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THE DISCOVERY OF SMALE HORSESHOE FOR DYNAMICAL SYSTEM GENERATED BY AUTOGENERATOR WITH TUNNEL DIODE

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The autogenerator with tunnel diode is described in [1] by the following system of ordinary differential equations:

$$\text{(1)} \qquad \dot{x} = y - \delta z, \quad \dot{y} = -x + 2\gamma y + \alpha z, \quad \mu \dot{z} = x - f(z),$$

where f is of type $z^3 - z$.

In [1], values of parameters α, γ, δ are discovered for which the limit dynamical system as $\mu \to +0$ is mixing. For various f and positive μ, numerical results show the strong scattering of trajectories. The chaotic behaviour has also been experimentally investigated; see [2].

In this paper we take for f the function given by formula (2) below and we look for a domain Δ_1 in the parameter space (α, γ, δ) such that the Poincaré mapping of the plane $z = -2/3$ has a smooth Smale horseshoe for all positive μ small enough. This domain will be described in geometrical terms. The existence of a Smale horseshoe implies the quasistochastic behaviour of oscillations. Moreover, the limit theorem proved in [3], [4] is applicable to this case. The results of this paper have been published in my previous paper [5] and [6] (Russian).

Let f be the continuous piecewise linear function

$$\text{(2)} \qquad \text{(A) } f(z) = 2(z-1), \quad \text{(B) } f(z) = -z, \quad \text{(C) } f(z) = 2(z+1),$$

the formulas holding in the domains

$$\text{(A) } z \geqslant 2/3, \quad \text{(B) } -2/3 \leqslant z \leqslant 2/3, \quad \text{(C) } z \leqslant -2/3,$$

respectively. Let $\mathscr{H}_A$ be the part of $\boldsymbol{R}^3$ defined by the inequality (A) for z. Parts $\mathscr{H}_B$ and $\mathscr{H}_C$ of $\boldsymbol{R}^3$ are defined by inequalities (B) and (C), respectively.

The solution of the system in the whole of $\boldsymbol{R}^3$ belongs to class C^{1+1}. In each part $\mathscr{H}_I$, $I = A, B, C$, the solution is an analytic function. The system for $\mathscr{H}_A$ has a fixed point $\vec{p}_A = (x_A, y_A, z_A)$ with coordinates

$$x_A = \frac{2(\alpha+2\gamma\delta)}{2-2\gamma\delta-\alpha}, \quad y_A = \frac{2\delta}{2-2\gamma\delta-\alpha}, \quad z_A = \frac{2}{2-2\gamma\delta-\alpha}.$$

The system for $\mathscr{H}_B$ has a fixed point $\vec{p}_B = (0, 0, 0)$. The point $\vec{p}_B$ belongs to $\mathscr{H}_B$. The system for $\mathscr{H}_C$ is symmetric to the system for $\mathscr{H}_A$, so $\vec{p}_C = -\vec{p}_A$, where $\vec{p}_C$ is a fixed point of the system for $\mathscr{H}_C$.

DEFINITION 1. The domain Δ is defined by the following conditions: 1° The fixed point $\vec{p}_A$ belongs to $\mathscr{H}_A$; 2° The fixed points $\vec{p}_A$, $\vec{p}_B$, $\vec{p}_C$ are unstable focuses. These conditions are equivalent to the inequalities:

$$-2\gamma < \delta < 4\gamma, \quad -1 < \alpha+2\gamma\delta < 2,$$

$$-1+(\gamma-\delta/2)^2 < \alpha < 2-2(\gamma+\delta/4)^2.$$

We assume these inequalities to be fulfilled.

The limit dynamical system as $\mu \to +0$ displays two kinds of motion: fast and slow. Trajectories of fast motion are parallel to the OZ-axis and trajectories of slow motion belong to the surface $x = f(z)$. The plane $x = -z$ in $\mathscr{H}_B$ is repelling, the plane $x = 2(z-1)$ in $\mathscr{H}_A$ is attracting. The slow motion in $\mathscr{H}_A$ is given by the system

$$\text{(3)} \qquad \dot{x} = y-\delta z(x), \quad \dot{y} = -x+2\gamma y+\alpha z(x), \quad z(x) = 1+x/2.$$

It is convenient to inspect it using the orthogonal projection onto the plane $z = 2/3$. The slow motion along the plane $x = 2(z-1)$ terminates on the boundary of $\mathscr{H}_B$, i.e. on the line $x = -2/3$, $z = 2/3$, when $\dot{x}(\tau) < 0$. The last inequeality is equivalent to $y(\tau) < 2\delta/3$. So the break loose ray is $L_A = \{(-2/3, y, 2/3)\colon y < 2\delta/3\}$. The fast motion begins on L_A and finishes on the plane $x = 2(z+1)$ in $\mathscr{H}_C$, where the slow motion begins. The boundary point M of the ray L_A with coordinates $(-2/3, 2\delta/3, 2/3)$ is the point of nonuniqueness: slow motion may continue either in the plane $x = 2(z-1)$ or in the plane $x = -z$. The slow motion in the repelling plane $x = -z$ may finish at any moment to give start to fast motion. The latter may finish attaining either the plane $x = 2(z-1)$ or the plane $x = 2(z+1)$. All motions mentioned above are limits as $\mu \to +0$ where the distance to the limit trajectory as a set is $O(\mu \ln 1/\mu)$ in order. It is convenient to study the limit dynamical system using the Poincaré mapping of the ray L_C into itself. ($\vec{p}$ is mapped into $\vec{g}$ if $\vec{p}$, $\vec{g}$ belong to the same trajectory and this trajectory consecutively intersects L_C in $\vec{p}$ and $\vec{g}$.) In view of central symmetry it is enough to consider the transformation π of the ray L_C into the ray L_A. This function has a gap at the point M. If $p_A \in \mathscr{H}_A$ and $x_A = 2/3$ then the transformation π is expanding for all y such that $\pi(y)$ is defined. This condition is equivalent to $\alpha+2\gamma\delta = 1/2$.

