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PARALLELIZABILITY IN BANACH SPACES: PARALLELIZABLE DYNAMICAL SYSTEMS WITH BOUNDED TRAJECTORIES

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1. Introduction and results

Let (X, d) be a metric space and let π be a dynamical system on X. π is called *parallelizable* if there is a global section S_π for π, i.e. a subset $S_\pi \subset X$ such that for every $x \in X$, there is a unique $\tau_\pi(x) \in R$ for which $x\pi\tau_\pi(x) \in S_\pi$ and that the mapping τ_π: $X \to R$, $x \mapsto \tau_\pi(x)$ is continuous.

The problem of parallelizability has a long history. For a survey of this development, see [3], [8], [6].

It is well known [3] that parallelizable dynamical systems cannot have compact invariant sets. Recall that the closure of an invariant set is invariant. Therefore, if a dynamical system in R^n has a bounded trajectory (or, more generally, if a dynamical system in an arbitrary metric space has a trajectory with compact closure) then it is not parallelizable.

The question naturally arises: given an infinite-dimensional Banach space $(E, \|\cdot\|)$, does there exist a parallelizable dynamical system π on E with bounded trajectories, i.e. a parallelizable dynamical system with the property that $\sup\{\|x\pi t - x\| \mid t \in R\}$ is finite for each $x \in X$?

The aim of the present paper is to show that the answer is affirmative:

THEOREM 1. *Let $(E, \|\cdot\|)$ be an infinite-dimensional Banach space. Then there exists a parallelizable dynamical system π on E with bounded trajectories.*

The proof of Theorem 1 is given in Section 2. The proof is based on a simple application of various homeomorphism (continuous onto bijection with continuous inverse) results of infinite-dimensional topology. More precisely, we make use of the following

LEMMA. *Let $(E, \|\cdot\|)$ be an infinite-dimensional Banach space. The origin and the unit sphere $\{e \in E \mid \|e\| = 1\}$ of E are denoted by O_E and S, respect-*

ively. Then

(a) [2, Prop. VI.6.1] $E \simeq S$, *i.e. there exists a homeomorphism* $J: E \to S$.

(b) [2, Cor. III.5.1] $E \simeq E \setminus \{O_E\}$.

Moreover [4, Cor. 1.], *there exists a homeomorphism* $K: E \to E \setminus \{O_E\}$ *such that* $K(e) = e$ *whenever* $\|e\| \geqslant 1$.

(c) [2, Th. VI.6.1] $E \simeq E \times R$, *i.e. there exists a homeomorphism* $L: E \to E \times R$.

In case of $E = L_p(-\infty, \infty)$, $1 \leqslant p < \infty$, constructive examples for parallelizable dynamical systems with bounded trajectories are given in Section 3. A preliminary version of these examples has its own interest: it is formulated here as

THEOREM 2. *For* $p \geqslant 1$ *arbitrarily chosen, let* $F = L_p(-\infty, \infty)$. *Then there exists a dynamical system* ϱ *on* F *satisfying the following conditions*:

(i) ϱ *is linear*;

(ii) ϱ *restricted to* $F \setminus \{O_F\}$ *is parallelizable*;

(iii) *for each* $O_F \neq f \in F$, *the function* $V_f: R \to R$ *defined by* $V_f(t) = \|f\varrho t\|_p$ *is strictly decreasing and*

(iv) *for each* $O_F \neq f \in F$, *there holds*

$$0 < \inf\{V_f(t) \mid t \in R\} < \sup\{V_f(t) \mid t \in R\} < \infty.$$

Theorem 2 points out that even for dynamical systems satisfying (i) and (ii), the existence of a strict Liapunov function need not imply any attraction and/or repulsion property of the origin nor that of the point at infinity.

It is worth to note that Theorem 2 with condition (i) removed, can be generalized to all infinite-dimensional separable Banach spaces:

COROLLARY. *Let* $(E, \|\cdot\|)$ *be an infinite-dimensional separable Banach space. Then there exists a dynamical system* δ *on* E *satisfying conditions* (ii), (iii), (iv).

Proof of the Corollary. By the famous Kadec–Anderson theorem, all infinite-dimensional separable Banach spaces are homeomorphic [2, Cor. VI.9.1]. Consequently, a twofold application of Lemma (a) yields the existence of a homeomorphism

$$H: E \to F = L_p(-\infty, \infty)$$

with the properties that

$$\|e\| = \|H(e)\|_p \quad \text{and} \quad H(\lambda e) = \lambda H(e) \quad \text{for all } e \in E,\ \lambda \geqslant 0.$$

The desired dynamical system can be defined by

$$e\sigma t = H^{-1}(H(e)\varrho t), \quad e \in E,\ t \in R.$$

For a similar application of the Kadec–Anderson theorem in topological dynamics, see [5].

