DYNAMICAL SYSTEMS AND ERGODIC THEORY
BANACH CENTER PUBLICATIONS, VOLUME 23
PWN – POLISH SCIENTIFIC PUBLISHERS
WARSZAWA 1989

ON A SINGULAR PERTURBED DIFFERENTIAL DELAY EQUATION

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Introduction

Consider differential-delay equation

$$\nu\dot{x}(t)+x(t) = f\big(x(t-1)\big), \qquad \nu > 0,\ t \geqslant 0 \tag{1}$$

with continuous real valued function $f(x)$.

Puting formally $\nu = 0$, we obtain the continuous time difference equation

$$x(t) = f\big(x(t-1)\big). \tag{2}$$

For the latter it is natural to consider the one-dimensional map

$$x \to f(x). \tag{3}$$

Let $X = C[-1, 0]$ be the space of continuous functions from $[-1, 0]$ into $\boldsymbol{R}$. For any $\varphi \in X$, there exists the unique solution $x_\varphi^\nu(t)$ of equation (1) defined on $[-1, \infty)$ with $x_\varphi^\nu(t)|_{[-1,0]} = \varphi(t)$. This solution can be easily built up by means of step-by-step integration procedure. Similarly, for any $\psi \in X$ there exists a unique solution $x_\psi(t)$ to equation (2) defined on $[-1, \infty)$ with $x_\psi(t)|_{[-1,0]} = \psi(t)$. In this case the solution is constructed by means of step-by-step iteration procedure given by the law (2). So, the two equations (1) and (2) both generate in a well-known way semiflows on X for $t \geqslant 0$.

The dynamics given by (2) can be studied using the one-dimensional dynamics of (3). This is done to a good and rather complete extent in [1, 2]. Much less is known about equation (1). Some of its properties can be derived from the properties of map (3) and may thus be regarded as inherited. The natural approach to the study of equation (1) is however to consider small values of parameter $\nu > 0$ and to expect some kind of closeness between solutions of equations (1) and (2). In fact the closeness takes place and this allows us to conclude that solutions to equation (1) behave in a similar way with for solutions of equation (2) but only within a finite time interval. In the limit as $t \to \infty$, the asymptotic properties of equation (1) may differ essentially from those ones to equation (2). The main task of this paper is to point out some of these principal differences.

Part 1 of the paper deals with inherited one-dimensional dynamics the equation (1). Closeness results are stated and discussed in Part 2. Principal differences and the main result (Theorem 3) are contained in Part 3.

1. Inherited one-dimensional dynamics

Stated below properties of solutions of equation (1) may be considered as retained from the one-dimensional dynamics (3) under singular perturbation of equation (2).

Invariance property [3, 4]. Suppose that the map $x \to f(x)$ has an invariant interval I ($f(x) \in I$ for any $x \in I$). Then any initial function $\varphi \in X$ satisfying $\varphi(s) \in I$, $s \in [-1, 0]$ generates the solution $x_\varphi^\nu(t)$ of equation (1) satisfying $x_\varphi^\nu(t) \in I$ for all $t \geqslant 0$ and any $\nu > 0$.

Invariance property says that the range domain for a solution to equation (1) is included within the invariant interval for the map (3) provided it is so for an initial time interval.

Stability property [3, 4, 5]. Suppose the map $x \to f(x)$ has an attracting fixed point x_0 with a domain I_0 of immediate attraction. Then any initial function $\varphi \in X$ satisfying $\varphi(s) \in I_0$, $s \in -[1, 0]$ generates solution $x_\varphi^\nu(t)$ of equation (1) satisfying $\lim_{t \to \infty} x_\varphi^\nu(t) = x_0$.

The stability property says that an attracting fixed point of the map (3) corresponds to the stable constant solution for equation (1) with "at least the same" domain of immediate attraction.

Instability property. Suppose the map $x \to f(x)$ has a fixed point x_0 such that $|f'(x_0)| > 1$. Then constant solution $x(t) = x_0$ to equation (1) is Liapunov unstable for all sufficiently small ν.

In the case when $f'(x_0) < -1$ and $f(x)$ satisfies the negative feedback condition $(x - x_0)[f(x) - f(x_0)] < 0$ for $x \neq x_0$, the instability property allows us to prove the existence of a nonconstant periodic solution to equation (1) [6].

2. Closeness results

We define the initial function spaces $X_1 = \{\varphi \in C[-1, 0] \mid |\varphi(s) - \varphi(t)| \leqslant L|s - t|$ for some positive constant $L\}$, $X_2 = L_1[-1, 0]$ with norms $\|\cdot\|_1 = \sup_{[-1,0]} |\cdot|$, $\|\cdot\|_2 = \int_{-1}^{0} |\cdot|\, ds$ respectively. For any fixed $T > 0$, $\varkappa > 0$ we introduce the corresponding distances between solutions of equations (1), (2) on $[0, T]$ as follows:

$$\|y(t)-z(t)\|_1 = \sup\left\{|y(t)-z(t)|,\ t\in[0, T]-\bigcup_{i=0}^{[T]}[i, i+\varkappa]\right\};$$

$$\|y(t)-z(t)\|_2 = \int_0^T |y(s)-z(s)|\,ds.$$

Suppose additionally that $f(x)$ satisfies the Lipschitz condition on I, i.e. $|f(x)-f(y)| \leqslant K|x-y|$ for some positive constant K.

