

ON THE LOCATION OF POLES OF RUELLE'S ZETA FUNCTION FOR GAUSS MAP

DIETER H. MAYER*

*Institut für Theoretische Physik
Aachen, F.R.G.*

We discuss examples of one-dimensional lattice spin systems of classical statistical mechanics whose generalized zeta function has all its poles and zeros on the real axis. The close relation between certain hyperbolic dynamical systems and these spin systems let one expect that similar things are true for some of these dynamical systems also. In fact we have found several one-dimensional expansive systems among them the Gauss map whose zeta functions have their zeros, respectively their poles, on the real axis. Such a behaviour is closely related to spectral properties of the systems transfer operator which in the cases considered is some positive nuclear operator in a Banach space of holomorphic functions. We formulate a general conjecture concerning the spectrum of this class of operators.

I. Zeta functions of one-dimensional spin systems

We consider classical spin systems on the one-dimensional lattice \mathbf{Z} . Let F be the set of values the classical spin σ can take. A configuration ξ of the system can be described by a two-sided infinite sequence $\xi = (\xi_i)_{i \in \mathbf{Z}}$ of spin values on every lattice site $i \in \mathbf{Z}$. It happens very often in physical applications that not all such configurations ξ are indeed realised but only certain of them, which we will call *allowed* ones. For simplicity we consider only the case where there are only restrictions on spin values on neighbouring sites. Then the allowed configurations can be described by a transition matrix t all of whose entries $t_{i,j}$, $i, j \in F$, are either zero or one: configuration ξ is said to be allowed if

$$t_{\xi_i, \xi_{i+1}} = 1 \quad \text{for all } i \in \mathbf{Z}, \text{ where } \xi = (\xi_i)_{i \in \mathbf{Z}}.$$

* Heisenberg Fellow.

If F_t^Z denotes the set of all allowed configurations the system (F_t^Z, τ) with

$$\tau(\xi)_i = \xi_{i+1}$$

defines a two sided subshift of finite type.

Let us then consider the numbers $N_n(\tau)$ which count the periodic points of the shift map τ of period n :

$$(1) \quad N_n(\tau) = \# \{ \xi \in F_t^Z : \tau^n \xi = \xi \}$$

and their generating function

$$(2) \quad \zeta_\tau(z) = \exp \sum_{n=1}^{\infty} z^n/n N_n(\tau).$$

It is a special case of the zeta function introduced by Artin and Mazur [1] and was calculated the first time by Bowen and Lanford [2] to be

$$(3) \quad \zeta_\tau(z) = 1/\det(1 - zt).$$

If the transition matrix t is symmetric, which at least physically is natural to assume, all the poles of ζ_τ lie on the real axis. A special case being $t_{ij} = 1$ for all $i, j \in F$ which leads to the zeta function

$$(4) \quad \zeta_\tau(z) = 1/(1 - nz)$$

where $n = |F|$.

Because the Artin-Mazur function for an arbitrary Axiom-A system can be written via the thermodynamic formalism of Bowen, Ruelle and Sinai [3]-[4] as a quotient of products of functions of the above kind (3) with different transition matrices t_k also their zeta functions have all zero's and poles on the real line as soon as the corresponding matrices t_k have only real eigenvalues. It would be certainly interesting to find a simple characterisation for this to happen.

From a physical point of view the function ζ_τ as defined in (2) is rather uninteresting: it can be interpreted as the generating function of the so called finite system partition functions Z_n with periodic boundary conditions of a lattice spin system without interactions at all. If ξ_n denotes a configuration on a finite interval of length n on Z then this configuration determines a unique periodic configuration of period n on the whole lattice Z which for brevity we denote again by ξ_n . If $H(\xi_n)$ is the energy of the finite configuration ξ_n , periodic boundary conditions taken into account, the partition function $Z_n(H)$ is given as

