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APPROXIMATE METHODS FOR LINEAR INTEGRAL EQUATIONS OF THE FIRST KIND

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1. Introduction

In this paper we investigate approximate methods for the linear integral equation of the first kind

$$\int_a^b k(x, s)\, y(s)\, ds = g(x), \qquad x \in [a, b]. \tag{1.1}$$

The kernel function $k(x, s)$ is a quadratic summable function. The equation (1.1) is considered in the Hilbert space $H = L_2[a, b]$ of quadratic summable functions in the interval $[a, b]$. The kernel function $k(x, s)$ produces a compact integral operator

$$Ky = \int_a^b k(x, s)\, y(s)\, ds$$

in H. Then the integral equation will have the following form

$$Ky = g. \tag{1.2}$$

The solutions can be determined from the transformed equation

$$K^* Ky = K^* g. \tag{1.3}$$

This is an integral equation with the positive compact operator $A = K^* K$. In this paper we will study approximate methods for the equation

$$Ay = f, \tag{1.4}$$

where A is a positive compact operator.

By the choice $A = K^* K$ and $f = K^* g$ we obtain approximate methods for the solution of equation (1.2). The general approximate method for the solution y of (1.4) has the following form

$$y = \lim_{\alpha \to 0} p(\alpha, A) f. \tag{1.5}$$

The general iteration method for the calculation of y is the iterative process

$$Q(\tau A)y_n = P(\tau A)y_{n-1} + S(\tau A)\tau f \tag{1.6}$$

with a starting element y_0. We study these general methods and their relations between them. The Padé-approximation for the exponential function can be used to establish such general iteration methods. The classical approximate methods:

1. Regularization method of Tikhonov

$$y = \lim_{\alpha \to 0} (\alpha I + A)^{-1} f. \tag{1.7}$$

2. Landweber iteration

$$y_n = (I - \tau A)y_{n-1} + \tau f, \quad y_0 \in H. \tag{1.8}$$

3. Iterative form of the Tikhonov regularization

$$(\alpha I + A)y_n = \alpha y_{n-1} + f, \quad y_0 \in H, \tag{1.9}$$

are the most important special cases of the general theory. For the numerical realization of iterative methods we will use Chebyshev expansion methods.

From the extensive literature for linear integral equations of the first kind the following papers are mentioned: H. Bialy [3], H. W. Engel [4], J. Graves and P. M. Prenter [5], L. Landweber [11], J. W. Lee and P. M. Prenter [12], M. Z. Nashed [14], H. J. J. te Riele [16], O. N. Strand [18]. Further references are to be found in the following books: C. W. Groetsch [7], G. Hämmerlin and K. H. Hoffmann [8], B. Hofmann [9], A. N. Tikhonov and W. J. Arsenin [19]. The theory of compact linear operators and linear integral operators can be found in the monographs: N. Achieser and I. M. Glasmann [1], B. L. Moiseiwitsch [13] and F. Riesz and B. Sz. Nagy [17].

2. Preliminaries

In this section we summarize the results concerning the solution of the equations (1.2), (1.3) and (1.4). $N(K)$ and $R(K)$ will denote the null space and the range of an operator K. The Hilbert space H is the orthogonal sum of the closure $\overline{R(K^*)}$ and the null space $N(K)$:

$$H = \overline{R(K^*)} \oplus N(K).$$

Similarly we have

$$H = \overline{R(K)} \oplus N(K^*).$$

1. The singular value system $\{u_n, v_n; s_n\}$ of the compact operator K has

the following properties: $s_n > 0$

$$Kv_n = s_n u_n,$$
$$K^* u_n = s_n v_n,$$
$$K^* K v_n = s_n^2 v_n,$$
$$K K^* u_n = s_n^2 v_n.$$

The systems $\{u_n;\ n \in N\}$ and $\{v_n;\ n \in N\}$ are orthonormal systems (see M. Z. Nashed [14], G. Hoheisel [10]).