THEOREM 1. *For any fixed positive constants* ε, T, L, $\varkappa$ *there exist positive constants* δ, ν_0 *such that any* $\varphi\in X$, $\psi\in X_1$ *generate solutions* $x_\varphi^\nu(t)$ *and* $x_\psi(t)$ *satisfying* $\|x_\varphi^\nu(t)-x_\psi(t)\|_1 < \varepsilon$ *for all* $0<\nu<\nu_0$ *provided* $\|\varphi-\psi\|_1 < \delta$.

Note that in the case of $\psi\in X_1^0 = \{\psi\in X_1 |\ \psi(0) = f(\psi(-1))\}$ we may put $\varkappa = 0$. This means uniform closeness on $[0, T]$ between the solutions considered.

THEOREM 2. *For any fixed positive constants* ε, T *there exist positive constants* δ, ν_0 *such that any* φ, $\psi\in X_2$ *generate solutions* $x_\varphi^\nu(t)$ *and* $x_\psi(t)$ *satisfying*

$$\|x_\varphi^\nu(t)-x_\psi(t)\|_2 < \varepsilon \textit{ for all } 0<\nu<\nu_0 \textit{ provided } \|\varphi-\psi\|_2 < \delta.$$

As was shown in [1, 2] solutions of three types are generic for the equation (2): asymptotically constant, relaxation, and turbulent ones. The latest type is characterized by increasing (exponentially) oscillation frequency and unbounded Lipschitz constant on any interval $[t-1, t]$ as $t\to\infty$, while the number of oscillations within unit time interval for relaxation solutions is constant. This points at a rather complicated behavior of solutions for continuous time difference equation (2). Details may be found in [1, 2].

Theorems 1 and 2 show that within finite time interval solutions to equation (1) may be as complicated as solutions to equation (2) are there.

3. Principal differences

By the invariance property for the equation (1) we see that every its solution has bounded derivative: $|\nu\dot{x}(t)| \leqslant |x(t)|+|f(x(t-1))| \leqslant \text{const}$. This means that equation (1) has no solutions of relaxation or turbulent type. In view of the stability property we can say that equation (1) may have asymptotically constant solutions. So, under a singular perturbation of equation (2) relaxation and turbulent type solutions disappear while asymptotically constant solutions still do occur.

We may however expect for small ν the presence of solutions of equation (1) which are "close at infinity" to corresponding relaxation or turbulent type solutions of equation (2). In general there may be no such closeness. A simple example constructed in [4] shows the case where asymptotically constant solutions and relaxation ones are generic for equation (2) while any solution of equation (1) with the same f is asymptotically constant. This is due to the fact that the immediate attracting domain for fixed point of the map (3) is essentially larger than the remaining part of the invariant interval. The following theorem shows this situation to be generic.

THEOREM 3 [4]. *Suppose the map $x \to f(x)$ has an invariant interval I ($fI \subseteq I$) and is Lipschitz on it. Let map f has a single fixed point x_0 with a domain I_0 of immediate attraction. Then, if* $\operatorname{mes}(I-I_0)$ *is small enough, all solutions to equation* (1) *satisfy* $\lim_{t\to\infty} x(t) = x_0$ *for arbitrary* $\nu > 0$.

Note that the map (3) may have such a form on $I-I_0$ that solutions of both relaxation or turbulent types will be present for equation (2). Then, due to the closeness, some solutions to equation (1) can be complicated enough within a finite time interval where ν is small. However, if $\operatorname{mes}(I-I_0)$ is small then after some transition regim all these solutions get into domain I_0 and thus become asymptotically constant.

I would like to thank Banach Center Officials for their hospitality, kindness and assistance during all the time of may stay in Warsaw.

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SOME NUMERICAL STUDIES OF DYNAMICAL SYSTEMS

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1. Introduction

This text gives some review of numerical studies of three well-known problems in the theory of dynamical systems, namely

1. Hyperbolicity conditions of the Poincaré mapping for the Lorenz system.

2. Unstable one-dimensional manifold of Feigenbaum's fixed point.

3. The construction of KAM-curves for the standard mapping with the help of renormalization group theory.

Each topic is presented in a separate section.