DEFINITION 2. We shall say that $(\alpha, \gamma, \delta) \in \Delta_1$ if $(\alpha, \gamma, \delta) \in \Delta$ and the following four conditions formulated in geometric terms are fulfilled; see [4]. Let us fix on the ray L_C the interval $I_C = \{(2/3, y, -2/3): d_1 < y < d_2\}$ in such a way that (I) $-6d_1 < 4\delta < 3d_1$. Assume that: (II) the projection Γ' of the spiral Γ finishing in M intersects the projection I'_C of I_C at least three times; (III) the turn of its continuation in $\mathscr{H}_A$ intersects the projection of L_C and comes to L_A at the point G with coordinates $p_G = (-2/3, y_*, 2/3)$ with $y_* < -d_2$. The inequality (I) ensures the transversal intersection of trajectory projections and intervals. The condition (III) and the left-hand inequality in (I) give $I_A \subset GM$. So the π-image of I_C covers the interval I_A at least twice. It is also demanded (IV) that the transformation π be strongly expanding on $I_C \cap \pi^{-1} I_A$.

The region Δ_1 is non-empty because, if $\alpha = \gamma = 1/4$, $\delta = 1/2$ and $I_C = \{(2/3, y, -2/3): 0.7 < y < 1.3\}$, all the four conditions are fulfilled. This proposition is verified by numerical calculations.

Let us assume that $(\alpha, \gamma, \delta) \in \Delta_1$ and μ is positive and small enough. We shall estimate the main terms and the order of the remainder terms as $\mu \to +0$. If $(\alpha, \gamma, \delta) \in \Delta$ and $\mu > 0$ is small enough then the fixed point O_A has two complex-conjugate eigenvalues $\lambda_{1,2}(A, \mu) = \varkappa_A(\mu) \pm \omega_A(\mu)$, where $\varkappa_A(\mu)$ and $\omega_A(\mu)$ are positive, and one real eigenvalue $\lambda_3(A, \mu)$, which is negative.

$$\varkappa_A(\mu) = \gamma - \delta/4 + O(\mu) > 0,$$

$$\omega_A(\mu) = (1 - \alpha/2 - (\gamma + \delta/4)^2)^{1/2} + O(\mu) > 0$$

$$\lambda_3(A, \mu) = -2/\mu + \delta/2 + (\delta^2 + 2\alpha)\mu/8 + o(\mu) < 0.$$

The invariant line $\Lambda_A = \{\vec{p} = \vec{p}_A + \zeta \cdot \vec{e}_3(A, \mu)\}$, where $\vec{e}_3(A, \mu)$ is the eigenvector corresponding to $\lambda_3(A, \mu)$ with coordinates

$$\vec{e}_3(A, \mu) = \langle \delta\mu/2 + o(\mu), -\alpha\mu/2 + o(\mu), 1 \rangle,$$

and the invariant plane $\Pi_A = \{\vec{p} = \vec{p}_A + \xi \vec{g}_1(A, \mu) + \eta \cdot \vec{g}_2(A, \mu)\}$, where $\vec{g}_1(A, \mu)$ and $\vec{g}_2(A, \mu)$ are the real and imaginary parts of the complex eigenvector $\vec{e}_1(A, \mu)$, respectively,

$$\vec{g}_1(A, \mu) = \langle 1 + \varkappa_A \cdot \mu/2, \varkappa_A + \delta/2 + (\varkappa_A^2 - \omega_A^2) \cdot \mu/2, 1/2 \rangle,$$

$$\vec{g}_2(A, \mu) = \langle \omega_A \cdot \mu/2, \omega_A + \omega_A \cdot \varkappa_A \cdot \mu, 0 \rangle,$$

pass through the fixed point O_A. The line Λ_A is almost parallel to the axis OZ, the plane Π_A is close to the plane $x = 2(z-1)$. In $\mathscr{H}_A$ the trajectory exponentially approaches the fixed point O_A along the line Λ_A. A trajectory starting in Π_A is a spiral belonging to Π_A with unstable focus O_A until it leaves $\mathscr{H}_A$. The motion in $\mathscr{H}_A$ may be decomposed in the direct sum of fast motion along Λ_A attracting to Π_A and slow motion along Π_A. Let us denote by I the matrix with columns formed by the coordinates of vectors $\vec{g}_1(I, \mu)$,

$\vec{g}_2(I, \mu)$. $\vec{e}_3(I, \mu)$, and introduce

$$G_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad G_2 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad G_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$E = G_1 + G_3$ and for $I = A$

$$N_I(t) = \big(\cos(\omega_I \cdot t)\, G_1 + \sin(\omega_I \cdot t)\, G_2\big) \cdot \exp(\varkappa_I \cdot t) + \exp\big(\lambda_3(I, \mu) \cdot t\big) G_3.$$