The Picard's criteria for a solution of equations (1.2) read

$$g \perp N(K^*), \tag{2.1}$$

$$\sum_{n=1}^{\infty} s_n^{-2} |(g, u_n)|^2 < +\infty. \tag{2.2}$$

Then there exists a unique normal solution y of (1.2) with the property $y \perp N(K)$. We have the series expansion

$$y = \sum_{n=1}^{\infty} s_n^{-1} (g, u_n) v_n. \tag{2.3}$$

2. The positive compact integral operator A may possess the sequence (μ_i) of positive eigenvalues. The corresponding eigenfunctions ϕ_i may be an orthonormal system. Then we have

$$A\phi_i = \mu_i \phi_i,$$
$$(\phi_i, \phi_n) = \delta_{in}, \quad i, n \in N.$$

The criteria for the existence of a solution for equation (1.4) are

$$f \perp N(A), \tag{2.4}$$

$$\sum_{i=1}^{\infty} \mu_i^{-2} |(f, \phi_i)|^2 < +\infty. \tag{2.5}$$

Then the normal solution y of (1.4) with the property $y \perp N(A)$ is uniquely determined. The series expansion of y becomes

$$y = \sum_{i=1}^{\infty} \mu_i^{-1} (f, \phi_i) \phi_i. \tag{2.6}$$

3. For the equation (1.2) may be valid the conditions (2.1) and (2.2). We consider the equation (1.3). Since $K^* g \in R(K^*)$ it follows $K^* g \perp N(K) = N(K^* K)$. Therefore we get the condition (2.4) for the operator $A = K^* K$. The operator A possesses the eigenvalues $\mu_i = s_i^2$ and the orthonormal eigenfunctions $\phi_i = v_i$. We obtain the equation

$$\sum_{i=1}^{\infty} \mu_i^{-2} |(K^* g, v_i)|^2 = \sum_{i=1}^{\infty} s_i^{-4} |(g, K v_i)|^2 = \sum_{i=1}^{\infty} s_i^{-2} |(g, u_i)|^2 < +\infty.$$

The corresponding normal solution of equation (1.3) is given by

$$y = \sum_{i=1}^{\infty} \mu_i^{-1}(K^*g, v_i)v_i = \sum_{i=1}^{\infty} s_i^{-2}(g, Kv_i)v_i = \sum_{i=1}^{\infty} s_i^{-1}(g, u_i)v_i.$$

This y is the solution (2.3) of equation (1.3).

3. The general approximate method

The general method uses operator functions of the positive compact operator A. The Hilbert–Schmidt theorem states the series expansion

$$Ay = \sum_{i=1}^{\infty} \mu_i(y, \phi_i)\phi_i \quad \text{for } y \in H. \tag{3.1}$$

The orthogonal projection P on the null space $N(A)$ is of the form

$$Py = y - \sum_{i=1}^{\infty} (y, \phi_i)\phi_i. \tag{3.2}$$

THEOREM 1. 1. *Let the sequence (a_n) be bounded. Then for every $y \in H$ exists the element*

$$z = \sum_{i=1}^{\infty} a_i(y, \phi_i)\phi_i. \tag{3.3}$$

2. *Let the family $(d_{\alpha i})$ of real numbers $0 < \alpha \leqslant \alpha_0$, $i = 1(1)\infty$, be such that $|d_{\alpha i}| \leqslant c$ for $0 < \alpha \leqslant \alpha_0$, $i = 1(1)\infty$,*

$$\lim_{\alpha \to 0} d_{\alpha i} = 0 \quad \textit{for every } i. \tag{3.4}$$

For the family (z_α) defined by

$$z_\alpha = \sum_{i=1}^{\infty} d_{\alpha i}(y, \phi_i)\phi_i \tag{3.5}$$

we have

$$\lim_{\alpha \to 0} z_\alpha = 0.$$

Proof. The existence of the sequence (z_α) follows from part 1. Now let $\varepsilon > 0$. Then there exists an m with the property

$$\Big\| \sum_{i=m+1}^{\infty} d_{\alpha i}(y, \phi_i)\phi_i \Big\|^2 \leqslant c^2 \sum_{i=m+1}^{\infty} |(y, \phi_i)|^2 < \frac{\varepsilon}{2} \tag{3.6}$$

for all α. The assumption (3.4) gives

$$\lim_{\alpha \to 0} \sum_{i=1}^{m} d_{\alpha i}(y, \phi_i)\phi_i = 0. \tag{3.7}$$

Then we have

$$\|\sum_{i=1}^{m} d_{\alpha i}(y, \phi_i)\phi_i\|^2 < \frac{\varepsilon}{2} \quad \text{for } 0 < \alpha < \alpha(\varepsilon). \tag{3.8}$$

