

## A NEW MODEL OF THE ERGODIC TRANSFORMATIONS

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We prove that every ergodic automorphism is isomorphic to an induced (in topological sense) automorphism of a minimal compact dynamical system with some invariant measure. In particular, for the additive machine in  $G = \varprojlim_n \mathbb{Z}/q_n \mathbb{Z}$ ,  $(q_n | q_{n+1})$  and a Markov compact in  $G$  we obtain what we have called an adic transformation, in the previous papers [1,2].

**1. DEFINITION.** Let  $X$  be a metric compact space and  $T$  its homeomorphism. If  $X_0 \subset X$  is a closed subset of  $X$ , we denote by  $T_{X_0}$  the induced (partial) transformation of  $X_0$  defined by the formula

$$T_{X_0} x = T^{h(x)} x$$

where

$$h(x) = \min \{k > 0: T^k x \in X_0\}$$

for  $x \in X_0$  if such a  $k$  exists and  $T_{X_0} x$  undefined otherwise.

We shall consider a closed subset  $X_0$  for which  $T_{X_0}$  is defined on a nonempty set of the second category.

For a measure space  $(X, \mu)$  and its automorphism  $T$ , it follows Poincaré's theorem that for every measurable set  $A \subset X$  of positive measure  $T_A$  is defined almost every. In Russian literature  $T_A$  is called the *derived automorphism*. There exist numerous of papers and theorems concerning this notion in the measurable case, and to my knowledge, none for the topological case.

### 2. A motivation

The previous definition appeared as a result of analysis of the notion of an adic transformation (see [1, 2]) and below) and with attempts to give an algebraic version of it. We shall start from shifts on abelian compact groups

which form by Von Neumann's theorem a class of ergodic systems with discrete spectrum. More exactly, if  $G$  is a compact abelian group, its Haar measure and  $g_0 \in G$  is an element with a dense subgroup of its powers  $\{g_0^n\}_{n \in \mathbf{Z}}$ , then the transformation  $L(g_0) L(g_0)g = g_0g$  is ergodic and has a discrete spectrum. Moreover every such transformation can be obtained in this way. Let us take a subset  $H \subset G$ ; if  $mH > 0$ , then the induced automorphism  $L(g_0)_H$  can be mixing (D. Ornstein) and hence does not even have a discrete component in the spectrum. But by Abramov's formula its entropy is zero in any case. If we omit the positivity assumption on the Haar measure for  $H$ , we can obtain an arbitrary automorphism, as we shall see below, all restrictions like the positivity of entropy or loosely Bernoulliness will be absent. Our main result asserts that the induced automorphisms for shifts on some abelian compact groups are universal models in ergodic theory.

**3. THEOREM I.** *Let  $G = \varprojlim \mathbf{Z}/2^n \mathbf{Z}$  be the additive group of integer 2-adic numbers, and let  $T$  be the shift by unit, i.e.  $Tg = g + 1$ , where 1 the unit of the multiplicative group. Then for every ergodic automorphism  $S$  of a Lebesgue space  $(Y, \mu)$  there exists closed subset  $G_0 \subset G$  and borel probability measure  $m_0$  on  $G_0$  such that  $T_0 \equiv T_{G_0}$  is defined for  $m_0$  - almost all points of  $G_0$  and  $(Y, \mu, S)$  is isomorphic to  $(G_0, m_0, T_0)$ .*

Indeed,  $G_0$  can be chosen to be special compact (namely, multi-Markov) as we shall see from the proof below.

Instead of  $G$  and  $T$  we can take an arbitrary abelian compact group and ergodic shift by its element - in particular the unit circle and an irrational rotation.

#### 4. Sketch of the proof

We will rely on our theorem about an adic representation of arbitrary ergodic automorphism [1, 2]. It states that for every ergodic triple  $(Y, \mu, S)$  we can construct a (nonstationary) Markov compact  $X$  and a measure  $\nu$  on it such that the adic transformation of  $(X, 0)$  is isomorphic to  $(Y, \mu, S)$ . Recall that a Markov compact  $X$  with a sequence of matrices  $\{M^n\}$ ,  $M^n = (m_{ij}^{(n)})$ ,  $i = 1, \dots, r_n > 1$ ,  $j = 1, \dots, r_{n+1}$ ,  $m_{ij}^{(n)} = 0 \vee 1$  is the space of all sequences  $\{x_n\}_{n=1}^{\infty}$ ,  $x_n = 1, \dots, r_n$ ,  $n \geq 1$ , with the property:

if  $x_n = a$ ,  $x_{n+1} = b$  then  $m_{ab}^{(n)} = 1$ , and  $M^{(n)}$  has no zero columns or rows.