Then the general solution of (1–2) in $\mathscr{H}_A$ is

$$\vec{p}(t) = \vec{p}_A + A \cdot N_A(t) \cdot A^{-1} \cdot (\vec{p}(0) - \vec{p}_A).$$

To follow the trajectories in $\mathscr{H}_A$ let us use their projections along the vector $\vec{e}_3(A, \mu)$ onto the plane $z = 2/3$

(4) $$\vec{r}(t, \mu) = (\hat{x}(t, \mu), \hat{y}(t, \mu), 2/3)$$
$$= \vec{p}_A + (2/3 - z_A)\vec{e}_3 + (E - AG_3) \cdot A \cdot N_A(t) \cdot G_1 \cdot A^{-1} \cdot (\vec{r}(0, \mu) - \vec{p}_A).$$

If the two trajectories project onto the same point at the initial moment t_0, then their projections will coincide until one of these trajectories leaves $\mathscr{H}_A$. Such two trajectories lie on the cylindric surface with element parallel to $\vec{e}_3(A, \mu)$ and directrix $\vec{r}(t, \mu)$. If $z_1(t_0, \mu) > z_2(t_0, \mu)$ then the inequality $z_1(t, \mu) > z_2(t, \mu)$ will be valid until one of these trajectories leaves $\mathscr{H}_A$. Let τ' be the moment of the arrival at $\mathscr{H}_A$ and τ'' be the moment of the departure from $\mathscr{H}_A$; then $\tau_1' < \tau_2' < \tau_2'' < \tau_1''$. In limit, as $\mu \to +0$, this projection becomes the orthogonal projection of the limit motion onto the plane $z = 2/3$. The projection $\vec{r}(t, \mu)$ and its derivatives are Lipschitzian, in μ, t and initial position $\vec{r}(0, \mu)$. If $z(0, \mu) \geqslant 2/3$ then $z(t, \mu) = l_0 + l_1 \cdot \hat{x}(t, \mu) + l_2 \hat{y}(t, \mu) + \zeta(0, \mu) \cdot \exp\big(\lambda_3(A, \mu) \cdot t\big)$ for all t: $z(t, \mu) \geqslant 2/3$.

If $(\alpha, \gamma, \delta) \in \Delta$ and $\mu > 0$ is small enough then the fixed point O_B has two complex-conjugate eigenvalues and one real eigenvalue $\lambda_3(B, \mu)$, which is positive:

$$\lambda_3(B, \mu) = 1/\mu - \delta + (\alpha - \delta^2) \cdot \mu + o(\mu) > 0.$$

The invariant line $\Lambda_B = \{\vec{p} = \zeta \vec{e}_3(B, \mu)\}$, where $\vec{e}_3(B, \mu)$ is the eigenvector corresponding to $\lambda_3(B, \mu)$ with coordinates

$$\vec{e}_3(B, \mu) = \langle -\delta\mu + o(\mu), \alpha\mu + o(\mu), 1 \rangle,$$

and the invariant plane $\Pi_B = \{\vec{p} = \xi \vec{g}_1(B, \mu) + \eta g_2(B, \mu)\}$, where $\vec{g}_1(B, \mu)$ and $\vec{g}_2(B, \mu)$ are the real and imaginary parts of the complex eigenvector, respectively,

$$\vec{g}_1(B, \mu) = \langle 1 - \varkappa_B \cdot \mu, \varkappa_B - \delta - (\varkappa_B^2 - \omega_B^2) \cdot \mu, -1 \rangle,$$
$$\vec{g}_2(B, \mu) = \langle -\omega_B \cdot \mu, \omega_B \cdot (1 - 2\varkappa_B \cdot \mu), 0 \rangle,$$

pass through the fixed point $O_B = (0, 0, 0)$. The line Λ_B is almost parallel to the axis OZ, the plane Π_B is close to the plane $x = -z$. The general solution of (1–2) in $\mathscr{H}_B$ is

$$\vec{p}(t) = B \cdot N_B(t) \cdot B^{-1} \cdot \vec{p}(0).$$

The typical trajectory departs from $\mathscr{H}_B$ very quickly repelling the invariant plane Π_B.

The frontier between $\mathscr{H}_A$ and $\mathscr{H}_B$ is the plane $z = 2/3$. Let us describe the transition. (I) A trajectory leaves $\mathscr{H}_B$ through the half-plane $\{x > -2/3, z = 2/3\}$. A trajectory leaves $\mathscr{H}_A$ through the half-plane $\{x < -2/3, z = 2/3\}$. In such a frontier point $\ddot{z}(t, \mu)$ has a discontinuity of the first type. (II) If a trajectory intersects the line $\{x = -2/3, z = 2/3\}$ then it is tangent to the plane $\{z = 2/3\}$ in this point. Such a trajectory stays in $\mathscr{H}_B$ if $y < 2\delta/3$ and it stays in $\mathscr{H}_A$ if $y > 2\delta/3$. (III) At the point M the trajectory leaves $\mathscr{H}_B$ and arrives in $\mathscr{H}_A$.

