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A NOTE ON A TWO-WEIGHTED SOBOLEV INEQUALITY

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In [3], Fabes, Kenig and Serapioni proved the following inequality

$$(1) \qquad \left(\frac{1}{w(B_R)}\int_{B_R}|u(x)|^{kp}w(x)\,dx\right)^{1/kp} \leqslant c\cdot R\left(\frac{1}{w(B_R)}\int_{B_R}|\nabla u(x)|^p w(x)\,dx\right)^{1/p}.$$

Here, the weight w belongs to the *Muckenhoupt-class* A_p $(1<p<\infty)$, $1\leqslant k\leqslant n/(n-1)+\delta$, u is any function in $C_0^\infty(B_R)$ and n is the dimension of the ball B_R. Further,

$$w(B_R)=\int_{B_R}w(x)\,dx.$$

In [2], Chiarenza and Frasca gave a simple proof of the weighted imbedding theorem (1) based on the properties of the Hardy–Littlewood maximal function.

The purpose of this note is to extend (1) to the case of two weights and thus to obtain a two-weighted imbedding theorem of the type

$$(2) \qquad W_0^{1,p}(\Omega;v)\hookrightarrow L^q(\Omega;w), \qquad 1<p<q<\infty,$$

which means that the inequality

$$\Big(\int_\Omega|u(x)|^q w(x)\,dx\Big)^{1/q}\leqslant c\Big(\int_\Omega|\nabla u(x)|^p v(x)\,dx\Big)^{1/p}$$

takes place for every $u\in C_0^\infty(\Omega)$. More precisely, we shall show that the following inequality holds:

$$(3) \qquad \left(\frac{1}{w(Q_R)}\int_{Q_R}|u(x)|^{kp}w(x)\,dx\right)^{1/kp} \leqslant c\cdot R\left(\frac{1}{w(Q_R)}\int_{Q_R}|\nabla u(x)|^p v(x)\,dx\right)^{1/p}$$

for every function $u\in C_0^\infty(Q_R)$, where $k=nr/(nr-p)$, $1<r<p<nr$ and Q_R is a cube $\prod_{i=1}^n(a_i,a_i+R)$. Moreover, it is assumed that the weight functions v, w satisfy

$$(4) \qquad \left(\frac{1}{|Q|}\int_Q w(x)\,dx\right)\left(\frac{1}{|Q|}\int_Q v^{1-r'}(x)\,dx\right)^{r-1}\leqslant c_0<\infty$$

for any cube $Q = \prod_{i=1}^{n}(a_i, a_i+\varrho)$ ($a_i \in \mathbf{R}$, $i = 1, \ldots, n$, $\varrho > 0$ arbitrary, $r' = r/(r-1)$) with c_0 independent of Q. This assumption will be briefly written in the form

$$(w, v) \in A_r.$$

The proof of (3) follows the ideas of [2] and is based on the following lemma:

LEMMA. *Let* $1 < r < p < nr$, $w, v \geqslant 0$, $(w, v) \in A_r$, *let* Q_R *be a cube with edges of length* R, $f \in L^p(Q_R, v)$. *If we denote*

$$If(x) = \int_{\mathbf{R}^N} \frac{|f(y)|}{|x-y|^{n-1}}\, dy,$$

then the inequality

$$\left(\frac{1}{w(Q_R)} \int_{Q_R} |If(x)|^{kp} w(x)\, dx\right)^{1/kp} \leqslant c \cdot R \left(\frac{1}{w(Q_R)} \int_{Q_R} |f(x)|^p v(x)\, dx\right)^{1/p} \tag{5}$$

holds with $k = nr/(nr-p)$ *and a constant* c *independent of* f.

Proof. To prove (5) set, for any $\varepsilon > 0$:

$$\begin{aligned} I^{(\varepsilon)} f(x) &= \int_{\{y: |x-y| \leqslant \varepsilon\} \cap Q_R} \frac{|f(y)|}{|x-y|^{n-1}}\, dy, \\ I_{(\varepsilon)} f(x) &= If(x) - I^{(\varepsilon)} f(x) \\ &= \int_{\{y: |x-y| > \varepsilon\} \cap Q_R} \frac{|f(y)|}{|x-y|^{n-1}}\, dy. \end{aligned}$$

It is easy to prove that

$$I^{(\varepsilon)} f(x) \leqslant c \cdot \varepsilon \cdot Mf(x), \qquad x \in Q_R, \tag{6}$$

where Mf is the Hardy–Littlewood maximal operator (see Hedberg [5]). Further, using the Hölder inequality we have

$$I_{(\varepsilon)} f(x) \leqslant \|f\|_{L^p(Q_R, v)} \Big\{ \int_{\{y: |x-y| > \varepsilon\} \cap Q_R} |x-y|^{(1-n)p'} v^{-1/(p-1)}(y)\, dy \Big\}^{1/p'}.$$

