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APPROXIMATION OF MIXING POINT CONFIGURATION SPACES OVER Z^d BY SPACES OF SET CONFIGURATIONS

K. HÄSLER

Sektion Mathematik, Martin-Luther-Universität Halle-Wittenberg
Halle, D.D.R.

Configuration spaces over countable basic sets are suitable to model atomic and molecular lattice systems as they are found e.g. in statistical physics or in theoretical chemics (D. Ruelle [6], O. J. Heilmann, E. H. Lieb [4]). In this paper two forms of such models are considered: spaces of point and of set configurations (in the sense of D. Ruelle [6] resp. H. Michel [5]) over the lattice Z^d (d – any natural number).

In [5] some basic results on relations between these two kinds of configuration spaces are proved, especially the fact that every set configuration space is homeomorphic to a point configuration space. Here it is shown by using a Hausdorff metric that for any lattice dimension d every mixing point configuration space can be approximated by a sequence of set configuration spaces. The given proof demonstrates how the problem for dimension $d \geqslant 2$ can be reduced – by a certain projection algorithm – to the one-dimensional result.

1. Notation. Let Z^d denote the d-dimensional product space of the set Z of the integers (d – any natural number) and let $(\sigma^a)_{a \in Z^d}$ be the set of translations on the lattice Z^d defined by

$$\sigma^a(x) = x + a \quad (a, x \in Z^d).$$

2. Definition (D. Ruelle [6]). Let S be a nonempty finite set, $\mathscr{F}$ a locally finite set of finite subsets of Z^d with the property

$$F \in \mathscr{F} \Rightarrow \sigma^a(F) \in \mathscr{F} \quad (a \in Z^d)$$

and let $(\Omega_F)_{F \in \mathscr{F}}$ be a system of nonempty finite sets such that

$$\Omega_F \subset \prod_{x \in F} S \quad (F \in \mathscr{F})$$

and

$$F, G \in \mathscr{F} \text{ with } F = \sigma^a G \text{ for a certain } a \in Z^d \Rightarrow \Omega_F = \Omega_G$$

are true. Then the set

$$\Lambda = \Lambda\big(Z^d, (\Omega_F)_{F\in\mathscr{F}}\big) := \Big\{\xi = (\xi_x)_{x\in Z^d} \in \prod_{x\in Z^d} S \,|\, F \in \mathscr{F} \Rightarrow \xi|_F \in \Omega_F\Big\}$$

is called a *point configuration space* (= PCS) on Z^d. (For the elements of Λ, the denotion 'configurations' is used.)

3. *Remark.* (1) Regarding the product topology on $\prod_{x\in Z^d} S$ which is induced by the discrete topology on S, one obtains that every PCS $\Lambda \subset \prod_{x\in Z^d} S$ is a compact topological space with respect to the trace topology determined on Λ by the topology on $\prod_{x\in Z^d} S$.

(2) By $\sigma^a(\xi) = \sigma^a\big((\xi_x)_{x\in Z^d}\big) = (\eta_x)_{x\in Z^d} \in \prod_{x\in Z^d} S$ $(\xi \in \prod_{x\in Z^d} S,\ a \in Z^d)$ with $\eta_x = \xi_{x-a}$ $(x \in Z^d)$ the system $(\sigma^a)_{a\in Z^d}$ of translations on $\prod_{x\in Z^d} S$ is well defined. Every PCS Λ is invariant under the translation group $(\sigma^a)_{a\in Z^d}$.

