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A NEW CONVERGENCE RATE FOR THE QUADRATURE METHOD FOR SOLVING SINGULAR INTEGRAL EQUATIONS

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1. Introduction

Let us consider a singular integral equation of the Cauchy type

$$a(x)u(x)+\frac{b(x)}{\pi}\int_{-1}^{1}\frac{u(t)}{t-x}dt+\frac{1}{\pi}\int_{-1}^{1}h(x,t)u(t)\,dt=f(x), \tag{1}$$

$-1 \leqslant x \leqslant 1$, where $a(x)$ and $b(x)$ are real-valued Hölder continuous coefficients with $a^2(x)+b^2(x) > 0$ for $-1 \leqslant x \leqslant 1$. The functions $h(x, t)$ and $f(x)$ are the given kernel of the regular part of the integral operator defined by the left-hand side of (1) and the given right-hand side of (1), respectively. Define a continuous function

$$g(x)=\frac{1}{2\pi i}\ln\frac{a(x)-ib(x)}{a(x)+ib(x)}$$

and a weight function

$$\sigma(x)=\frac{(1-x)^{\lambda_0}(1+x)^{\mu_0}}{r(x)}\exp\int_{-1}^{1}\frac{g(t)}{t-x}dt,$$

where $r(x)=\left(a^2(x)+b^2(x)\right)^{1/2} > 0$ and λ_0 and μ_0 are integers such that

$$-1 < \alpha := \lambda_0+g(1), \quad \beta := \mu_0-g(-1) < 1.$$

Then the representation

$$\sigma(x)=(1-x)^{\alpha}(1+x)^{\beta}w_0(x) \tag{2}$$

is valid, where $w_0(x)$ is continuous and positive on $[-1, 1]$. Furthermore, we make the following assumptions (cf. [1], [2]):

1° There is a function

$$c(x) = c_0(x) \prod_{j=0}^{N+1} |x - x_j|^{-\alpha_j},$$

$-1 = x_{N+1} < x_N < \ldots < x_1 < x_0 = 1$, $-1 \leqslant x \leqslant 1$, such that

$$B(x) = c(x) b(x), \quad -1 \leqslant x \leqslant 1,$$

is a polynomial.

2° The exponents α_j $(j = 1, \ldots, N)$ are greater than -1, and, furthermore, $\alpha_0 - \alpha$, $\alpha_{N+1} - \beta$, $\alpha_0 + \alpha$, $\alpha_{N+1} + \beta > -1$, and $c_0(x) > 0$ is continuous on $[-1, 1]$.

Now, for $-1 < x < 1$, equation (1) is equivalent to

$$a(x)\sigma(x)v(x) + \frac{B(x)}{\pi} \int_{-1}^{1} \frac{v(t)}{t-x} \sigma_1(t)\,dt + \frac{1}{\pi} \int_{-1}^{1} h_1(x, t) v(t) \sigma_1(t)\,dt = f_1(x), \tag{3}$$

where

$$u(t) = \sigma_1(t) v(t), \quad \sigma_1(t) = \sigma(t)/c(t),$$

$$h_1(x, t) = c(x) h(x, t), \quad f_1(x) = c(x) f(x).$$

If we define a linear bounded operator A in the pair of Hilbert spaces $(L^2_{\sigma_1}, L^2_{\mu_1})$, where

$$\mu_1(x) = \mu(x)/c(x), \quad \mu(x) = 1/\sigma(x) r^2(x),$$

by

$$(Av)(x) = a(x)\sigma(x)v(x) + \frac{B(x)}{\pi} \int_{-1}^{1} \frac{v(t)}{t-x} \sigma_1(t)\,dt, \quad -1 < x < 1,$$

then $A^{(-1)}$ given by

$$(A^{(-1)} g)(t) = a(t)\mu(t)g(t) - \frac{B(t)}{\pi} \int_{-1}^{1} \frac{g(x)}{x-t} \mu_1(x)\,dx, \quad -1 < t < 1,$$

is at least a one-sided inverse of A.