Let us say that $\vec{p} \in \mathscr{H}_A$ is situated below the plane Π_A, if

$$\zeta(\vec{p}) = a_{31}^{(-1)} \cdot (x - x_A) + a_{32}^{(-1)} \cdot (y - y_A) + a_{33}^{(-1)} \cdot (z - z_A) < 0$$

where $a_{3i}^{(-1)}$ is an element of the matrix A^{-1}.

A trajectory which has reached $\mathscr{H}_A$ below the invariant plane Π_A cannot intersect it inside $\mathscr{H}_A$. Such a trajectory leaves $\mathscr{H}_A$ trough the angle

$$\Phi_A = \{(x, y, 2/3)\colon (y - 2\delta/3) \cdot \mu - (\varkappa_A^2 + \omega_A^2) \cdot (z_A - 2/3) \cdot \mu^2 < 2(1 + \varkappa_A \cdot \mu) \cdot (x + 2/3) < 0\}.$$

This angle Φ_A is the intersection of the plane $z = 2/3$ and the two half-spaces: $\{\vec{p}\colon \zeta(\vec{p}) < 0\}$, $\{\vec{p}\colon x(\vec{p}) < -2/3\}$. Let us denote by $M_A(\mu)$ the vertex of Φ_A. Then

$$M_A(\mu) = (-2/3,\ 2\delta/3 + (\varkappa_A^2 + \omega_A^2)(z_A - 2/3) \cdot \mu,\ 2/3).$$

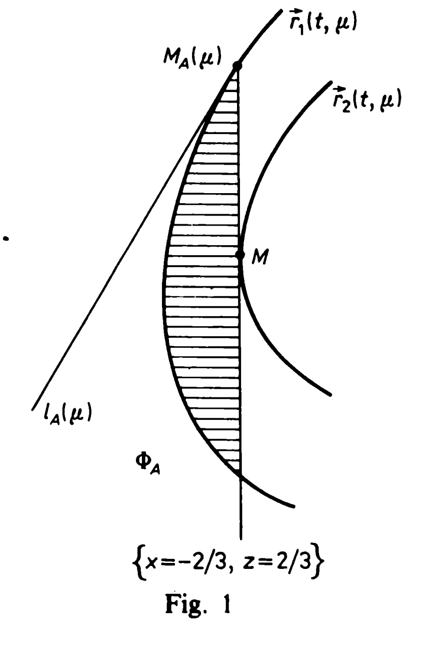

Fig. 1

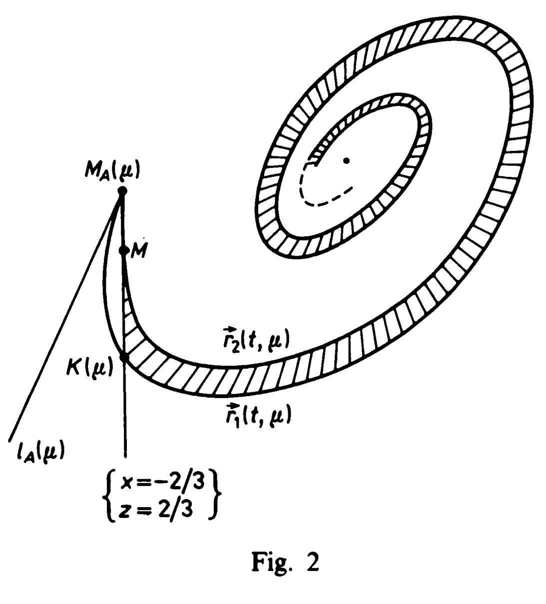

Fig. 2

Let us denote by $l_A(\mu)$ the line of intersection of the plane $z = 2/3$ and the invariant plane Π_A. Denote by $\vec{r}_1(t, \mu)$ the projection (4) which is tangent to the line $l_A(\mu)$ at the point $M_A(\mu)$ and by $\vec{r}_2(t, \mu)$ the projection (4) which is tangent to the line $\{x = -2/3, z = 2/3\}$ at the point M.

Let $D(\mu)$ be the set on the plane $z = 2/3$ lying between $\vec{r}_1(t, \mu)$ and the line $\{x = -2/3, z = 2/3\}$ (the shaded area in Figure 1). Let us parametrize $\vec{r}_1(t, \mu)$ and $\vec{r}_2(t, \mu)$ in such a way that $\vec{r}_1(0, \mu) = K(\mu)$ and $\vec{r}_2(0, \mu) = M$. Let us parametrize every projection $\vec{r}(t, \mu)$ intersecting the interval $MK(\mu)$ in such a way that $\vec{r}(0, \mu)$ is a point of this intersection. Consider the projections $\vec{r}(t, \mu)$ for $t < 0$. The ambiguous corridor $\mathfrak{M}(\mu)$ is thus introduced (the shaded area in Figure 2).

If a trajectory enters $\mathscr{H}_A$ through $\mathfrak{M}(\mu)$ below the invariant plane Π_A then it is impossible to predict in terms of its projection where this trajectory will leave $\mathscr{H}_A$. If a projection $\vec{r}(t, \mu)$ intersects the angle Φ_A out of $D(\mu)$ at first time then all trajectories lying below the plane Π_A and projecting into $\vec{r}(t, \mu)$ leave $\mathscr{H}_A$ before $\vec{r}(t, \mu)$ intersects $l_A(\mu)$.