4. Example. On the lattice Z^2, let be given the following system $\mathscr{F}$ of finite sets:

$$\mathscr{F} := \{\sigma^a(F) | F = \{(0, 0), (1, 0), (1, -1)\},\ a \in Z^2\}.$$

Considering the set $S = \{0, 1\}$, let the family $(\Omega_F)_{F\in\mathscr{F}}$ be determined by the condition

$$\Omega_F := \left\{ \begin{matrix} 1 & 0 \\ & 1 \end{matrix},\ \begin{matrix} 0 & 1 \\ & 1 \end{matrix},\ \begin{matrix} 1 & 1 \\ & 0 \end{matrix} \right\} \quad (F \in \mathscr{F}).$$

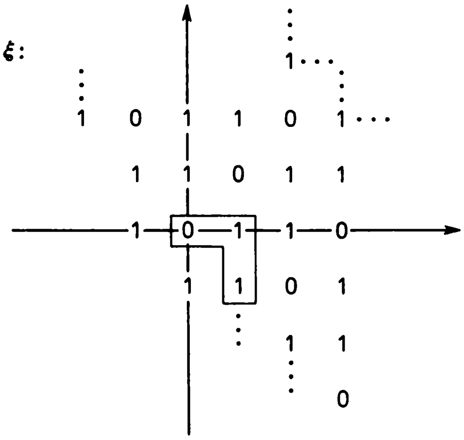
Then the PCS $\Lambda = \Lambda\big(Z^2, (\Omega_F)_{F\in\mathscr{F}}\big)$ is defined.

It is easy to see that there exists a configuration $\xi \in \Lambda$ such that $\Lambda = \{\sigma^a(\xi)\}_{a\in Z^2}$ is true. For this fact the following two properties are essential:

– If a (with respect to $(\Omega_F)_{F\in\mathscr{F}}$) allowed coloration of (e.g.) the lattice points (0, 0), (1, 0), (1, −1) is given, then there exists one and only one configuration η in Λ which has this coloration over $F = \{(0, 0), (1, 0), (1, -1)\}$ $(\subset Z^2)$. That means that Λ contains exactly 3 configurations.

– Each of these 3 configurations is converted into each other under a suitable translation σ^a.

(E.g.) ξ:



5. Definition (H. Michel [5]). Let $\mathscr{G}$ be a nonempty locally finite system of finite subsets of Z^d with the following properties:

(1) $$\pi(\mathscr{G}) := \{\mathscr{P} := (G_i)_{i\in N} \mid [G_i \in \mathscr{G}\ (i \in N)] \wedge [G_i \cap G_j \neq \emptyset\ (i, j \in N,\ i \neq j)] \wedge [\bigcup_{i\in N} G_i = Z^d]\}$$

is a nonempty set,

(2) $$G \in \mathscr{G} \Rightarrow \exists\ \mathscr{P} \in \pi(\mathscr{G}):\ G \in \mathscr{P},$$

(3) $$G \in \mathscr{G} \Rightarrow \sigma^a(G) = \{x+a \mid x \in G\} \in \mathscr{G} \quad (a \in Z^d),$$

and let $(S_G)_{G\in\mathscr{G}}$ be a system of nonempty finite sets such that

$$S_{\sigma^a(G)} = S_G \quad (a \in Z^d,\ G \in \mathscr{G})$$

is fulfilled. Then the set configuration space (= SCS) Λ^* on Z^d is defined by

$$\Lambda^* = \Lambda^*(Z^d, (S_G)_{G\in\mathscr{G}}) := \bigcup_{\mathscr{P}\in\pi(\mathscr{G})} \prod_{G\in\mathscr{P}} S_G$$

$$= \{\xi^* = (\xi^*_G)_{G\in\mathscr{P}} \mid \mathscr{P} \in \pi(\mathscr{G}) \wedge \xi^*_G \in S_G \quad (G \in \mathscr{P})\}.$$

(The elements of a SCS are called *set configurations*.)