$$(5) \quad Z_n(H) = \sum_{\xi_n} \exp -H(\xi_n)$$

where the sum is over all allowed configurations ξ_n . The trivial case is certainly where $H(\xi_n) \equiv 0$ for all ξ_n . Then $Z_n(H) = N_n(\tau)$ as defined in (1)

and therefore $\zeta_\tau(z)$ is indeed the generating function of all these free partition functions. It is clear that boundary conditions do not play any role in this case. They play however a role in the interacting case where $H \neq 0$ which alone is of interest in physics. This lead Ruelle [6] to an at least from the physical point of view very natural generalisation of the zeta function of Artin and Mazur. Let us start from a finite system with periodic boundary conditions. The energy $H(\xi_n)$ of any configuration ξ_n can then be decomposed into the different contributions coming from the interaction of the spins ξ_i at the lattice sites say $i = 0, \dots, n-1$ with all the other spins of the periodically extended configuration ξ_n on Z as follows

$$(6) \quad H(\xi_n) = \sum_{i=0}^{n-1} A(\tau^i \xi_n)$$

where $A(\xi_n)$ describes the interaction energy of the spin ξ_0 at the lattice site $i = 0$ with all other spins $\xi_j, j \in Z$, self interaction included. The partition function $Z_n = Z_n(H) = Z_n(A)$ can therefore be written as

$$(7) \quad Z_n(A) = \sum_{\xi_n} \exp - \sum_{i=0}^{n-1} A(\tau^i \xi_n).$$

Inserting this instead of $N_n(\tau)$ into the definition of the function ζ_τ in (2), we arrive at the function

$$(8) \quad \zeta_\tau(z, A) = \exp \sum_{n=1}^{\infty} z^n/n \sum_{\xi \in \text{Fix} \tau^n} \exp - \sum_{i=0}^{n-1} A(\tau^i \xi).$$

This can be generalized immediately to any dynamical system $T: M \rightarrow M$ and $\varphi: M \rightarrow \mathbf{R}$ some function to give Ruelle's generalized zeta function for a dynamical system

$$(9) \quad \zeta_T(z, \varphi) = \exp \sum_{n=1}^{\infty} z^n/n \sum_{x \in \text{Fix} T^n} \prod_{k=0}^{n-1} \varphi(T^k x).$$

Analyticity properties of this function both in z and φ have been studied both for spin systems and dynamical systems by several authors [7]–[13]. Contrary to the Artin–Mazur function the function $\zeta_T(z, \varphi)$ is for instance not meromorphic in z for arbitrary Axiom A systems [14] or systems of statistical mechanics with even exponentially decaying interactions [15]. Our main concern here is a point which in the above mentioned works was completely ignored namely the location of zero's and poles of this function in cases where meromorphy in the z plane could be established. This is a rather obvious question in the study of zeta functions of the different kinds [6]. Meromorphy of $\zeta_T(z, \varphi)$ is closely related to the existence of nuclear transfer operators L in some Banach space of holomorphic functions which allows one to express the numbers $\sum_{x \in \text{Fix} T^n} \prod_{k=0}^{n-1} \varphi(T^k x)$ as traces of such operators L^n

and therefore the function $\zeta_T(z, \varphi)$ as quotients of Fredholm determinants generalizing this way formula (3). There are now several examples available where this program could be carried through both for spin systems and dynamical systems. Let us consider first spin systems.

For both discrete and continuous spin systems on a lattice with finite range interactions the partition functions $Z_n(A)$ can be written in terms of the so called Kramers–Wannier transfer matrix L [16] as

$$Z_n(A) = \text{trace } L^n$$

such that for these systems the function $\zeta_\tau(z, A)$ is again simply given as

$$(10) \quad \zeta_\tau(z, A) = 1/\det(1 - zL).$$

In the case of continuous spin variables this matrix L is an integral operator in a Hilbert space of square-integrable functions [16]. The general form of the transfer operator L of a spin system is as follows: if Z_+ denotes the positive half lattice $Z_+ = \{i \in \mathbf{Z}: i \geq 0\}$ and F^{Z_+} the allowed configurations on Z_+ (we assume $t_{ij} = 1$ for all $i, j \in F$) then L is a linear operator on the space $\mathcal{C}(F^{Z_+})$ of all continuous observables on F^{Z_+} defined as

