has an invariant torus $x = u(\varphi)$, but the Green function (4) for the system (1) does not exist. The following assertions hold.

THEOREM 1. *Suppose that the system* (1) *admits a Green function* (4) *satisfying estimate* (5). *Then there exists an n-dimensional matrix-valued function* $S(\varphi) = S^T(\varphi) \in C^1(\mathcal{T}_m; a)$ *satisfying the condition*

$$\langle (\dot{S}(\varphi) - S(\varphi)A^T(\varphi) - A(\varphi)S(\varphi))x, x \rangle \leqslant -\|x\|^2 \tag{7}$$

for all $x \in R^n$. Moreover, if the Green function (4) is unique then

$$\det S(\varphi) \neq 0, \quad \forall \varphi \in R^m. \tag{8}$$

As an example of such a function one can choose the following

$$S(\varphi) = 2\int_0^{\infty} \Omega_\tau^0(\varphi; A)(C(\varphi_\tau(\varphi)) - I_n)[\Omega_\tau^0(\varphi; A)(C(\varphi_\tau(\varphi)) - I_n)]^T d\tau$$
$$- 2\int_{-\infty}^{0} \Omega_\tau^0(\varphi; A)C(\varphi_\tau(\varphi))[\Omega_\tau^0(\varphi; A)C(\varphi_\tau(\varphi))]^T d\tau.$$

THEOREM 2. *Let n-dimensional symmetric matrix-valued function* $S(\varphi) \in C^1(\mathcal{T}_m; a)$ *satisfying* (7), *then the Green function* (4) *with estimate* (5) *exists. Moreover, if* $S(\varphi)$ *satisfies condition* (8) *then the Green function* (4) *is* *unique and the following identities hold*

$$C^2(\varphi) \equiv C(\varphi), \quad C(\varphi_t(\varphi)) = \Omega_0^t(\varphi; A)\, C(\varphi)\, \Omega_t^0(\varphi; A) \tag{9}$$

If the matrix $S(\varphi)$ is degenerate for some $\varphi = \varphi_0 \in R^m$ then there are infinitely many Green functions (4) *with estimate* (5).

THEOREM 3. *Assume that there exist two symmetric matrix-valued functions $S_i(\varphi) \in C^1(\mathcal{T}_m; a)$, $i = 1, 2$, satisfying the conditions*

$$\langle (\dot S_1(\varphi) - S_1(\varphi) A^T(\varphi) - A(\varphi) S_1(\varphi))x, x\rangle \leqslant -\gamma_1 \|x\|^2,$$

$$\langle (\dot S_2(\varphi) + S_2(\varphi) A(\varphi) + A^T(\varphi) S_2(\varphi))x, x\rangle \leqslant -\gamma_2 \|x\|^2, \quad \gamma_i > 0.$$

Then $\det S_i(\varphi) \neq 0\ \forall \varphi \in R^m$ *and the Green function* (4) *exists.*

THEOREM 4. *Assume that the assumptions of Theorem* 2 *are and* $\exists \varphi = \varphi_0 \in R^m$, $\det S(\varphi_0) = 0$. *Then*:

1. *There exists a unique n-dimensional symmetric matrix valued function $\mathcal{H}(\varphi) \in C^0(\mathcal{T}_m)$ with the following properties*:

(a) *the operator $\mathfrak{M}$, acting on n-dimensional functions by the formula*

$$\mathfrak{M}g = \int_{-\infty}^{\infty} \mathcal{H}(\varphi)(\Omega_0^t(\varphi; A))^T g(\varphi_t(\varphi))\, dt$$

turns out to be projective, i.e. $\mathfrak{M}^2 = \mathfrak{M}$;

(b) *the identity*

$$\mathcal{H}(\varphi_t(\varphi)) \equiv \Omega_0^t(\varphi; A)\, \mathcal{H}(\varphi)(\Omega_0^t(\varphi; A))^T$$

and the estimate

$$\|\Omega_0^t(\varphi; A)\, \mathcal{H}(\varphi)(\Omega_0^\tau(\varphi; A))^T\| \leqslant K_0 \exp\{-\gamma_0 |t-\tau|\}$$

hold for all $t, \tau \in R$, $\varphi \in R^m$; K_0, γ_0-const > 0;

(c) *every invariant torus of the system* (1) *is defined by the equation* $x = \mathfrak{M}g$;

(d) *the identity*

$$\mathcal{H}(\varphi) \equiv \int_{-\infty}^{\infty} \mathcal{H}(\varphi)(\Omega_0^t(\varphi; A))^T\, \Omega_0^t(\varphi; A)\, \mathcal{H}(\varphi)\, dt$$

holds for all $\varphi \in R^m$;

(e) *every restriction to R of the solution of the linear system $dx/dt = A(\varphi_t(\varphi_0))x$ is defined by the equality*

$$x = \Omega_0^t(\varphi_0; A)\, \mathcal{H}(\varphi_0)\, \eta, \quad \eta \in R^n.$$

2. *Every invariant torus of the system* (2) *is defined by the equality*

$$x = \mathfrak{M}g + \int_{-\infty}^{\infty} G_0(\tau, \varphi) f(\varphi_\tau(\varphi))\, d\tau,$$

where $g(\varphi)$ is an arbitrary vector-valued function from the space $C^0(\mathcal{T}_m)$.