6. *Remark*. For every SCS Λ^* is a topology generated by the subbasic sets

$$G^{[s]} := \{\xi^* = (\xi^*_H)_{H\in\mathscr{P}} \in \Lambda^* \mid G \in \mathscr{P} \wedge \xi^*_G = s\}.$$

7. Example. Given the family

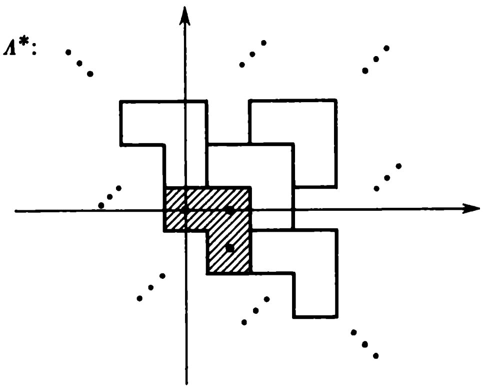
$$\mathscr{G} := \{\sigma^a(G) \mid G = \{(0, 0), (1, 0), (1, -1)\},\ a \in Z^2\}$$

and a class $(S_G)_{G\in\mathscr{G}}$ with card $(S_G) = 1$ and $S_G = S_H$ for all $G, H \in \mathscr{G}$, a SCS Λ^* on Z^2 is defined.

Every of the set configurations η^* of Λ^* generates the SCS:

$$\eta^* \in \Lambda^* \Rightarrow \Lambda^* = \{\sigma^a(\eta^*)\}_{a\in Z^2}.$$

(E.g.) $\eta^* \in \Lambda^*$:



8. *Remark.* (1) It is known that there exists a connection between the notions of point- and set-configuration spaces on a given lattice Z^d (H. Michel [5]). Using the denotions

$$\mathscr{L}(S^{Z^d}) := \{\mathrm{PCS}\,\Lambda(Z^d, (\Omega_F)_{F\in\mathscr{F}}) | \Omega_F \subset \prod_{x\in F} S\},$$

$$\mathscr{L}^*(S^{Z^d}) := \{\Lambda \in \mathscr{L}(S^{Z^d}) | \exists\, \mathrm{SCS}\ \Lambda^* = \Lambda^*(Z^d, (S_G)_{G\in\mathscr{G}}) \wedge$$

$$\exists \text{ homeomorphism } \varphi\colon \Lambda^* \to \Lambda \text{ with } \sigma^a\varphi = \varphi\sigma^a \quad (a \in Z^d)\},$$

one can prove the following relations (see [5]):

– If Λ^* is a given SCS on Z^d then there exists a finite set S with the property: There exist a PCS $\Lambda \in \mathscr{L}(S^{Z^d})$ and a homeomorphism $\varphi\colon \Lambda^* \to \Lambda$ such that $\sigma^a\varphi = \varphi\sigma^a$ $(a \in Z^d)$ is true.

– For every nonempty finite set S the following connection holds:

$$\mathscr{L}^*(S^{Z^d}) \subsetneqq \mathscr{L}(S^{Z^d}).$$

(2) For the comparison of the sets $\mathscr{L}(S^{Z^d})$ and $\mathscr{L}^*(S^{Z^d})$ (where S is any given nonempty set) it is useful to regard a suitable metric. For this in the following a Hausdorff metric $\tilde{d}$ is considered on $\mathscr{L}(S^{Z^d})$:

$$\tilde{d}(\Lambda_1, \Lambda_2) := \max\big(\max_{\xi\in\Lambda_1}\min_{\eta\in\Lambda_2} d(\xi, \eta), \max_{\eta\in\Lambda_2}\min_{\xi\in\Lambda_1} d(\xi, \eta)\big)(\Lambda_1, \Lambda_2 \in \mathscr{L}(S^{Z^d}))$$

where

$$d(\xi, \eta) = d\big((\xi_x)_{x\in Z^d}, (\eta_x)_{x\in Z^d}\big) := \sum_{l=1}^{\infty} \operatorname{card}(\mathscr{Z}_l)^{-l} \sum_{x\in\mathscr{Z}_l} \delta(\xi_x, \eta_x)$$

with

$$\mathscr{Z}_l := \{x = (x_1, \ldots, x_d) \in Z^d | \max_{i=1,\ldots,d} |x_i| = l\}$$

and

$$\delta(\xi_x, \eta_x) = \begin{Bmatrix} 1 & \text{if } \xi_x \neq \eta_x, \\ 0 & \text{if } \xi_x = \eta_x \end{Bmatrix}.$$

It is easy to prove that this metric is compatible with the product topology on S^{Z^d}.