LEMMA 1 ([7], §5, 5°–7°). *Assume that a function $\psi(x, t)$ is Hölder continuous with exponent γ, $0 < \gamma < 1$, uniformly in the variables x and t, i.e.*

$$|\psi(x', t') - \psi(x'', t'')| \leqslant C_1(|x' - x''|^\gamma + |t' - t''|^\gamma)$$

(the C_k denote positive constant numbers), x', $x'' \in [a_1, b_1]$, t', $t'' \in [a_2, b_2]$,

$-\infty < a_j, b_j < \infty$ $(j = 1, 2)$. *Let* $x_0 \in [a_1, b_1]$, $0 < \delta < \gamma$, *and*

$$\varphi(x, t) = \frac{\psi(x, t) - \psi(x_0, t)}{|x - x_0|^\delta}.$$

Then there exists a constant C_2 *such that*

$$|\varphi(x', t') - \varphi(x'', t'')| \leqslant C_2(|x' - x''|^{\gamma - \delta} + |t' - t''|^{\gamma - \delta})$$

for x', x'', t', t'' *as above.*

Corollary 1. *Assume that* $\psi(x, t)$ *possesses the same property as in Lemma* 1 $(a_j = -1, b_j = 1)$, $0 < \delta_k < \gamma$, $\psi(x_k, t) = 0$ $(k = 0, 1, \ldots, N+1;$ $t \in [-1, 1])$, *and*

$$\Phi(x, t) = \frac{\psi(x, t)}{\prod_{k=0}^{N+1} |x - x_k|^{\delta_k}}.$$

Then

$$|\Phi(x', t') - \Phi(x'', t'')| \leqslant C_{31}(|x' - x''|^{\gamma - \delta} + |t' - t''|^{\gamma - \delta}),$$

$x', x'', t', t'' \in [-1, 1]$, *where* $\delta = \max\limits_{0 \leqslant k \leqslant N+1} \delta_k$.

Proof. We apply Lemma 1 to the intervals

$$[-1, \tfrac{1}{2}(x_N + x_{N-1})], [\tfrac{1}{2}(x_N + x_{N-1}), \tfrac{1}{2}(x_{N-1} + x_{N-2})], \ldots, [\tfrac{1}{2}(x_2 + x_1), 1].$$

Lemma 2. *If* $|f(x') - f(x'')| \leqslant C_4 |x' - x''|^\gamma$, $x', x'' \in [a_1, b_1]$, $0 < \gamma < 1$, $\delta > 0$, *and* $x_0 \in [a_1, b_1], f(x_0) = 0$, *then there is a constant* C_5 *such that*

$$|g(x') - g(x'')| \leqslant C_5 |x' - x''|^\gamma, \qquad x', x'' \in [a_1, b_1],$$

where $g(x) = f(x)|x - x_0|^\delta$.

Proof. In the case $\delta \geqslant \gamma$ the validity of the assertion is clear. Now, assume $\delta < \gamma$. Let $x_0 \leqslant x' < x''$. If $x_0 = x'$, then

$$\begin{aligned}|g(x') - g(x'')| = |f(x'')(x'' - x_0)^\delta| &\leqslant C_4(x'' - x_0)^{\gamma + \delta} \\ &\leqslant C_{51}(x'' - x')^\gamma.\end{aligned}$$

In the other case $(x_0 < x' < x'')$ it follows that

$$\begin{aligned}|g(x') - g(x'')| &= |(x' - x_0)^\delta f(x') - (x'' - x_0)^\delta f(x'')| \\ &= ((x'' - x_0)^\delta - (x' - x_0)^\delta)|f(x')| + (x'' - x_0)^\delta |f(x') - f(x'')|.\end{aligned}$$

First, let $x'-x_0 \leqslant x''-x'$. Thus,

$$\begin{aligned}|g(x')-g(x'')| &\leqslant C_{52}|f(x')|+C_{53}|f(x')-f(x'')| \\ &\leqslant C_{52}C_4(x'-x_0)^\gamma+C_{53}C_4(x''-x')^\gamma \\ &\leqslant C_{54}(x''-x')^\gamma.\end{aligned}$$

Secondly, if $x''-x' < x'-x_0$, then there is an $\tilde{x}$, $x' < \tilde{x} < x''$, with

$$\begin{aligned}|g(x')-g(x'')| &\leqslant \delta(\tilde{x}-x_0)^{\delta-1}(x''-x')\cdot C_4(x'-x_0)^\gamma+C_{53}C_4(x''-x')^\gamma \\ &\leqslant (C_4\delta(x'-x_0)^\delta+C_{53}C_4)(x''-x')^\gamma \leqslant C_{55}(x''-x')^\gamma.\end{aligned}$$

In the case $x' < x'' \leqslant x_0$, we proceed analogously, and (4) follows.