Remark. The problem of existence of parallelizable dynamical systems with uniformly bounded trajectories (i.e. of parallelizable dynamical systems with the property that

$$\sup\{\sup\{\|x\pi t - x\| \mid t \in R\} \mid x \in X\}$$

is finite) seems to be much more complicated.

In c_0^r, the Banach space of real sequences converging to zero, we have constructed [7] a parallelizable dynamical system with uniformly bounded trajectories (and with global section homeomorphic to c_0^r). This particular example gives rise to similar examples in those separable Banach spaces $(F, |\cdot|)$ for which there exists a homeomorphism $H\colon c_0^r \to F$ that satisfies a Lipschitz-condition for large distances, i.e.

$$|H(x) - H(\tilde{x})| \leqslant M\|x - \tilde{x}\|$$

for some constants M, $L > 0$ whenever x, $\tilde{x} \in c_0^r$, $\|x - \tilde{x}\| \geqslant L$. Unfortunately, to the best of our knowledge, it is not much what is known [10] about the existence of such homeomorphisms.

2. The proof of Theorem 1

(A) For $(e, u) \in E \times R$, $t \in R$, define

$$(e, u)\alpha t = (e, u + t).$$

It is clear that α is a parallelizable dynamical system on $E \times R$. In fact, $S_\alpha = \{(e, u) \in E \times R \mid u = 0\}$ is a global section for α and $\tau_\alpha((e, u)) = -u$.

(B) For $e \in E$, $t \in R$, in virtue of Lemma (c), define

$$e\beta t = L^{-1}(L(e)\alpha t).$$

It is clear that β is a parallelizable dynamical system on E. In fact, $S_\beta = L^{-1}(S_\alpha)$ is a global section for β and $\tau_\beta(e) = \tau_\alpha(L(e))$.

(C) Recall that $S = \{e \in E \mid \|e\| = 1\}$. For $s \in S$, $t \in R$, in virtue of Lemma (a), define

$$s\gamma t = J(J^{-1}(s)\beta t).$$

It is clear that γ is a parallelizable dynamical system on S. In fact, $S_\gamma = J(S_\beta)$ is a global section for γ and $\tau_\gamma(s) = \tau_\beta(J^{-1}(s))$.

(D) For $e \in E$, $t \in R$, define

$$e\delta t = \begin{cases} \|e\| \cdot ((e/\|e\|)\gamma t) & \text{if } e \neq O_E, \\ O_E & \text{if } e = O_E. \end{cases}$$

It is easy to see that δ is a dynamical system on E. The continuity of δ at O_E follows from the invariance of the spheres

$$\{\lambda s \in E \mid s \in S\}, \quad \lambda > 0.$$

Further,

$$S_\delta = \{\lambda e \in E \mid e \in S_\gamma, \lambda > 0\}$$

is a global section for δ restricted to $E \setminus \{O_E\}$ and $\tau_\delta(e) = \tau_\gamma(e/\|e\|)$. Thus, δ restricted to $E \setminus \{O_E\}$ is parallelizable.

(E) For $e \in E$, $t \in R$, in virtue of Lemma (b), define

$$e\pi t = K^{-1}(K(e)\delta t).$$

It is easy to see that π is a parallelizable dynamical system on E with bounded trajectories. In fact, $S_\pi = K^{-1}(S_\delta)$ is a global section for π and $\tau_\pi(e) = \tau_\delta(K(e))$. Further, for arbitrary $c > 0$, we have that

$$\|e\pi t\| \leqslant \max\{1, c\} \quad \text{whenever } \|e\| = c, \ t \in R.$$

Concluding the proof of Theorem 1, we remark that $S_\pi \simeq E$. In fact; an easy application of Lemma (c) yields that

$$E \simeq S_\alpha \simeq S_\beta \simeq S_\gamma \simeq S_\delta \simeq S_\pi.$$

3. Examples. The proof of Theorem 2

Throughout this section, for $p \geqslant 1$ arbitrarily chosen, let $F = L_p(-\infty, \infty)$, the Banach space of Lebesgue measurable functions f: $(-\infty, \infty) \to R$ such that

$$\|f\|_p = \Big(\int_{-\infty}^{\infty} |f(x)|^p dx\Big)^{1/p} < \infty.$$

Let $c_1, c_2 \in R$, $0 < c_1 < c_2 < \infty$ and let w: $R \to (c_1, c_2) \subset R$ be a continuous function.