3. *The system of equations*

$$\frac{d\varphi}{dt} = a(\varphi), \quad \frac{dy}{dt} = -A^T(\varphi)y + F(\varphi) \tag{10}$$

has an invariant torus not for all vector-valued functions $F(\varphi) \in C^0(\mathcal{T}_m)$. A necessary and sufficient condition for the existence of an invariant torus for the system (10) is the identity

$$\int_{-\infty}^{\infty} \mathcal{H}(\varphi)(\Omega_0^\tau(\varphi; A))^T F(\varphi_\tau(\varphi))\, d\tau \equiv 0,$$

it is equivalent to the identity

$$\int_{-\infty}^{\infty} \langle u(\varphi_t(\varphi)), F(\varphi_t(\varphi))\rangle\, dt \equiv 0$$

for every nontrivial invariant torus $x = u(\varphi)$ of the system (1).

Remark. Under the conditions of Theorem 4 the extended system of equations

$$\frac{d\varphi}{dt} = a(\varphi), \quad \frac{dx}{dt} = A(\varphi)x, \quad \frac{dy}{dt} = B(\varphi)x - A^T(\varphi)y, \qquad y \in R^n$$

with arbitrary matrix-valued function $B(\varphi) \in C^0(\mathcal{T}_m)$ satisfying

$$\langle B(\varphi)x, x\rangle \leqslant -\beta_0 \|x\|^2, \ \beta_0 = \text{const} > 0,$$

has a unique Green function for the problem of invariant tori.

THEOREM 5. *Assume that the system of equations*

$$\frac{d\varphi}{dt} = a(\varphi), \quad \frac{dx}{dt} = A(\varphi)x + B(\varphi)y,$$
$$\frac{dy}{dt} = D(\varphi)x - A^T(\varphi)y, \qquad x, y \in R^n, \tag{11}$$

is such that

$$\langle B(\varphi)y, y\rangle \geqslant \beta_0 \|y\|^2, \quad \langle D(\varphi)x, x\rangle \geqslant d_0 \|x\|^2, \quad \beta_0, d_0\text{-const} > 0. \tag{12}$$

Then for every fixed matrix-valued function $A(\varphi) \in C^0(\mathcal{T}_m)$ the system (11) has a unique Green function.

Remark. In Theorem 5 the quadratic matrices $B(\varphi)$ and $D(\varphi)$ can be non-symmetric. In this case conditions (12) should be replaced by the following ones

$$\langle B(\varphi)y, y\rangle \leqslant -\beta_0 \|y\|^2, \quad \langle D(\varphi)x, x\rangle \leqslant -d_0 \|x\|^2, \quad \beta_0, d_0\text{-const} > 0.$$

To conclude this note we consider the example

$$
(13)\qquad
\begin{aligned}
&\frac{d\varphi_1}{dt} = p_1 \sin \varphi_1 + p_2 \cos 2\varphi_2, \qquad \frac{d\varphi_2}{dt} = p_3 \sin 3\varphi_1 + p_4 \cos \varphi_2,\\
&\frac{dx}{dt} = (p_5 \cos \varphi_1 + p_6 \sin \varphi_2)\, x + f(\varphi_1, \varphi_2).
\end{aligned}
$$

The problem is to choose nonzero values of parameters p_i, $i = 1, 6$, for which the system (13) will have an invariant torus for every function $f(\varphi_1, \varphi_2) \in C^0(\mathcal{T}_2)$.

If we choose $S(\varphi_1, \varphi_2)$ to be the scalar function $\cos \varphi_1$ then we obtain the conditions

$$
(14)\qquad
\begin{aligned}
&p_1 p_5 > 0,\\
&\min(|p_1|, 2|p_5|) > |p_2| + 2|p_6|, \qquad p_3, p_4 \in R.
\end{aligned}
$$

If we put $S(\varphi_1, \varphi_2) = \sin \varphi_2$, we get the conditions

$$
(15)\qquad
\begin{aligned}
&p_4 p_6 <\\
&\min(|p_4|, 2|p_6|) > p_3 + 2|p_5|, \qquad p_1, p_2 \in R
\end{aligned}
$$

Therefore, the conditions (14), (15) are sufficient for the existence of an invariant torus of the system (13) for every function $f \in C^0(\mathcal{T}_2)$.

Remark. The problem of choosing the optimal function $S(\varphi)$ (in order to obtain the largest set of parameters p) remains open.

References

[1] N. N. Bogdinbov, Yu. A. Mitropolski, A. M. Samoilenko, *A method of fasting the convergence in nonlinear mechanics*, Nauk. Dumka, Kiev 1969 (in Russian).

[2] A. M. Samoilenko, *On the preservation of an invariant torus under perturbations*, Izv. Akad. Nauk. USSR, Ser. Mat. 34 (6) (1970), 1219–1240 (in Russian).

[3] A. M. Samoilenko, W. L. Kulik, *Exponential dyhotomy of invariant torus of dynamical systems*, Diff. Uravnenija 15 (8) (1979), 1434–1443, (in Russian).

[4] W. L. Kulik, *Quadratic forms and dyhotomy of solutions of linear systems of differential equations*, Ukrain. Mat. Zh. 34 (1) (1982), 43–49 (in Russian).