Using this metric, one obtains that from the property

$$\tilde{d}(\Lambda_1, \Lambda_2) \leqslant \frac{1}{2^{n-1}}$$

it follows that for the configurations of Λ_1 over the central cube $Q := \{x = (x_1, \ldots, x_d) \in Z^d \mid -n \leqslant x_i \leqslant +n \ (i = 1, \ldots, d)\}$ of the lattice Z^d the same colorations and only these are allowed as for the configurations of Λ_2, i.e.: for every element $\xi \in \Lambda_k$ there exists a configuration $\eta \in \Lambda_{k+1 (\mathrm{mod} 2)}$ $(k = 1, 2)$ such that $\xi_x = \eta_x$ for all $x \in Q$.

With respect to the topology induced by the metric $\tilde{d}$, $\mathscr{L}(S^{Z^d})$ is a compact metric space.

9. *Remark*. For the one-dimensional lattice Z^1 one can prove that for any finite set S the space $\mathscr{L}^*(S^{Z^1})$ is dense in $\mathscr{L}(S^{Z^1})$: $\mathscr{L}(S^{Z^1}) = \overline{\mathscr{L}^*(S^{Z^1})}$. (For the proof see [1].) In the case of dimension $d > 1$ one can find an approximation result for the relation between PCS's and SCS's under the additional condition that the regarded PCS's are mixing.

10. Definition ([5]). A PCS $\Lambda = \Lambda(Z^d, (\Omega_F)_{F \in \mathscr{F}}) \in \mathscr{L}(S^{Z^d})$ is called to be *mixing* if the dynamical system $(\Lambda, (\sigma^a)_{a \in Z^d})$ is mixing, i.e. if for any two open sets U, V in Λ the following relation is true:

$$U \cap \sigma^a V \cap \Lambda \neq \emptyset \quad \text{for almost all } a \in Z^d.$$

11. Example. (1) The PCS Λ of Example 4 is an element of $\mathscr{L}^*(\{0, 1\}^{Z^2})$, because the SCS Λ^* of Example 7 is homeomorphic to Λ under the transformation $\varphi\colon \Lambda \to \Lambda^*$ defined by $\varphi(\xi) = \eta^*$ and $\varphi(\sigma^a \xi) = \sigma^a \varphi(\xi)$ $(a \in Z^2)$. Since (e.g.) $G^{[s]} \cap \sigma^{(3n+2,0)}(G^{[s]}) \cap \Lambda^* = \emptyset$ is true for all integers n, the system Λ^* and, therefore, also Λ are not mixing.

(2) Considering the space $\Lambda' \in \mathscr{L}(\{0, 1\}^{Z^2})$ which is generated by $\mathscr{F}' := \mathscr{F}$ (in the sense of Example 4) and

$$\Omega'_F := \Omega_F \cup \left\{\begin{matrix} 0 & 0 \\ & 0 \end{matrix}\right\}$$

$(F \in \mathscr{F}')$, one obtains a PCS of much more rich structure than in the case of the system Λ regarded in Example 4. It is possible to prove that Λ' is mixing. It is not clear whether there exists a homeomorphic SCS to Λ' or not. But the following theorem shows that Λ' can be approximated at least by a sequence of SCS's.