$$(11) \quad Lf(\xi^+) = \sum_{\sigma \in F} \exp -A(\eta_\sigma^+) f(\eta_\sigma^+)$$

where η_σ^+ denotes the configuration $(\eta_\sigma^+)_0 = \sigma$, $(\eta_\sigma^+)_i = \xi_{i-1}$, $i = 1, 2, \dots$ on Z_+ . The sum in the above definition of L has to be replaced by an integral for continuous spins. It was shown in [16] how L reduces to the transfer matrix L for finite range interactions. For symmetric interactions like the Ising system the transfer matrix has only real eigenvalues and the poles of the corresponding zeta function all are located on the real axis.

A much more interesting case arises when the interaction is of long range but decays exponentially fast with the distance on the lattice. This is just the kind of interaction appearing in the thermodynamic formalism of hyperbolic dynamical systems. A well known model in statistical mechanics is the so called Kac-model [17]: It is an Ising like model leading to the following function A :

$$(12) \quad A(\xi^+) = -J\xi_0^+ \sum_{i=1}^{\infty} \lambda^i \xi_i^+$$

with some constant J , $0 < \lambda < 1$ and $F = \{+1, -1\}$. It was shown in [9], respectively [16], that the corresponding transfer operator can be restricted to the Banach space $B(D_R)$ of holomorphic functions over the disc $D_R = \{z \in \mathbf{C}: |z| < R\}$ for some $R > \lambda/(1 - \lambda)$ leading to a nuclear operator

$$(13) \quad Lf(z) = e^{Jz} f(\lambda + \lambda z) + e^{-Jz} f(-\lambda + \lambda z).$$

The partition functions $Z_n(A)$ can be expressed via the traces of this operator as

$$(14) \quad Z_n(A) = (1 - \lambda^n) \text{trace } L^n$$

which leads to the zeta function

$$(15) \quad \zeta_\tau(z, A) = \frac{\det(1 - z\lambda L)}{\det(1 - zL)}.$$

To say something on the location of the zeros and poles of this function we have to know the eigenvalues of the operator L . It was shown in [16] that symmetry and positivity properties of L allow one to prove that at least the two leading eigenvalues λ_1 and λ_2 are positive and simple. What about the other eigenvalues? In fact we expect, even if we cannot prove it at the moment, that all eigenvalues are positive.⁽¹⁾ This is based on the following argument: there exists an at least at the first glance completely different approach to the transfer operator for this system by Kac [17]. He found that the partition functions $Z_n(A)$ with free boundary conditions can be expressed as the traces of some positive definite symmetric integral operator K in some Hilbert space of square summable functions. All its eigenvalues therefore are positive. Even if the exact relation between the two operators is not clear at the moment, especially for the different boundary conditions, it nevertheless supports our conjecture. In fact Baker treating the same model with free boundary conditions as Kac did arrived at a transfer operator very similar to our one [18]. Another support to our conjecture comes from the limiting behaviour of the operator L in (13) for $J = 0$. In this case the spectrum is explicitly known and consists of the numbers $2\lambda^i$, $i = 0, 1, 2, \dots$. Therefore our conjecture is true in this limit. Indeed, because boundary conditions do not play any role for $J = 0$ Kac's integral operator has in this limit up to the factor 2 exactly the same spectrum, as one expects from physical grounds.