Corollary 1 and Lemma 2 imply

COROLLARY 2. *Let $\psi(x, t)$ satisfy the assumptions of Lemma 1. Furthermore, assume $\delta_k < \gamma$, $\psi(x_k, t) = 0$ $(k = 0, 1, \ldots, N+1;\ t \in [-1, 1])$, and*

$$\Phi(x, t) = \frac{\psi(x, t)}{\prod_{k=0}^{N+1} |x-x_k|^{\delta_k}}.$$

Then

$$|\Phi(x', t')-\Phi(x'', t'')| \leqslant C_{32}(|x'-x''|^{\gamma-\delta}+|t'-t''|^{\gamma-\delta}),$$

$x', x'', t', t'' \in [-1, 1]$, *where* $\delta = \max\{0, \delta_0, \ldots, \delta_{N+1}\}$.

Proof. Analogous to the proof of Corollary 1.

LEMMA 3 ([7], §18, 3°). *Let $\psi(x, t)$ fulfil the assumptions of Lemma 1 with $a_1 = a_2 = -1$, $b_1 = b_2 = 1$. Then the function*

$$G(t) = \int_{-1}^{1} \frac{\psi(x, t)}{x-t}\,dx$$

is Hölder continuous with exponent γ on $[-1, 1]$ if $\psi(-1, t) = \psi(1, t) = 0$ $(t \in [-1, 1])$.

The fact that a function $f(t)$, $-1 \leqslant t \leqslant 1$, has a derivative of order m (m a nonnegative integer), Hölder continuous with exponent γ, $0 < \gamma < 1$, will be denoted by $f \in C^{m,\gamma}$.

LEMMA 4 ([2], §3; [5], p. 91). *For each polynomial $p(x)$, $(A^{(-1)}p)(t)$ is again a polynomial.*

2. A regularity property of the solution of a singular integral equation

LEMMA 5. *If* $a, b \in C^{0,\gamma}$, *then* $w_0 \in C^{0,\gamma}$ (cf. (2)).

Proof. Note that $r, g \in C^{0,\gamma}$ and write

$$\int_{-1}^{1} \frac{g(t)}{t-x} dt = \int_{-1}^{1} \frac{g(t)-((1+t)g(1)+(1-t)g(-1))/2}{t-x} dt$$

$$+\frac{1}{2}\int_{-1}^{1}\left\{\frac{(1+x)g(1)+(1-x)g(-1)}{t-x}+g(1)-g(-1)\right\} dt.$$

Denote the first integral on the right-hand side by $\tilde{g}(x)$. Then $\tilde{g} \in C^{0,\gamma}$ in view of Lemma 3. It follows that

$$\sigma(x) = \frac{(1-x)^{\lambda_0}(1+x)^{\mu_0}}{r(x)}\left(\frac{1-x}{1+x}\right)^{(1+x)g(1)/2+(1-x)g(-1)/2}$$
$$\times \exp(\tilde{g}(x)+g(1)-g(-1)),$$

or

$$w_0(x) = (1-x)^{(1-x)(g(-1)-g(1))/2}(1+x)^{(1+x)(g(1)-g(-1))/2}$$
$$\times \frac{\exp(\tilde{g}(x)+g(1)-g(-1))}{r(x)}.$$

Because $e^{s \ln s} \in C^{0,\varepsilon}$ for $s \in [0, 1]$ and all $\varepsilon \in (0, 1)$ the assertion follows immediately.

THEOREM 1. *Let* $a, b, c_0 \in C^{0,\eta}$, *and*

$$\max\{0, \alpha-\alpha_0, \beta-\alpha_{N+1}, -\alpha_1, \ldots, -\alpha_N\} < \gamma < 1.$$

Assume $f \in C^{m,\gamma}$. *Then*

$$A^{(-1)} f \in C^{m,\delta},$$

where $\delta = \min\{\eta, \gamma, \gamma+\alpha_0-\alpha, \gamma+\alpha_{N+1}-\beta, \gamma+\alpha_1, \ldots, \gamma+\alpha_N\}$.

Proof. On taking into account the equality

$$(A^{(-1)} f)(t) = f(t)\left[a(t)\mu(t)-\frac{B(t)}{\pi}\int_{-1}^{1}\frac{\mu_1(x)}{x-t}dx\right]$$
$$-\frac{B(t)}{\pi}\int_{-1}^{1}\frac{f(x)-f(t)}{x-t}\mu_1(x)\,dx$$

and Lemma 4 it remains to prove that

$$BF \in C^{m,\delta}, \tag{4}$$

where

$$F(t) = \frac{1}{\pi} \int_{-1}^{1} \frac{f(x)-f(t)}{x-t} \mu_1(x)\, dx.$$

By induction it can be proved that

$$\frac{d^k}{dt^k}\left[\frac{f(x)-f(t)}{x-t}\right] = \frac{k!\left[f(x) - \sum_{j=0}^{k} \frac{f^{(j)}(t)}{j!}(x-t)^j\right]}{(x-t)^{k+1}},$$

