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SOME NUMERICAL STUDIES OF DYNAMICAL SYSTEMS

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1. Introduction

This text gives some review of numerical studies of three well-known problems in the theory of dynamical systems, namely

1. Hyperbolicity conditions of the Poincaré mapping for the Lorenz system.

2. Unstable one-dimensional manifold of Feigenbaum's fixed point.

3. The construction of KAM-curves for the standard mapping with the help of renormalization group theory.

Each topic is presented in a separate section.

2. Hyperbolic properties of the Lorenz attractor

The famous Lorenz system is the system of three ordinary differential equations (see [1])

$$\frac{dx}{dt} = -\sigma x + \sigma y, \qquad \frac{dy}{dt} = rx - y - xz, \qquad \frac{dz}{dt} = -bz + xy. \tag{1}$$

There exists an open domain in the space of parameters σ, r, b such that for each point of this domain the corresponding flow S^t has a strange attractor. We follow closely the analysis of the Lorenz system presented in the paper by Afraimovich, Bykov, Shilnikov (see [1]). In particular, for $r = 28$, $b = 8/3$ and σ around 5.8, a neighbourhood of the strange attractor can be constructed with the help of the stable manifold of the hyperbolic periodic orbit which appears as a result of the bifurcation occurring at the value of the parameter σ where the unstable one-dimensional manifold of the origin is contained in the two-dimensional stable manifold of the origin. In the paper [2] the following general problem was discussed. Assume that we have found by computer a numerical trajectory (x_i, y_i, z_i), $i = 0, \ldots, n$ such that the distance between (x_0, y_0, z_0) and (x_n, y_n, z_n) is small. The question is, under what conditions the flow $\{S^t\}$ has a periodic orbit passing near (x_0, y_0, z_0). In [2] the corresponding criterium was proposed, which took into account the round-off errors and numerical estimations of the norm of the monodromy matrix. Later it was extended by Hibnik (Pushino) and some other people. The method of [2] can be considered as one of the first computer — assisted proofs in the theory of dynamical systems.

In [3] the results of [2] were used for numerical checking of the so-called hyperbolicity conditions for the strange attractor of (1). These conditions guarantee the stochasticity of the attractor. Thereby we mean the following. Denote by A the attractor and by O a neighbourhood of it such that almost every trajectory starting in O tends to the attractor as $t \to \infty$. Take an initial probability distribution μ_0 concentrated in O and having a density ϱ_0 with respect to the Lebesgue measure.

DEFINITION 1 (see [4]). The attractor A is called *stochastic* if the shift μ_t of μ_0 tends to a limit, $\bar{\mu}$ which does not depend on μ_0. The flow $\{S^t\}$ with the invariant measure $\bar{\mu}$ is mixing.

We shall not give here the precise formulations of the hyperbolicity conditions. A reader can find the definitions in [4], [5]. Remark that these conditions are formulated in terms of properties of Jacobi matrices of the corresponding Poincaré mappings.

The Jacobi matrices were constructed in [2] numerically with some step in x, y coordinates for $\sigma = 6$, $r = 28$, $b = 8/3$. The results show that hyperbolicity really does occur. However it is worthwhile mentioning that in the case considered in [2] the hyperbolicity conditions are valid only in a very narrow and small neighbourhood of the attractor and the expanding coefficient is at the boundary close to 1 exceeding 1 of course (it is equal approximately to 1.05). This fact can be seen also from the first analysis of Lorenz [6].

3. Unstable one-dimensional manifold of the Feigenbaum's fixed point

The doubling equation in Feigenbaum's theory of universality of period-doubling bifurcations takes the form

$$\varphi(x) = -\frac{1}{\alpha}\varphi(\varphi(\alpha x)), \quad x \in [-1, 1] \tag{2}$$

Here φ is an even function satisfying the normalization condition $\varphi(0) = 1$. The existence of the solution of (2) was the subject of many papers we mention only some of them ([7]–[9]).

The equation (2) can be considered as an equation for the fixed point of the non-linear mapping defined by the right-hand part of (2). The whole universality theory of Feigenbaum is based upon some properties of the one-dimensional unstable manifold of the fixed point.

In [10] this manifold was constructed numerically. The main tool was the functional equation for it. The needed unstable manifold is a stable fixed point of this equation. An one-parameter family of one-dimensional mappings obeys Feigenbaum's universality if it is close enough to the one-dimensional manifold in question.

4. Renormalization group approach to the construction of KAM-curves

Consider the famous standard mapping T acting on the two-dimensional cylinder C with the coordinates z, $-\infty < z < \infty$ $\varphi \in 0, 1 \pmod 1$. It has the form $T(z, \varphi) = (z', \varphi')$ where

$$z' = z + \lambda \sin 2\pi\varphi, \qquad \varphi' = \varphi + z' \pmod 1.$$

The KAM-theory yields the existence of invariant curves of the form $z = f(\varphi)$ where f is a smooth periodic function (see [11]). The corresponding rotation number must satisfy some diophantine conditions. One of the appealing problems is the bifurcation of KAM-curves into cantori. The study of this bifurcation was started by J. Greene [12] and continued by R. MacKay in his dissertation with use of the renormalization group theory. This theory is still too difficult for a rigorous treatment.

In [13], the renormalization group theory was applied to the construction of KAM-curves. It turns out that the KAM-curves correspond to the "trivial" fixed point of the renormalization group which is linear and can be written in an explicit form. The stability of this fixed point has been also investigated explicitly. A statement of KAM-theory turns out to be a statement of a convergence of renormalization group transformations to the stable fixed point of the group. The conditions for such convergence are formulated in terms of closeness of the initial family to the fixed point.

Precise formulations of these conditions given in [13] have a rather complicate form. Their advantage is that they can be checked numerically. The corresponding work is under progress. One can hope that using this approach it will be possible to get better estimations from below of values of λ for which the golden KAM-curve exists.

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