II. Zeta functions for expanding maps

The simplest dynamical systems with zeta function meromorphic in the whole complex plane are the piecewise monotone Markov maps T of the unit interval I . Let us consider especially the piecewise linear ones under them. For such a T there exists a finite partition $I = \bigcup_j I_j$ into subintervals I_j with $\text{int } I_j \cap \text{int } I_k = \emptyset$ for $j \neq k$ such that $T(I_j) = I$ and $T|_{I_j}$ is linear for all j . If φ denotes any holomorphic function on the disc D_R of radius $R > 1$ in \mathbb{C} then the zeta function $\zeta_T(z, \varphi)$ defined as in (9) for this system turns out to be given by

⁽¹⁾ *Added in proof:* This we can show now.

$$(16) \quad \zeta_T(z, \varphi) = \frac{\det(1 - zL_1)}{\det(1 - zL_2)}$$

where the operators $L_i: B(D_R) \rightarrow B(D_R)$ are nuclear operators whose explicit form is given as

$$(17) \quad L_1 f(z) = \varphi(z) \sum_j \frac{1}{p_j} f \circ \psi_j(z),$$

respectively

$$(18) \quad L_2 f(z) = \varphi(z) \sum_j f \circ \psi_j(z).$$

Thereby ψ_j denotes the local inverse $T|_{I_j}^{-1}$ and p_j^{-1} its slope. The explicit form of ψ_j therefore is

$$\begin{aligned} \psi_1(z) &= p_1^{-1} z, \\ \psi_j(z) &= p_j^{-1} z + (p_1^{-1} + \dots + p_{j-1}^{-1}). \end{aligned}$$

Even for this rather simple operators almost nothing is known about their spectra. If the function φ however is positive on $D_R \cap \mathbf{R}$ then at least the leading eigenvalue of the operator L_2 is positive saying just that the pole nearest to the origin is real.

A rather trivial but nevertheless quite informative case is when the function $\varphi = 1$. Then the Ruelle zeta function is identical with the Artin-Mazur function: in fact the eigenvalues of the operators L_1 and L_2 are

$$\lambda_k^{(1)} = \sum_j p_j^{-k}, \quad k = 1, 2, \dots,$$

respectively

$$\lambda_k^{(2)} = \sum_j p_j^{-k}, \quad k = 0, 1, 2, \dots,$$

leading to the zeta function

$$(19) \quad \zeta_T(z, 1) = 1/(1 - nz)$$

where n denotes the number of intervals I_j in the partition of I . If one considers the simplest case $n = 2$ and $p_1 = p_2 = 2$ and takes for φ the function $\varphi(z) = \exp(Jz)$ then the operators L_1 and L_2 are rather similar to the transfer operators of the Kac model discussed earlier and we therefore expect them to have real eigenvalues only. Presumably this is true for all φ 's which are real on $D_R \cap \mathbf{R}$ as will be discussed later.

Let us next discuss a slightly more complicated system which is known to have a meromorphic zeta function. It is also locally expanding but allows for much stronger results as we will see. It is the Gauss map $T: I \rightarrow I$ with $Tx = 1/x \pmod{1}$ for $x \neq 0$. Its generalized zeta function was discussed in some detail in [12] where it was shown that it can be written as

$$(20) \quad \zeta_T(z, \varphi) = \frac{\det(1 - zL_1)}{\det(1 - zL_2)}$$

where φ is any holomorphic function in the disc $D = \{z \in \mathbb{C}: |z-1| < 3/2\}$ and vanishes at least like z^2 for $z \rightarrow 0$. The corresponding transfer operators $L_i: B_2(D) \rightarrow B_2(D)$ are again nuclear and given by

$$(21) \quad L_1 f(z) = \varphi(z) \sum_{i=1}^{\infty} \psi'_i(z) f \circ \psi_i(z),$$

respectively

$$(22) \quad L_2 f(z) = \varphi(z) \sum_{i=1}^{\infty} f \circ \psi_i(z),$$

for $f \in B_2(D) = \{f: f(z) = z^2 g(z), g \in B(D)\}$, where $\psi_i: D \rightarrow D$ are the holomorphic mappings

$$(23) \quad \psi_i(z) = (z+i)^{-1}.$$

Consider now the case $\varphi(z) = z^2$. The spectrum of the operator L_2 obviously coincides with the spectrum of the following operator L in the Banach space $B(D)$:

$$(24) \quad Lg(z) = \sum_{i=1}^{\infty} (z+i)^{-2} g((z+i)^{-1}).$$

In a recent paper with Roepstorff [19] we showed that this operator L has indeed only real eigenvalues. This was achieved by relating the spectrum of L to the spectrum of the following operator V

$$(25) \quad Vf(z) = \sum_{i=1}^{\infty} (z+i)^{-1} (z+i+1)^{-1} f((z+i)^{-1})$$

acting in a certain Hardy space $H^2(\nu)$ of holomorphic functions f in the half plane $H_{-1/2} = \{z = x+iy: x > -1/2\}$:

$$H^2(\nu) = \{f: f \text{ holomorphic in } H_{-1/2}, |(1+z)^{-1}f(z)| \text{ bounded in every half plane } H_{-1/2+\varepsilon} \text{ for } \varepsilon > 0 \text{ such that } \|f\|_{\nu}^2 = \int d\nu(z) |f(z)|^2 < \infty\}.$$

Thereby $d\nu$ denotes the measure

$$d\nu(z) = 1/\pi((1+x)^2 + y^2)^{-1} dx dy$$

on the strip $-1/2 < x < 0$.

It turns out that the operator V in this space is isomorphic to the following integral operator $K: L_2(\mathbb{R}_+, dm) \rightarrow L_2(\mathbb{R}_+, dm)$:

$$(26) \quad K\psi(s) = \int_0^{\infty} dm(t) J_1(2(st)^{1/2})(st)^{-1/2} \psi(t)$$

where m denotes the measure $dm(t) = (e^t - 1)^{-1} t dt$ on the positive real axis \mathbf{R}_+ . Since K is symmetric, all eigenvalues of K are real and, therefore, also those of V respectively L . But then all the poles of $\zeta_T(z, \varphi)$ are again on the real axis. Unfortunately, we cannot say much about the location of the function's zeros. Positivity properties which we will discuss a little bit more in detail later show that the leading eigenvalue of the operator

$$(27) \quad L_1 g(z) = - \sum_{i=1}^{\infty} (z+i)^{-4} g((z+i)^{-1})$$

is negative and therefore the zero next to the origin of the function $\zeta_T(z, z^2)$ is real. Up to now⁽²⁾ we did not succeed in relating the spectrum also of this operator L_1 to some symmetric Hilbert space operator as it was the case for L_2 . It is interesting to remark that the above zeta function for $\varphi(z) = z^2$ has been used quite recently by Pollicott to prove a prime number like theorem for the distribution of closed geodesics on the modular surface [20].

III. A conjecture about the spectrum of a class of composition operators in Banach spaces of holomorphic functions

In all cases where meromorphy of a zeta function in the whole complex plane could be established for a system to the present day this property derived from nuclearity properties of the corresponding transfer operators respectively analyticity properties of their Fredholm determinant as shown by Grothendieck. In all the examples discussed above these transfer operators act in a Banach space of holomorphic functions in one or several complex variables over some bounded domain D in \mathbf{C}^n with $D \cap \mathbf{R}^n \neq \emptyset$ and are of the quite general form

$$(28) \quad Lf(z) = \varphi(z) \sum_{i \in J} f \circ \psi_i(z)$$

where $\psi_i: D \rightarrow D$ are contracting holomorphic mappings with their unique fixed point z_i^* in $D \cap \mathbf{R}^n$. The function φ is also a holomorphic function on D . If the index set J consists of exactly one element then we showed in [21] that the spectrum of the operator L in (28) is determined by the value $\varphi(z^*)$ and the eigenvalues of the linear operator $D\psi(z^*)$ that means the derivative of ψ at the fixed point. If, therefore, φ is real on $D \cap \mathbf{R}^n$ and all these eigenvalues