$k = 0, 1, \ldots, m$. Thus,

$$F^{(k)}(t) = \frac{k!}{\pi} \int_{-1}^{1} \frac{f(x) - \sum_{j=0}^{k} \frac{f^{(j)}(t)}{j!}(x-t)^j}{(x-t)^{k+1}} \mu_1(x)\, dx,$$

and (4) is valid if

$$BF^{(m)} \in C^{0,\delta}. \tag{5}$$

Setting

$$\Phi(x, t) = m!\left[f(x) - \sum_{j=0}^{m-1} \frac{f^{(j)}(t)}{j!}(x-t)^j\right] / (x-t)^m$$

we obtain

$$\begin{aligned} \Phi(x, t) &= \frac{m}{(x-t)^m} \int_t^x f^{(m)}(y)(x-y)^{m-1}\, dy \\ &= m \int_0^1 f^{(m)}\big(st + (1-s)x\big) s^{m-1}\, ds. \end{aligned} \tag{6}$$

Using the inequality

$$(r_1 + r_2)^\gamma \leqslant 2^{1-\gamma}(r_1^\gamma + r_2^\gamma), \quad r_1 \geqslant 0, r_2 \geqslant 0$$

([7], §5), we conclude that

$$|\Phi(x', t') - \Phi(x'', t'')| \leqslant C_6 (|x' - x''|^\gamma + |t' - t''|^\gamma),$$

$x', x'', t', t'' \in [-1, 1]$ (cf. [3], p. 138). Let

$$\Psi(x, t) = \Phi(x, t) - f^{(m)}(t).$$

We define

$$P(x, t) = \sum_{j=0}^{N+1} \Psi(x_j, t) \prod_{\substack{k=0 \\ k \neq j}}^{N+1} \frac{x - x_k}{x_j - x_k},$$

which is a polynomial with respect to the variable x. Now, we can write $BF^{(m)}$ as follows:

$$B(t)F^{(m)}(t) = \frac{B(t)}{\pi}\int_{-1}^{1}\frac{(\Psi(x,t)-P(x,t))\mu_1(x)}{x-t}dx \tag{7}$$
$$-\left[a(t)\mu(t)P(t,t)-\frac{B(t)}{\pi}\int_{-1}^{1}\frac{P(x,t)}{x-t}\mu_1(x)dx\right]$$
$$+a(t)\mu(t)P(t,t).$$

In view of Lemma 5 the representation

$$\mu_1(x) = (1-x)^{\alpha_0-\alpha}(1-x)^{\alpha_{N-1}-\beta}\prod_{j=1}^{N}|x-x_j|^{\alpha_j}c_0(x)/(w_0(x)r^2(x))$$

holds with $c_0/(w_0r^2)\in C^{0,\eta}$. According to Corollary 2 and Lemma 3,

$$\int_{-1}^{1}\frac{(\Psi(x,t)-P(x,t))\mu_1(x)}{x-t}dx \in C^{0,\delta}.$$

Furthermore, by Lemma 4 the term in square brackets in (7) also belongs to $C^{0,\delta}$. Finally, from (6) we obtain $\Phi(t,t) = f^{(m)}(t)$, from which we conclude that $P(-1,-1) = \Psi(-1,-1) = 0$, $P(1,1) = \Psi(1,1) = 0$, and (by Corollary 2)

$$a(t)\mu(t)P(t,t)\in C^{0,\delta}.$$

This proves (5).

3. The Gauss quadrature method and rates of convergence

By definition, $\varkappa = -(\lambda_0+\mu_0)$ is the *index* of equation (3). In case $\varkappa > 0$ we require that $v(t)$ fulfils the additional conditions

$$\frac{1}{\pi}\int_{-1}^{1} v(t)t^l\sigma_1(t)dt = 0 \quad (l = 0, 1, \ldots, \varkappa-1). \tag{8}$$