If  $M^{(n)} \equiv M$ ,  $r_n \equiv r$ , we have a usual stationary (one sided) Markov compact (topological Markov chain). Notice that

$$X \subset \prod_{n=1}^{\infty} \{1, r_n\} \equiv \chi = \varprojlim \mathbf{Z}/q_n \mathbf{Z} \quad \text{where } q_n = r_1 \dots r_n, n \geq 1.$$

By the definition an adic transformation  $T$  on  $X$  is induced by the rotation on  $X = \varprojlim \mathbb{Z}/q_n \mathbb{Z}: x \rightarrow x+1$ . There exists an invariant measure  $\nu$  for  $T$ , moreover  $X$  can be chosen to be minimal and strictly ergodic for  $T$ .

It was proved in [1] that this definition is correct and  $\{X, \nu, T\}$  as a universal model as claimed above. Another proof had been found by Livšic [3].

Now we may suppose that  $r_n = 2^{l_n}$  for some  $l_n, n = 1, 2, \dots$  (see proof in [2]), then

$$X = \varprojlim \mathbb{Z}/q_n \mathbb{Z} \xrightarrow{h} \varprojlim \mathbb{Z}/2^n \mathbb{Z} \equiv G,$$

where  $h$  is the group isomorphism between  $X$  and  $G$ , and the rotation in  $X$  goes to a rotation in  $G$ .

Thus, we have some compact  $G_0$  in  $G$  defined as the image of  $X$  under  $h$  and a transformation which is induced by this shift. If  $m_0$  is the image of the measure  $\nu$  under  $h$  then we have:

$$(G_0, m_0, T_0) \simeq (X, \nu, T) \simeq (Y, \mu, S).$$

We see that our theorem is an easy corollary of the theorem about adic realizations, but it seems to be important. Notice that  $G_0$  is a compact of a special type (multi-Markov) because it is the image of a Markov compact under an isomorphism.

### 5. A generalization

If  $Y$  is a compact,  $P$  is its homeomorphism and there exists a homeomorphism of the group  $G$  (treated as a compact space) into  $Y$  such that the diagram

$$\begin{array}{ccc} Y & \xrightarrow{P} & Y \\ h \downarrow & & \downarrow h \\ G & \xrightarrow{T} & G \end{array}$$

is commutative then our theorem is valid for  $(Y, P)$  instead of  $(G, T)$ . Of course method of proof of the theorem admits this modification. For example, our theorem holds for a Markov compact  $Y$  and an adic transformation  $P$  of  $Y$  such that  $(Y, P)$  is minimal. By a slight change of the argument we can prove the theorem for the unit circle and irrational rotations instead of  $(G, T)$  as we have claimed.

These facts open a new possibility in ergodic theory, namely, to consider a compact subsets of a minimal dynamical systems (in particular, of the unit circle) and the induced shifts. Classes of subsets correspond to the classes of ergodic transformations. This approach is close to our investigations on the scale and orbit theory (see [4]).

## 6. The stationary case

An adic transformation of a stationary Markov compact is called a stationary adic transformation. Such transformation is defined by a single 0-1 matrix. The class of stationary adic transformations includes substitutions in the sense of Morse and et al. The spectral properties of these automorphisms are being intensely studied (See [3, 5] and papers about Morse systems [6]). Now we can give a definition of a *stationary induced transformation* which generalizes the stationary adic transformations. A subset  $X_0$  in  $G = \varprojlim \mathbb{Z}/p^n \mathbb{Z}$  where  $p$  is a prime number will be called *stationary* if  $pX_0 = X_0$  (multiplication in the ring of  $p$ -adic numbers). The automorphism which is induced by  $T (Tg = g + 1)$  on a stationary subset will be called a *stationary induced automorphism*. If  $X_0$  is stationary Markov set in  $G$  we obtain a stationary adic transformation. It would be interesting to describe this class similar to the standard description of substitutions.

It is easy to generalize this definition to the unit circle or to an arbitrary compact abelian group with a given endomorphism instead of multiplication by  $p$ . (For unit circle we have to take the be endomorphism  $z \rightarrow z^n$ .)

## References

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