⁽²⁾ Added in proof: We can show now: L_1 is related to the kernel $J_3(2(st)^{1/2})$ replacing J_1 in (26).

of $D\psi(z^*)$ are real then also the eigenvalues of L are real. This is trivially true for $n = 1$ and φ real on $D \cap \mathbf{R}^n$. Surprisingly enough the examples discussed above show that this property seems to be valid also in the case where J consists of several elements. Such a property would follow immediately when we could show that under the above mentioned assumptions the operator L in (28) can be related to a symmetric Hilbert space operator as it was the case for the operator L in (24) and is presumably the case for the operator L corresponding to the Kac model. Unfortunately we do not have any idea if something like this can be true. Indeed, the following example and also the fact that in some of the examples discussed before we could prove reality of the two leading eigenvalues by restoring to positivity properties of the corresponding transfer operator in the Banach space, can throw some light on these problems from a different side.

Consider the map $Tx = kx \bmod 1$ for any integer $k > 2$. For simplicity restrict the discussion to the case $k = 2$ but all we have to say is also true for arbitrary k . The transfer operator for the function $\varphi \equiv 1$ reads

$$(29) \quad Lf(z) = \frac{1}{2}f(z/2) + \frac{1}{2}f((z+1)/2).$$

This operator is u_0 -positive in the sense of Krasnoselskii [22] in the space $B(D_{\mathbf{R}})$ with respect to the cone $K_1 = \{f: f \geq 0 \text{ on } D \cap \mathbf{R}\}$. Therefore there exists a unique eigenfunction f_1 of L in K_1 with dominant eigenvalue $\lambda_1 > 0$. The operator L can then be written as

$$(30) \quad L \equiv \lambda_1 P_1 + N_1$$

with P_1 the projector onto f_1 and N_1 a linear operator with spectral radius strictly smaller than λ_1 such that $P_1 N_1 = N_1 P_1 = 0$. Denote by $\text{Ke } P_1$ the kernel of the operator P_1 . It turns out to be given by

$$\text{Ke } P_1 = \left\{ f: \int_0^1 f(x) dx = 0 \right\}.$$

On this space the operator N_1 is again u_0 -positive now with respect to the cone $K_2 = \{f \in \text{Ke } P_1: f' \geq 0 \text{ on } D \cap \mathbf{R}\}$. Therefore also N_1 can be written as

$$N_1 = \lambda_2 P_2 + N_2$$

where λ_2 is now the leading eigenvalue of N_1 and P_2 the projection onto the corresponding eigenfunction f_2 in K_2 and $P_2 N_2 = N_2 P_2 = 0$. This can be continued for general i : the operator N_{i-1} is u_0 -positive with respect to the cone $K_i = \left\{ f \in \bigcap_{j=1}^{i-1} \text{Ke } P_j: f^{(i-1)}(x) \geq 0 \text{ on } D \cap \mathbf{R} \right\}$. Therefore there exists a positive leading eigenvalue of N_{i-1} in this cone which furthermore is even simple. This way one can prove simplicity and positivity of any eigenvalue λ of the operator L in (29) in a rather complicated way. But the hope is that

this way one could prove reality of all the eigenvalues of the transfer operators discussed in the examples before and also quite generally the following conjecture.

Conjecture. If all the eigenvalues of the operators $D\psi_i(z_i^*)$ in the operator L in (28) are real and $\varphi(z) > 0$ on $D \cap \mathbf{R}^n$ then the operator L has only real eigenvalues.

Presumably it would be even enough that the function φ is real on the set $D \cap \mathbf{R}^n$.

There seems to be no theory equivalent to the theory of symmetric operators in Hilbert spaces for operators in Banach spaces which could be applied in the above case. What one is looking for is an extension of the work of Krein and Gantmacher on positive matrices of the oscillatory type and generalizations thereof [23] which ensures reality of the spectra of such matrices. Progress in this direction would certainly help us in understanding better, if our examples are very exceptional or belong to a class of operators with such nice spectral properties.