12. Theorem. *For any finite set S and for any dimension $d \geqslant 1$, the following relation between the set $\mathscr{L}_m(S^{Z^d})$ of the mixing systems of $\mathscr{L}(S^{Z^d})$ and*

the set $\mathscr{L}^*(S^{Z^d})$ *is true*:

$$\mathscr{L}_m(S^{Z^d}) \subset \overline{\mathscr{L}^*(S^{Z^d})}.$$

(*Remark*. One can even prove that $\overline{\mathscr{L}^*(S^{Z^d}) \cap \mathscr{L}_m(S^{Z^d})} = \mathscr{L}_m(S^{Z^d})$ is true (see [2]). The renunciation of the mixing property of the approximating systems from $\mathscr{L}^*(S^{Z^d})$ allows an essential simplification of the idea of the proof. Therefore, here the reduced form of the relation is regarded.)

Proof. (1) In the case $d = 1$, one can deduce the result directly from the proof of the relation $\mathscr{L}(S^{Z^1}) = \overline{\mathscr{L}^*(S^{Z^1})}$ which is given in [1].

(2) In the following, the assertion will be proved for $d = 2$. One can generalize this proof by induction for any dimension $d > 2$. Let be $d = 2$. The content of this proof is to construct for any fixed PCS $\Lambda \in \mathscr{L}_m(S^{Z^2})$ a sequence $(\Lambda_n^*)_{n\in N} \subset \mathscr{L}^*(S^{Z^2})$ with the property

$$\tilde{d}(\Lambda, \Lambda_n^*) \leqslant \frac{1}{2^{n-1}} \qquad (n \in N).$$

(3) Let now n be a fixed natural number. The following scheme illustrates the steps of the proof:

$$
\begin{array}{ll}
\Lambda \in \mathscr{L}_m(S^{Z^2}) & \\
\quad \downarrow (3) & \\
\left.\begin{array}{l} \exists\, \xi \in \Lambda:\ (1)\ \xi \text{ is periodic}, \\ \qquad\quad (2)\ \tilde{d}\left(\Lambda, \overline{\{\sigma^a \xi\}_{a\in Z^2}}\right) \leqslant \dfrac{1}{2^{n-1}} \end{array}\right\} & \xrightarrow[(4)]{p:M\to Z^1} \vartheta \in S^{Z^1} \\
 & \quad \downarrow (5) \\
 & \tilde{\Lambda} := \overline{\{\sigma^r \vartheta\}_{r\in Z^1}} \\
 & \quad \downarrow (5) \\
 & \exists \tilde{\Lambda}_{i_0}^* \in \mathscr{L}^*(S^{Z^1}): \\
 & \tilde{d}(\tilde{\Lambda}, \tilde{\Lambda}_{i_0}^*) \leqslant \dfrac{1}{2^{n-1}} \\
 & \quad \downarrow (5) \\
 & \exists \text{ a set configu-} \\
 & \text{ration } \vartheta^* \text{ on } Z^1: \\
\text{set configuration } \eta^* \text{ on } Z^2 & \xleftarrow[(6)]{p^{-1}} \tilde{\Lambda}_{i_0}^* = \overline{\{\sigma^r \vartheta^*\}_{r\in Z^1}} \\
\quad \downarrow (6) & \\
\Lambda^* := \overline{\{\sigma^a \eta^*\}_{a\in Z^2}}: & \\
\tilde{d}(\Lambda, \Lambda^*) \leqslant \dfrac{1}{2^{n-1}}. &
\end{array}
$$

One regards all sets of the configurations

$$_{(-n,n)}[(s_{(i_1,i_2)})_{i_j=-n,\ldots,n(j=1,2)}]$$
$$:= \{\xi \in S^{Z^2} \mid \xi_{(i_1,i_2)} = s_{(i_1,i_2)} \ (i_j = -n, \ldots, n \ (j = 1, 2))\} \ (s_{(i_1,i_2)} \in S)$$

which have a nonempty intersection with Λ. (Let us denote these sets by $Q_1, \ldots, Q_r$.)