By letting

$$(\varrho(t,f))(x) = f(t+x) \cdot w(t+x)/w(x)$$

for $t \in R$, $f \in F$, $x \in R$, we define a mapping

$$\varrho = \varrho(w): \ R \times F \to F.$$

By definition,

$$\varrho(t, f+g) = \varrho(t,f) + \varrho(t, g), \ \varrho(t, \lambda f) = \lambda \cdot \varrho(t,f), \ \varrho(0,f) = f$$

and

$$\varrho(t, \varrho(s,f)) = \varrho(t+s,f) \quad \text{for all } f, g \in F,\ \lambda \in R,\ t, s \in R.$$

It follows easily from basic integration theory that ϱ is continuous. Thus, ϱ is a linear dynamical system and we are justified in writing $\varrho(t,f) = f\varrho t$. In case of $w(x) = 1$ for all $x \in R$, ϱ is the well-known [9], [11] shift dynamical system on $F = L_p(-\infty, \infty)$

By elementary computations,

$$(1) \qquad c_1 c_2^{-1} \|f-g\|_p \leqslant \|f\varrho t - g\varrho t\|_p \leqslant c_1^{-1} c_2 \|f-g\|_p$$

for all $f, g \in F$, $t \in R$.

Since $O_F \varrho t = O_F$ for all $t \in R$, inequality (1) implies that ϱ has bounded trajectories.

PROPOSITION 1. *ϱ restricted to $F \setminus \{O_F\}$ is parallelizable.*

Proof. In virtue of [8, Cor. 11.], it is sufficient to prove that ϱ restricted to $F \setminus \{O_F\}$ is Poisson-unstable as well as Liapunov-stable.

A dynamical system is called *Poisson-unstable* if no point of the phase space belongs to its alpha– or omega– limit set. The Poisson instability of ϱ restricted to $F \setminus \{O_F\}$ follows from the inequality

$$\underline{\lim}_{t \to \pm\infty} \|f - f\varrho t\|_p \geqslant (1 + c_1^p \cdot c_2^{-p})^{1/p} \cdot \|f\|_p$$

which, in turn, is a consequence of the simple fact that

$$\lim_{N \to \infty} \|f - f_N\|_p = 0$$

where

$$f_N(x) = \begin{cases} f(x) & \text{if } x \in [-N, N], \\ 0 & \text{otherwise} \end{cases}$$

for all $f \in F$, $N = 1, 2, \ldots$

On the other hand, as a consequence of inequality (1), ϱ is Liapunov-stable on F (in the sense that, for every $\varepsilon > 0$, there exists a $\delta > 0$ with $\|f\varrho t - g\varrho t\|_p < \varepsilon$ whenever $\|f-g\|_p < \delta$, $t \in R$).

Thus, by [8, Cor. 11], ϱ restricted to $F \setminus \{O_F\}$ is parallelizable. In particular, the usual shift dynamical system [9], [11] on $F = L_p(-\infty, \infty)$, $p \geqslant 1$, restricted to $F \setminus \{O_F\}$ is parallelizable. (The case $p = \infty$ must be excluded. In fact, the usual shift dynamical system [9], [11] on $F = L_\infty(-\infty, \infty)$ restricted to $F \setminus \{O_F\}$ is not Poisson-unstable and therefore [3], [8], it is not parallelizable.) It is worth to mention that we are not able to give S_ϱ, the global section of ϱ restricted to $F \setminus \{O_F\}$, in an explicite form, although ϱ is constructed explicitely.

From now on, assume, in addition, that w is strictly decreasing. Then $\varrho = \varrho(w)$ has the additional property:

$$\|f\varrho t\|_p < \|f\|_p \quad \text{for all } f \in F\setminus\{O_F\},\ t > 0. \tag{2}$$

PROPOSITION 2. *If w is strictly decreasing, then S_ϱ, the global section of ϱ restricted to $F\setminus\{O_F\}$ can be given explicitely. Namely, one can put*

$$\begin{aligned} S_\varrho &= \{f \in F\setminus\{O_F\} \mid \|f\|_p = 2^{-1}\cdot((\inf w)^{-1} + (\sup w)^{-1})\cdot\|fw\|_p\} \\ &= \{f \in F\setminus\{O_F\} \mid \|f\|_p = 2^{-1}\cdot(\lim_{t\to-\infty}\|f\varrho t\|_p + \lim_{t\to\infty}\|f\varrho t\|_p)\}. \end{aligned}$$