2. Hyperbolic properties of the Lorenz attractor

The famous Lorenz system is the system of three ordinary differential equations (see [1])

$$\frac{dx}{dt} = -\sigma x+\sigma y,$$
$$\frac{dy}{dt} = rx-y-xz, \tag{1}$$
$$\frac{dz}{dt} = -bz+xy.$$

There exists an open domain in the space of parameters σ, r, b such that for each point of this domain the corresponding flow S^t has a strange attractor. We follow closely the analysis of the Lorenz system presented in the paper by Afraimovich, Bykov, Shilnikov (see [1]). In particular, for $r = 28$, $b = 8/3$ and σ around 5.8, a neighbourhood of the strange attractor can be constructed with the help of the stable manifold of the hyperbolic periodic orbit which appears as a result of the bifurcation occurring at the value of the parameter σ where the unstable one-dimensional manifold of the origin is contained in the two-dimensional stable manifold of the origin. In the paper [2] the following general problem was discussed. Assume that we have found by computer a numerical trajectory (x_i, y_i, z_i), $i = 0, \ldots, n$ such that the distance between (x_0, y_0, z_0) and (x_n, y_n, z_n) is small. The question is, under what conditions the flow $\{S^t\}$ has a periodic orbit passing near (x_0, y_0, z_0). In [2] the corresponding criterium was proposed, which took into account the round-off errors and numerical estimations of the norm of the monodromy matrix. Later it was extended by Hibnik (Pushino) and some other people. The method of [2] can be considered as one of the first computer – assisted proofs in the theory of dynamical systems.

In [3] the results of [2] were used for numerical checking of the so-called hyperbolicity conditions for the strange attractor of (1). These conditions guarantee the stochasticity of the attractor. Thereby we mean the following. Denote by A the attractor and by O a neighbourhood of it such that almost every trajectory starting in O tends to the attractor as $t \to \infty$. Take an initial probability distribution μ_0 concentrated in O and having a density ϱ_0 with respect to the Lebesgue measure.

DEFINITION 1 (see [4]). The attractor A is called *stochastic* if the shift μ_t of μ_0 tends to a limit, $\bar{\mu}$ which does not depend on μ_0. The flow $\{S^t\}$ with the invariant measure $\bar{\mu}$ is mixing.

We shall not give here the precise formulations of the hyperbolicity conditions. A reader can find the definitions in [4], [5]. Remark that these conditions are formulated in terms of properties of Jacobi matrices of the corresponding Poincaré mappings.

The Jacobi matrices were constructed in [2] numerically with some step in x, y coordinates for $\sigma = 6$, $r = 28$, $b = 8/3$. The results show that hyperbolicity really does occur. However it is worthwhile mentioning that in the case considered in [2] the hyperbolicity conditions are valid only in a very narrow and small neighbourhood of the attractor and the expanding coefficient is at the boundary close to 1 exceeding 1 of course (it is equal approximately to 1.05). This fact can be seen also from the first analysis of Lorenz [6].

3. Unstable one-dimensional manifold of the Feigenbaum's fixed point

The doubling equation in Feigenbaum's theory of universality of period-doubling bifurcations takes the form

$$\varphi(x) = -\frac{1}{\alpha}\varphi(\varphi(\alpha x)), \quad x \in [-1, 1] \tag{2}$$

Here φ is an even function satisfying the normalization condition $\varphi(0) = 1$. The existence of the solution of (2) was the subject of many papers we mention only some of them ([7]–[9]).

The equation (2) can be considered as an equation for the fixed point of the non-linear mapping defined by the right-hand part of (2). The whole universality theory of Feigenbaum is based upon some properties of the one-dimensional unstable manifold of the fixed point.

In [10] this manifold was constructed numerically. The main tool was the functional equation for it. The needed unstable manifold is a stable fixed point of this equation. An one-parameter family of one-dimensional mappings obeys Feigenbaum's universality if it is close enough to the one-dimensional manifold in question.

4. Renormalization group approach to the construction of KAM-curves

Consider the famous standard mapping T acting on the two-dimensional cylinder C with the coordinates z, $-\infty < z < \infty$ $\varphi \in 0, 1 \pmod 1$. It has the form $T(z, \varphi) = (z', \varphi')$ where

$$z' = z + \lambda \sin 2\pi\varphi, \qquad \varphi' = \varphi + z' \pmod 1.$$

The KAM-theory yields the existence of invariant curves of the form $z = f(\varphi)$ where f is a smooth periodic function (see [11]). The corresponding rotation number must satisfy some diophantine conditions. One of the appealing problems is the bifurcation of KAM-curves into cantori. The study of this bifurcation was started by J. Greene [12] and continued by R. MacKay in his dissertation with use of the renormalization group theory. This theory is still too difficult for a rigorous treatment.

In [13], the renormalization group theory was applied to the construction of KAM-curves. It turns out that the KAM-curves correspond to the "trivial" fixed point of the renormalization group which is linear and can be written in an explicit form. The stability of this fixed point has been also investigated explicitly. A statement of KAM-theory turns out to be a statement of a convergence of renormalization group transformations to the stable fixed point of the group. The conditions for such convergence are formulated in terms of closeness of the initial family to the fixed point.

Precise formulations of these conditions given in [13] have a rather complicate form. Their advantage is that they can be checked numerically. The corresponding work is under progress. One can hope that using this approach it will be possible to get better estimations from below of values of λ for which the golden KAM-curve exists.

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