As a consequence of the mixing property of Λ, there exists a configuration $\xi \in \Lambda$ with the following two properties:

(i) There exists a natural number m such that for all $k \in \{1, \ldots, r\}$ one can find a point $a_k = (a_{k_1}, a_{k_2}) \in Z^2$ with

$$Q := {}_{(-m,m)}[(\xi_{(i_1,i_2)})_{i_j=-m,\ldots,m(j=1,2)}] \subset \sigma^{a_k}(Q_k),$$

i.e.

$$Q|_{\{x \in Z^2 \mid a_{k_i} - n \leqslant x_i \leqslant a_{k_i} + n (i=1,2)\}} = \sigma^{a_k}(Q_k).$$

(ii) The configuration ξ is periodic with respect to at least one of the two directions of the coordinate axes. (In the following it is assumed that (e.g.) $\xi = \sigma^{(0,(2m+1)l)}(\xi)$ is true for all $l \in Z$.)

(4) One can prove that there exists a transformation p which assigns to every point of the set $M := \{x = (x_1, x_2) \in Z^2 \mid -m \leqslant x_2 \leqslant m\}$ one and only one point of the lattice Z^1 such that every translation on Z^1 corresponds under p^{-1} to a translation σ^a over M where $a \in Z^2 \bmod (1, 2m+1)$.

(For the definition and for the properties of this transformation p see [3].)

Using the transformation p, one can assign to ξ a configuration $\vartheta \in S^{Z^1}$.

(5) The PCS $\tilde{\Lambda} \in \mathscr{L}(S^{Z^1})$ which is generated by $\{\sigma^r(\vartheta)\}_{r \in Z^1}$ can be approximated by a sequence $(\tilde{\Lambda}_l^*)_{l \in N} \subset \mathscr{L}^*(S^{Z^1})$. (See Remark 9.)

Let the natural number h be fixed such that

$$p(Q|_M) \supset \{\varrho \in \tilde{\Lambda} \mid \varrho|_{[-h,h]} = \vartheta|_{[-h,h]}\}$$

is true. One chooses now a natural number l_0 with the property

$$\tilde{d}(\tilde{\Lambda}, \tilde{\Lambda}_{l_0}^*) \leqslant \frac{1}{2^{h-1}},$$

i.e. there exists a generating configuration ϑ^* of $\tilde{\Lambda}_{l_0}^*$ which has (at least) over the central part $[-h, h]$ of Z^1 the same coloration as ϑ (= the generating configuration of $\tilde{\Lambda}$) and which corresponds (because of $\tilde{\Lambda}_{l_0}^* \in \mathscr{L}^*(S^{Z^1})$) to a set configuration on Z^1.

(6) Regarding the transformation p^{-1}, the configuration $\vartheta^* \in S^{Z^1}$ corre-

sponds to a unique coloration of the points of the set M and – by periodic iteration of this coloration in the direction of the second coordinate axe – to a unique configuration $\eta^* \in S^{Z^2}$.

One considers now the PCS Λ^* on Z^2 which is generated by $\{\sigma^a(\eta^*)\}_{a \in Z^2}$. From the construction it follows directly that $\Lambda^* \in \mathscr{L}^*(S^{Z^2})$ is true: by p^{-1} and by the periodic iteration in the direction of the second coordinate axe one assigns to the set configuration which corresponds to ϑ^* a set configuration on Z^2 which is in correlation with η^*.

Moreover, one obtains:

$$\xi|_{\{x=(x_1,x_2)\in Z^2 | -m \leqslant x_i \leqslant m (i=1,2)\}} = \eta^*|_{\{x=(x_1,x_2)\in Z^2 | -m \leqslant x_i \leqslant m (i=1,2)\}},$$

because ϑ and ϑ^* have the same coloration over $[-h, h]$. That is enough to show that $\tilde{d}(\Lambda, \Lambda^*) \leqslant \frac{1}{2^{m-1}} < \frac{1}{2^{n-1}}$ is true. Therefore, the system Λ^* can be used as the wanted PCS Λ_n^*.

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