Proof. For $f \in F\setminus\{O_F\}$, $t \in R$, let $N(f, t) = \|f\varrho t\|_p$. For all $f \in F\setminus\{O_F\}$, the monotonicity hypothesis on w implies that N is strictly decreasing in t – this is equivalent to inequality (2) – and, by basic integration theory, it follows that

$$L^-(f) = \lim_{t\to-\infty} N(f, t) = (\inf w)^{-1}\cdot\|fw\|_p,$$

$$L^+(f) = \lim_{t\to\infty} N(f, t) = (\sup w)^{-1}\cdot\|fw\|_p.$$

Consequently, $L^-(f)$ and $L^+(f)$ depend continuously on f.

Since $L^-(f) = L^-(f\varrho t_0)$ and $L^+(f) = L^+(f\varrho t_0)$ for all fixed $t_0 \in R$, it follows that, given $f \in F\setminus\{O_F\}$ arbitrarily, there exists a unique $\tau = \tau_\varrho(f) \in R$ such that $f\varrho\tau_\varrho(f) \in S_\varrho$ or, equivalently, $N(f, \tau_\varrho(f)) = 2^{-1}(L^-(f) + L^+(f))$. Recall that N is strictly decreasing in its second variable. By elementary real analysis, the continuity of L^- and of L^+ implies that the mapping $\tau_\varrho\colon F\setminus\{O_F\} \to R$, $f \mapsto \tau_\varrho(f)$ is continuous.

PROPOSITION 3. *Let $(E, \|\cdot\|)$ be an infinite-dimensional Banach space and let η be a dynamical system on E. Assume that η is Liapunov-stable on E (in the sense that, for every $\varepsilon > 0$, there exists a $\delta > 0$ with $\|e\eta t - \tilde{e}\eta t\| < \varepsilon$ whenever $\|e - \tilde{e}\| < \delta$, $t \in R$) and satisfies conditions* (iii) *and* (iv) *of Theorem* 2. *Then ($E\setminus\{O_E\} \subset E$ is invariant and) η restricted to $E\setminus\{O_E\}$ is parallelizable and*

$$S_\eta = \{f \in E\setminus\{O_E\} \mid \|f\| = 2^{-1}\cdot(\lim_{t\to-\infty}\|f\eta t\| + \lim_{t\to\infty}\|f\eta t\|)\}$$

is a global section for η restricted to $E\setminus\{O_E\}$.

Proof. The proof is similar to the one of Proposition 2. No essential changes are needed. (Applying the method used in [6, Section 3.1.], it is not hard to show that, in general, the requirement "η is Liapunov-stable on E" cannot be dropped.)

Proof of Theorem 2. Summing up the properties of ϱ, especially inequalities (1) and (2), we conclude that our dynamical system ϱ satisfies conditions (i)–(iv). Thus, Theorem 2 is proved. In virtue of Proposition 3, the proof does not require [8, Cor. 11.].

Now we turn back to Theorem 1 and, in case of $E = L_p(-\infty, \infty)$, $p \geqslant 1$, we construct examples for parallelizable dynamical systems with bounded trajectories. Starting from Theorem 2, we apply the homeomorphism method used in Section 2.

In virtue of Lemma (b), there exists a homeomorphism $K\colon E \to E\setminus\{O_E\}$ with the property that $K(e) = e$ whenever $\|e\| \geqslant 1$. It is emphasized that, in case of $E = l_2$, K can be constructed [1, Lemma 8.2] explicitely. Since l_2 is linearly isometric to $L_2(-\infty, \infty)$, the Mazur homeomorphism

$$M_p\colon L_2(-\infty, \infty) \to L_p(-\infty, \infty), \quad f \mapsto |f|^{2/p}\cdot \operatorname{sgn}(f)$$

gives rise to an explicite construction of K in case of $E = L_p(-\infty, \infty)$, $p \geqslant 1$ as well. Finally, given $p \geqslant 1$ arbitrarily, for $f \in E = L_p(-\infty, \infty)$, $t \in R$, define

$$f\pi t = K^{-1}\big(K(f)\varrho t\big).$$

As in Section 2, it is easy to show that π is a parallelizable dynamical system with bounded trajectories.

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