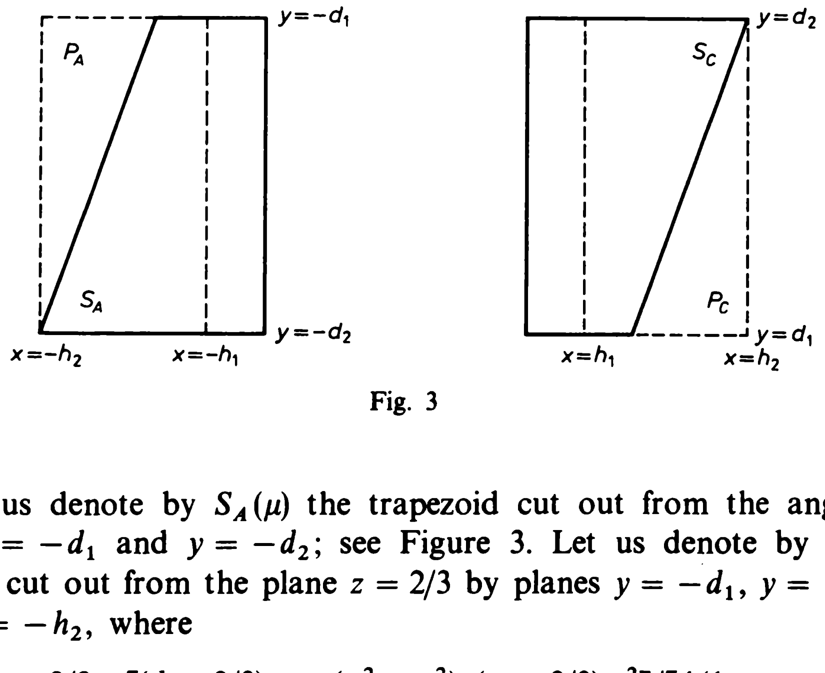

Fig. 3

Let us denote by $S_A(\mu)$ the trapezoid cut out from the angle Φ_A by planes $y = -d_1$ and $y = -d_2$; see Figure 3. Let us denote by $P_A(\mu)$ the rectangle cut out from the plane $z = 2/3$ by planes $y = -d_1$, $y = -d_2$, $x = -h_1$, $x = -h_2$, where

$$h_1 = 2/3 + [(d_1 + 2/3)\cdot\mu + (\varkappa_A^2 + \omega_A^2)\cdot(z_A - 2/3)\mu^2]/[4(1 + \varkappa_A\cdot\mu)],$$

$$h_2 = 2/3 + [(d_2 + 2/3)\cdot\mu + (\varkappa_A^2 + \omega_A^2)\cdot(z_A - 2/3)\mu^2]/[2(1 + \varkappa_A\cdot\mu)].$$

Let us denote by $S_C(\mu)$ the trapezoid centrally symmetric to $S_A(\mu)$ and by $P_C(\mu)$ the rectangle centrally symmetric to $P_A(\mu)$.

Trajectories of the system (1–2) generate a mapping of one plane into another plane. It is convenient to choose planes $z = -2/3$ and $z = 2/3$. Let us introduce the Poincaré mapping σ_{CC} of the plane $z = -2/3$. We look for two consecutive moments t_1 and t_2 when a trajectory $\vec{p}(t, \mu)$ leaves $\mathscr{H}_C$ to generate σ_{CC}: $\vec{p}(t_1, \mu) \to \vec{p}(t_2, \mu)$. By consecutive moments we mean t_1, t_2 such that $t_1 < t_2$ and there is no moment t_3 between t_1 and t_2 at which this trajectory $\vec{p}(t, \mu)$ leaves $\mathscr{H}_C$. This mapping is defined not for all points. Let us introduce the mapping σ_{AC} of the plane $z = -2/3$ into the plane $z = 2/3$. σ_{AC} is generated by two consecutive moments of time, t_1 when a trajectory leaves $\mathscr{H}_C$ and t_2 when this trajectory leaves $\mathscr{H}_A$. σ_{CA} is introduced analogously with the roles of A and C reversed. There is a domain such that $\sigma_{CC} = \sigma_{CA} \cdot \sigma_{AC}$. Moreover, $\sigma_{CC}(x, y, -2/3) = -\sigma_{AC}(-\sigma_{AC}(x, y, -2/3))$, by central symmetry.

Let us study the trajectories beginning in $P_C(\mu)$, crossing $\mathscr{H}_B$, entering $\mathscr{H}_A$ and leaving $\mathscr{H}_A$ through $P_A(\mu)$ for the first time. Such trajectories generate the mapping σ_{AC} of a subset of $P_C(\mu)$ into the corresponding subset of $P_A(\mu)$.

Let us consider the transition through $\mathscr{H}_B$. For sufficiently small positive μ the plane Π_B does not intersect $P_C(\mu)$ and the distance between them is about μ in order. Thus for sufficiently small positive μ the behaviour of trajectories beginning in $P_C(\mu)$ is almost the limit one. If $p(0, \mu) = (u, v, -2/3)$ belongs to $P_C(\mu)$ then there exists a value $T(u, v)$ such that

$$z(T(u, v), \mu) = 2/3 \quad \text{and} \quad \mu \ln 1/\mu - \mu C_1 < T(u, v) < \mu \ln 1/\mu + \mu C_2$$

Moreover,

$$x(t, \mu) = u + O(\mu \ln 1/\mu), \; y(t, \mu) = v + O(\mu \ln 1/\mu)$$

and $z(t, \mu)$ monotonically increases in t for all $0 < t \leqslant T(u, v)$. Let us write

$$U = x(T(u, v), \mu), \qquad V = y(T(u, v), \mu).$$

Thus we have a mapping σ_{BC}: $(u, v) \to (U, V)$ defined in $P_C(\mu)$.