In what follows we prove that (3) (with (8) in case $\varkappa > 0$) is uniquely solvable in $L^2_{\sigma_1}$. Denote by t_k ($k = 1, \ldots, n$) and x_j ($j = 1, \ldots, n-\varkappa$) the zeros of the orthogonal polynomials of degree n and $n-\varkappa$ with respect to the weights $\sigma_1(t)$ and $\mu_1(x)$, respectively. We seek an approximate solution of (3) in the form

$$v_n(t) = \sum_{k=1}^{n} v_n(t_k)\prod_{\substack{i=1\\ i\neq k}}^{n}\frac{t-t_i}{t_k-t_i}$$

by solving the algebraic equations

$$\sum_{k=1}^{n} \lambda_k \left[\frac{B(x_j)}{t_k - x_j} + h(x_j, t_k)\right] v_n(t_k) = f_1(x_j) \quad (j = 1, \ldots, n-\varkappa), \tag{9}$$

where the weights λ_k are given via the Gauss quadrature rule

$$\frac{1}{\pi} \int_{-1}^{1} v(t)\sigma_1(t)\,dt \sim \sum_{k=1}^{n} \lambda_k v(t_k). \tag{10}$$

In case $\varkappa > 0$ we complete (9), for example, by

$$\sum_{k=1}^{n} \lambda_k v_n(t_k) t_k^l = 0 \ (l = 0, 1, \ldots, \varkappa - 1). \tag{11}$$

If $\varkappa < 0$ we modify (9) so that the vector

$$(v_n(t_1), \ldots, v_n(t_n), \xi_1, \ldots, \xi_{-\varkappa})^T$$

is to be determined by solving

$$\sum_{k=1}^{n} \lambda_k \left| \frac{B(x_j)}{t_k - x_j} + h(x_j, t_k) \right| v_n(t_k) + \sum_{l=1}^{-\varkappa} a_{jl}\xi_l = f_1(x_j), \tag{9'}$$

where the columns $(a_{1l}, \ldots, a_{n-\varkappa,l})^T$ are chosen in such a way that

$$\sum_{j=1}^{n-\varkappa} \gamma_j \lambda_k \left| \frac{B(x_j)}{t_k - x_j} + h(x_j, t_k) \right| a_{jl} = 0 \quad (k = 1, \ldots, n;\ l = 1, \ldots, -\varkappa),$$

and

$$\sum_{j=1}^{n-\varkappa} \gamma_j a_{jl} a_{jm} = \delta_{lm} \quad (l, m = 1, \ldots, -\varkappa).\ (^1)$$

THEOREM 2 ([3], Theorem 3.1). *Assume* $c(x) \equiv 1$ (i.e. $b(x)$ *is a polynomial*), $\alpha, \beta < 0$, $\sigma = \sigma_1 \in C^{0,\varepsilon_1}$, *and also* $h(x, \cdot)$, $h(\cdot, t) \in C_t^{m,\gamma}$ *(uniformly with respect to the variables x and t, respectively). Then for sufficiently large n the equations* (9) *(with* (11) *in case* $\varkappa > 0$) *or* (9′) *are uniquely solvable, and*

$$\max_{1 \leq k \leq n} |v(t_k) - v_n(t_k)| \leq C_7 n^{-(m+\varepsilon)} \ln n, \tag{12}$$

where $\varepsilon = \min(\gamma, \varepsilon_1)$.

Using Theorem 1 and the same arguments as in [3] one can prove

THEOREM 3. *Let* f_1 *and* h_1 *fulfil the same conditions as f and h in Theorem 2. Furthermore, assume* $a, b, c_0 \in C^{0,\eta}$ *and* $\alpha_0 - \alpha, \alpha_{N-1} - \beta, \alpha_2, \ldots, \alpha_N \geq 0$. *Then for sufficiently large n the equations* (9) *or* (9) *with* (11) *or* (9′) *are*

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(1) γ_j are the Christoffel numbers with respect to $\mu_1(x)$.

uniquely solvable, and

(13) $$\max_{1 \leqslant k \leqslant n} |v(t_k) - v_n(t_k)| \leqslant C_8\, n^{-(m+\delta)} \ln n,$$

where $\delta = \min(\gamma, \eta)$.

Let us consider the case of constant real coefficients a and b with $a^2 + b^2 = 1$ and $b > 0$. Then, by applying Theorems 2 and 3 it is only possible to investigate the index case $\varkappa = 1$, where

$$\sigma(x) = (1-x)^{-\varepsilon'}(1-x)^{\varepsilon'-1},$$

$a + ib = e^{i\pi\varepsilon'}$, $0 < \varepsilon' < 1$. From Theorem 2 we obtain (12) with $\varepsilon = \min(\gamma, \varepsilon', 1-\varepsilon')$, while Theorem 3 gives (13) with $\delta = \gamma$.

Remark. With the help of Theorem 3 the estimate of the global error

$$\max_{-1 \leqslant t \leqslant 1} |v(t) - v_n(t)|$$

can also be improved using the method of [3], §4.

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