If we again use the Hölder inequality we get the estimate

$$I_{(\varepsilon)} f(x) \leqslant c \|f\|_{L^p(Q_R, v)} \Big(\int_{Q_R} [v(y)]^{1-r'}\, dy \Big)^{(r-1)/p} \varepsilon^{1-nr/p}. \tag{7}$$

From (6) and (7) we obtain

$$If(x) \leqslant c \cdot \varepsilon \cdot Mf(x) + c \|f\|_{L^p(Q_R, v)} \Big(\int_{Q_R} v^{1-r'}(y)\, dy \Big)^{(r-1)/p} \varepsilon^{1-nr/p}. \tag{8}$$

We minimize the right-hand side of (8) with respect to ε to get

$$(9) \qquad If(x) \leqslant c\,[Mf(x)]^{1-p/nr}\,\|f\|^{p/nr}_{L^p(Q_R,v)}\Big(\int_{Q_R} v^{1-r'}(y)\,dy\Big)^{(r-1)/nr}.$$

From the assumption $(w, v) \in A_r$, using the Hölder inequality we easily get that $(w, v) \in A_{\tilde{r}}$ for all $\tilde{r} > r$. By Muckenhoupt [8], Th. 8, and the Marcinkiewicz interpolation theorem (see [1]) we get the inequality

$$(10) \qquad \Big(\int_{Q_R} |Mf(x)|^p w(x)\,dx\Big)^{1/p} \leqslant c\cdot\Big(\int_{Q_R} |f(x)|^p v(x)\,dx\Big)^{1/p}$$

with a constant c independent of f, $r < p < \tilde{r}$. Using this inequality we obtain from (9)

$$\Big(\int_{Q_R} |If(x)|^{kp} w(x)\,dx\Big)^{1/kp} \leqslant c\cdot\Big(\int_{Q_R} v^{1-r'}(y)\,dy\Big)^{(r-1)/nr}\Big(\int_{Q_R} |f(x)|^p v(x)\,dx\Big)^{1/p},$$

where $k = nr/(nr-p)$. To complete the proof we divide both sides by $[w(Q_R)]^{(nr-p)/nrp}$ and use the condition $(w, v) \in A_r$.

Since $|u(x)| \leqslant cI(\nabla u)(x)$ for any sufficiently smooth function u with a compact support, (3) is an immediate consequence of (5).

Remark. Condition (4) provides a rather simple criterion for the pair w, v, under which the imbedding (2) takes place. Other criteria which are more complicated and in general cannot be easily verified have been derived by Lizorkin and Otelbaev [7], by Gurka [4] (the case $n = 1$ only) and by Opic and Kufner [9], [10] (for the case $p = q$ only). For comparison let us present some examples.

If we assume that $w(x) = [\operatorname{dist}(x, \partial\Omega)]^\alpha$, $v(x) = [\operatorname{dist}(x, \partial\Omega)]^\beta$, where $\partial\Omega$ is the boundary of a bounded domain Ω in $\mathbf{R}^n$, $\alpha, \beta \in \mathbf{R}$, then it can be easily shown that

$$(w, v) \in A_r(\Omega) \qquad \Big(\text{i.e. that } \Big(\frac{1}{|Q|}\int_{Q\cap\Omega} w\,dx\Big)\Big(\frac{1}{|Q|}\int_{Q\cap\Omega} v^{1-r'}\,dx\Big)^{r-1} \leqslant c_0\Big)$$

if and only if

$$(11) \qquad \alpha > -1, \quad \beta < r-1, \quad \alpha - \beta \geqslant 0.$$

Consequently, we have the imbedding

$$(12) \qquad W_0^{1,p}(\Omega; d^\beta) \hookrightarrow L^{kp}(\Omega; d^\alpha) \quad (d = \operatorname{dist}(x, \partial\Omega))$$

for α, β from (11), and the Hölder inequality yields the imbedding

$$(13) \qquad W_0^{1,p}(\Omega; d^\beta) \hookrightarrow L^p(\Omega; d^\alpha).$$

The imbedding (13) was derived earlier (see Kufner [6]) for $\alpha - \beta \geqslant -p$, $\beta \neq p-1$, i.e. for a wider class of admissible α, β. On the other hand, using the criteria of Lizorkin and Otelbaev [7] we can show that the imbedding

(12) holds for α, β, $\beta \neq p-1$ if and only if

$$\alpha \geqslant \beta \frac{nr}{nr-p} - p\frac{nr-n}{nr-p}.$$

These examples show that using other criteria we can obtain (at least for special weights of the form given above) a greater number of admissible weight functions. The main advantage of our approach is the simplicity of condition (4).

Remark. When proving (5) we have used the fact that the maximal operator Mf maps a weighted L^p-space into another weighted L^p-space. Using the result by Sawyer [11] who showed that Mf maps a weighted L^p-space into a weighted L^q-space with $q > p$ we can further improve our results, but the criteria for the weights are again very complicated.

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