THEOREM. *σ_{BC} is a diffeomorphism of the rectangle $P_C(\mu)$ into the curvilinear rectangle on the plane $z = 2/3$ lying below the invariant plane Π_A and we have*

$$\frac{\partial U}{\partial u} = (u - 2/3)/[u - 2/3 + \mu(v + 2\delta/3)] + O(\mu \ln 1/\mu),$$

$$\frac{\partial V}{\partial u} = \mu(-2\gamma v + 2\alpha/3 + 2/3)/[u - 2/3 + \mu(v + 2\delta/3)] + O(\mu \ln 1/\mu)$$

$$\frac{\partial U}{\partial v} = O(\mu \ln 1/\mu), \qquad \frac{\partial V}{\partial v} = 1 + O(\mu \ln 1/\mu),$$

$$Jac\,\sigma_{BC} = (u - 2/3)/[u - 2/3 + \mu(v + 2\delta/3)] + O(\mu \ln 1/\mu) > 0.$$

The partial derivatives $\dfrac{\partial U}{\partial u}$ *and* $\dfrac{\partial V}{\partial u}$ *are uniformly bounded. The partial derivatives of the second and higher order exist and are bounded in* $P_C(\mu)$, *but their values increase as* μ *decreases.*

We assume till the end of the paper that (α, γ, δ) belongs to Δ_1. Let us consider the trajectories entering $\mathscr{H}_A$ below the plane Π_A and leaving $\mathscr{H}_A$ through the the trapezoid $S_A(\mu)$. Their projections $\vec{r}(t, \mu)$ onto the plane $z = 2/3$ parallel to $\vec{e}_3(A, \mu)$ cover some spiral domain $\mathfrak{R}(\mu)$. Let us call $\mathfrak{R}(\mu)$ the strip region of departure; see Figure 4.

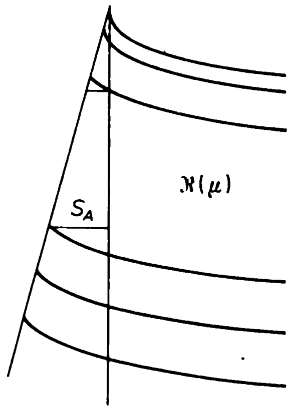

Fig. 4

Let us study the limit strip region $\mathfrak{R}(0)$ of departure covered by projections $\vec{r}(t, 0)$ of trajectories of the system (3) ending in points of the segment I_A.

Let us call $\vec{r}(t, 0)$ ending in the point $(-2/3, Y, 2/3)$ the accompanying limit spiral for the trajectory $\vec{p}(t, \mu)$ leaving $\mathscr{H}_A$ through $S_A(\mu)$ at the point $(X, Y, 2/3)$. As follows from the conditions (I) and (III), the projection Γ' of the spiral Γ does not intersect the segment I_A. The limit strip region of departure $\mathfrak{R}(0)$ lies between the turns of Γ' and thus intersects the segment I'_C twice at any rate. Let us parametrize $\vec{r}(t, 0)$ belonging to $\mathfrak{R}(0)$ in such a way that $\vec{r}(0, 0)$ belongs to I_A. If t_1 and t_2 are two consecutive moments of intersection of $\vec{r}(t, 0)$ and I'_C then there is a natural k such that:

$$-2(k-1)\cdot\pi > t_1\cdot\omega_A(0) > -2k\pi, \qquad -2k\pi > t_2\cdot\omega_A(0) > -2(k+1)\cdot\pi.$$

This number k is the same for $\vec{r}(t, 0)$ belonging to $\mathfrak{R}(0)$. Let us denote these turns by $\mathfrak{R}_{-k}(0)$ and $\mathfrak{R}_{-k-1}(0)$.

Lemma. *For all positive* μ *small enough the strip region of departure* $\mathfrak{R}(\mu)$ *behaves as the limit strip region of departure* $\mathfrak{R}(0)$, *where* $\mathfrak{M}(0) = \Gamma'$.

(a) *The strip region* $\mathfrak{R}(\mu)$ *lies between turns of the region* $\mathfrak{M}(\mu)$.