References

- [1] E. Artin, B. Mazur, *On periodic points*, Ann. Math. (2) 81 (1965), 82–99.
- [2] R. Bowen, O. Lanford, *Zeta functions of restrictions of the shift transformation*, Proc. Symp. Pure Math. 14 (1970), 43–49.
- [3] —, *Equilibrium States and Ergodic Theory of Anosov Diffeomorphisms*, Lecture Notes in Math. 470, Springer Verlag, Berlin 1975.
- [4] D. Ruelle, *Thermodynamic Formalism*, Addison Wesley, Reading, Mass. (1978).
- [5] A. Manning, *Axiom-A diffeomorphisms have rational zeta functions*, Bull. Lond. Math. Soc. 3 (1971), 215–220.
- [6] D. Ruelle, *Zeta functions and statistical mechanics*, Soc. Math. France, Asterisque 40 (1976), 167–176.
- [7] —, *Generalized zeta functions for Axiom-A basic sets*, Bull. Amer. Math. Soc. 82 (1976), 153–156.
- [8] —, *Zeta functions for expanding maps and Anosov flows*, Invent. Math. 34 (1976), 231–242.
- [9] K. S. Viswanathan, *Statistical mechanics of a one dimensional lattice gas with exponential polynomial interactions*, Commun. Phys. Math. 47 (1976), 131–141.
- [10] D. Mayer, *On Ruelle's zeta function for exponentially decreasing potentials*, Lett. Math. Phys. 3 (1979), 335–341.
- [11] —, *Ruelle's zeta function for a one sided subshift of finite type with interaction decreasing faster than exponentially*, Proc. of Coll. on Random Fields, Bolyai Math. Soc., Budapest (1979), 747–754.
- [12] —, *On a zeta function related to the continued fraction transformation*, Bull. Soc. Math. France 104 (1976), 195–203.
- [13] F. Hofbauer, G. Keller, *Zeta functions and transfer operators for piecewise linear transformations*, J. Reine Angew. Math. 352 (1984), 100–113.
- [14] G. Gallavotti, *Funzioni zeta ed insiemi basilari*, Acad. Lincei Rend. Sci. Fis. Mat. e Nat. 61 (1976), 309–317.

- [15] D. Mayer, K. S. Viswanathan, *On the zeta function of a one dimensional classical system of hard rods*, Commun. Phys. Math. 52 (1977), 175–189.
 - [16] D. Mayer, *The Ruelle–Araki Transfer Operator in Classical Statistical Mechanics*, Lecture Notes in Phys. 123, Springer Verlag, Berlin 1980.
 - [17] M. Kac, *Mathematical mechanism for phase transitions*, Brandeis Univ. Summer Institute in Theor. Phys. 1966, vol. 1, 245–303. Eds. Chretien, Groos, Deser. Gordon and Breach, New York 1968.
 - [18] G. Baker, *One dimensional order-disorder model which approaches a second order phase transition*, Phys. Rev. 122 (1961), 1477–1484.
 - [19] D. Mayer, G. Roepstorff, *On the relaxation time of Gauss’s continued fraction map*, I, *The Hilbert space approach*, II, *The Banach space approach*, RWTH Aachen Preprints (1986), J. Statist. Phys. 47 (1987), 149–171; 50 (1988), 331–344.
 - [20] M. Pollicott, *Distribution of closed geodesics on the modular surface and quadratic irrationals*, IHES Preprint M 85/24 (1985).
 - [21] D. Mayer, *On composition operators on Banach spaces of holomorphic functions*, J. Funct. Anal. 35 (1980), 191–206.
 - [22] M. Krasnoselskii, *Positive Solutions of Operator Equations*, Ch. 2, P. Noordhoff, Groningen 1964.
 - [23] F. Gantmacher, M. Krein, *Oszillationsmatrizen, Oszillationskerne und kleine Schwingungen mechanischer Systeme*, Ch. 5. Akademie Verlag, Berlin 1960.
-