(b) *The ambiguous corridor* $\mathfrak{M}(\mu)$ *intersects* $\sigma_{BC} P_C(\mu)$ *three times at any rate.*

(c) *The strip region of the departure* $\mathfrak{R}(\mu)$ *intersects* $\sigma_{BC} P_C(\mu)$ *twice by* $\mathfrak{R}_{-k}(\mu)$ *and* $\mathfrak{R}_{-k-1}(\mu)$ *at any rate.*

Each trajectory $\vec{p}(t, \mu)$ beginning at a point $(U, V, 2/3)$ either in $\mathfrak{R}_{-k}(\mu) \cap \sigma_{BC} P_C(\mu)$ or in $\mathfrak{R}_{-k-1}(\mu) \cap \sigma_{BC} P_C(\mu)$ leaves $\mathscr{H}_A$ through $\mathfrak{R}(\mu) \cap \Phi_A(\mu) \supset S_A(\mu)$ at a point $(X, Y, 2/3)$ for the first time at $t = T(U, V)$. These trajectories define a mapping σ_A of the plane $z = 2/3$ into itself,

$$\sigma_A\colon (U, V) \to (X, Y).$$

LEMMA. *Let a trajectory* $\vec{p}(t, \mu)$ *begin in* $\mathfrak{R}_{-k}(\mu) \cap \sigma_{BC} P_C(\mu)$. *Then it leaves* $\mathscr{H}_A$ *at a moment* T *such that*

$$2k \cdot \pi < T \cdot \omega_A(\mu) < 2(k+1) \cdot \pi.$$

If $T_1(U, V)$ *is the time moment when the analytic continuation of its projection* $\vec{r}(t, \mu)$ *intersects the line* $l_A(\mu)$ *then*

$$|T_1(U, V) - T(U, V)| = O\big(\exp[-4k\pi/(\mu\omega_A(\mu))]\big).$$

The accompanying limit spiral begins in the point $\big(2/3, V^*(U, V), 2/3\big)$ *and makes the same natural number* k *of rounds about* O'_A *before attaining the point* $(-2/3, Y, 2/3)$. *If* $T_2(U, V)$ *is the moment of attaining this point then*

$$|T_2(U, V) - T(U, V)| = O(\mu \ln 1/\mu).$$

For $\mu > 0$ *small enough, trajectories beginning in* $\mathfrak{R}_{-k}(\mu) \cap \sigma_{BC} P_C(\mu)$ *leave* $\mathscr{H}_A$ *for the first time in the narrow strip region along* $l_A(\mu)$ *belonging to the neighbourhood of* $S_A(\mu) \cap P_A(\mu)$. *The width of this strip region is*

$$O\big(\exp[-4k \cdot \pi/(\mu \cdot \omega_A(\mu))]\big).$$

THEOREM. *For* $\mu > 0$ *smal enough the following inequalities are fulfilled uniformly in the domain* $\mathfrak{R}_{-k}(\mu) \cap \sigma_{BC} P_C(\mu)$:

$$\frac{\partial X}{\partial U} = O(\mu), \quad \frac{\partial X}{\partial V} = O(\mu), \quad \frac{\partial Y}{\partial U} = O(1),$$

$$\frac{\partial Y}{\partial V} = \frac{d\pi}{dy}(V^*(U, V)) + O(\mu \ln 1/\mu),$$

$$Jac\, \sigma_A = -4 \cdot \lambda_3^4 \cdot \exp(\lambda_3^4 + 2\varkappa_A) \times (1 + O(\mu \ln 1/\mu))/(3Y - 2\delta) < 0.$$

The mapping $\pi\colon L_C \to L_A$ *is generated by the limit trajectories. The point* $\big(2/3, V^*(U, V), 2/3\big)$ *is the origin of the accompanying limit spiral for the trajectory* $\vec{p}(t, \mu)$.

MAIN THEOREM. *If $(\alpha, \gamma, \delta) \in \Delta_1$ and μ is positive and small enough, then the Poincaré mapping σ_{CC} of the plane $z = -2/3$ into itself produces the generalized Smale horseshoe. The corresponding dynamical system is quasi-stochastic, according to Alekseev [7].*

To prove the main theorem let us note that the rectangle $P_C(\mu)$ contains two curvilinear rectangles $Q^{(k)}$ and $Q^{(k+1)}$. The y-sides of these two sets lie on the y-sides of $P_C(\mu)$ and the σ_{AC}-images of their x-sides lie on the x-sides of $P_A(\mu)$. $\sigma_{BC}\, Q^{(i)} \subset \mathfrak{R}_{-i}(\mu) \cap \sigma_{BC}\, P_C(\mu)$ for $i = k,\ k+1$.

See Figures 5, 6, 7.

The elements of the Jacobian matrix of the mapping σ_{AC}: $(u, v) \to (X, Y)$ for each $Q^{(i)}$ are

$$\frac{\partial X}{\partial u} = O(\mu),. \quad \frac{\partial X}{\partial v} = O(\mu), \quad \frac{\partial Y}{\partial u} = O(1),$$

$$\frac{\partial Y}{\partial v} = \frac{d\pi}{dy}(V^*) + O(\mu \ln 1/\mu), \qquad Jac\, \sigma_{AC} < 0.$$

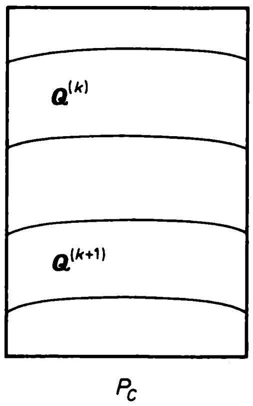

Fig. 5

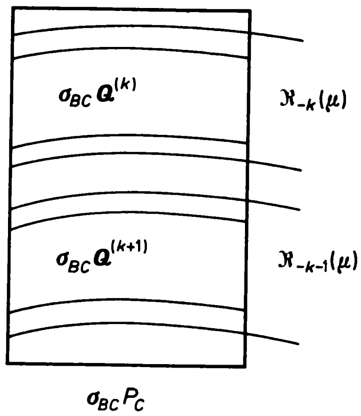

Fig. 6

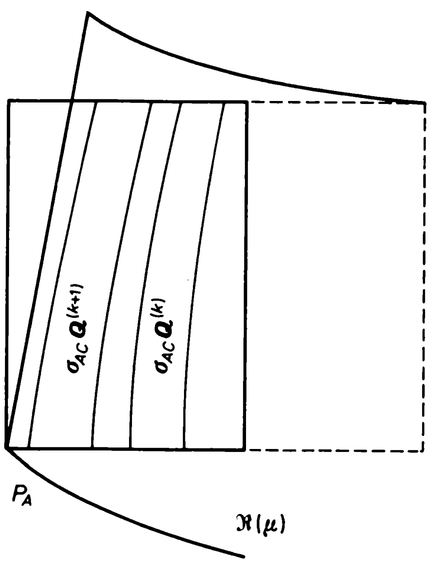

Fig. 7

For $i = k,\ k+1$ the mapping σ_{AC} for each region $Q^{(i)}$ is a diffeomorphism and satisfies there the sufficient conditions for uniform hyperbolicity. It follows from paper [4] that the mapping $\sigma = -\sigma_{AC}$ gives rise to the generalized Smale horseshoe with $S^{(-1)}(1) = Q^{(k)}$ and $S^{(-1)}(2) = Q^{(k+1)}$; see Figure 8.

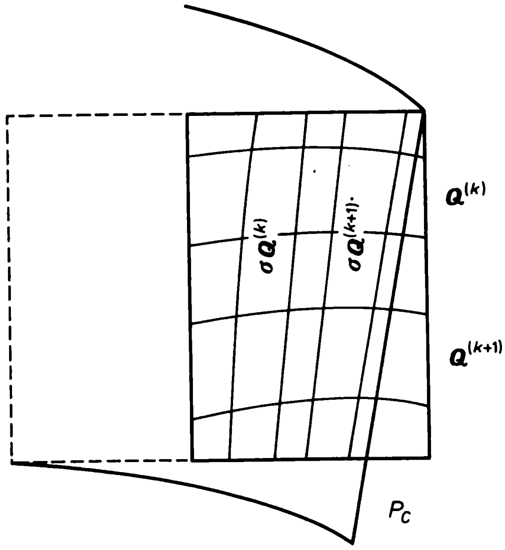

Fig. 8

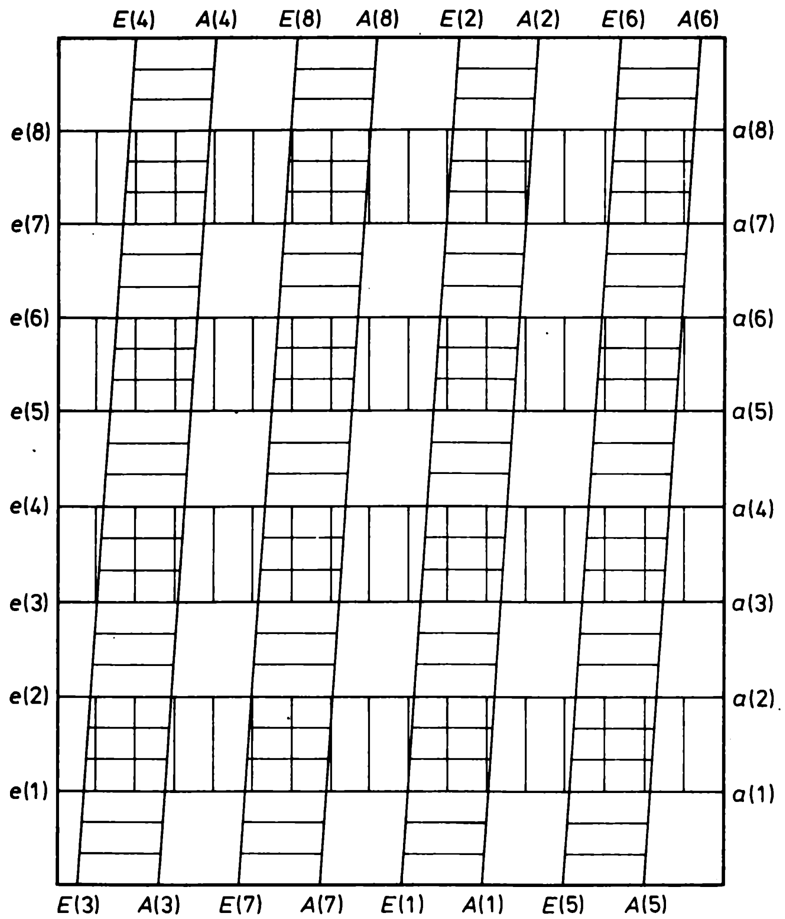

Fig. 9. $\sigma_{CC}(a(i)) = A(i)$, $\sigma_{cc}(e(i)) = E(i)$ for $i = 1.8$

The mapping σ_{CC} is equal to σ^2 on each $Q^{(i)} \cap \sigma Q^{(j)}$ for $i, j = k, k+1$. Thus σ_{CC} makes generalized Smale horseshoe with four components; see Figure 9.

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