Hence from the inequalities (3.6) and (3.8) we obtain

$$\|\sum_{i=1}^{\infty} d_{\alpha i}(y, \phi_i)\phi_i\|^2 < \varepsilon \quad \text{for } 0 < \alpha < \alpha(\varepsilon).$$

Therefore part 2 holds. ∎

THEOREM 2. 1. *The polynomial operator $p(A)$ corresponding to the polynomial*

$$p(t) = 1 + \sum_{k=1}^{n} a_k t^k$$

with real coefficients a_i, $i = 1(1)n$, is defined by

$$p(A) = I + \sum_{k=1}^{n} a_k A^k. \tag{3.9}$$

Then we have the expansion

$$p(A)y = Py + \sum_{i=1}^{\infty} p(\mu_i)(y, \phi_i)\phi_i. \tag{3.10}$$

2. *Let $p(t)$ be a power series*

$$p(t) = 1 + \sum_{k=1}^{\infty} a_k t^k \tag{3.11}$$

with real coefficients and the radius of convergence $r > 0$. Suppose the inequality

$$\|A\| < r.$$

Then the operator function

$$p(A) = I + \sum_{k=1}^{\infty} a_k A^k$$

exists, and we have the expansion

$$p(A) = Py + \sum_{i=1}^{\alpha} p(\mu_i)(y, \phi_i)\phi_i. \tag{3.12}$$

3. *For the polynomial*

$$q(t) = 1 + \sum_{k=1}^{m} b_k t^k$$

with real coefficients $b_k k = 1(1)m$ we assume $q(\mu_i) \neq 0$ for $i = 1(1)\infty$.

Then the operator $[q(A)]^{-1}$ *exists. We have the expansion*

$$[q(A)]^{-1}y = Py + \sum_{i=1}^{\infty} [q(\mu_i)]^{-1}(y, \phi_i)\phi_i. \tag{3.13}$$

Proof. 1. By use of (3.1) and (3.2) we obtain

$$A^k y = \sum_{i=1}^{\infty} \mu_i^k (y, \phi_i)\phi_i \quad \text{for } k = 1(1)\infty, \tag{3.14}$$

$$Iy = \sum_{i=1}^{\infty} (y, \phi_i)\phi_i + Py.$$

Then equation (3.9) gives the expansion (3.10).

2. Since $\|A\| < r$ the series $1+\sum_{k=1}^{\infty} |a_k| \|A\|^k$ is convergent. Hence exists

$$p(A) = I + \lim_{n\to\infty} \sum_{k=1}^{n} a_k A^k = I + \sum_{k=1}^{\infty} a_k A^k \tag{3.15}$$

(convergence concerning the operator norm). The inequality $\mu_i \leqslant \|A\| < r$ guarantees the existence of a constant c with

$$|p(\mu_i)| \leqslant c \quad \text{for } i = 1(1)\infty.$$

We consider the operator $\tilde{p}(A)$ given by the definition

$$\tilde{p}(A)y = Py + \sum_{i=1}^{\infty} p(\mu_i)(y, \phi_i)\phi_i. \tag{3.16}$$

For the partial sum $p_n(t) = 1+\sum_{k=1}^{n} a_k t^k$ we have

$$p_n(A)y = Py + \sum_{i=1}^{\infty} p_n(\mu_i)(y, \phi_i)\phi_i.$$

Hence

$$p_n(A)y = \tilde{p}(A)y + \sum_{i=1}^{\infty} [p_n(\mu_i) - p(\mu_i)](y, \phi_i)\phi_i.$$

We have the convergence property

$$\lim_{n\to\infty} p_n(\mu_i) = p(\mu_i) \quad \text{for every } i.$$

Since $p_n(t)$ convergers uniformly to $p(t)$ for $|t| \leqslant \|A\| < r$, there exists a constant d such that

$$|p_n(\mu_i)| \leqslant d \quad \text{for } n = 1(1)\infty,\ i = 1(1)\infty.$$

Now we get

$$|p_n(\mu_i) - p(\mu_i)| \leqslant |p_n(\mu_i)| + |p(\mu_i)| \leqslant d + c$$

for $n = 1(1)\infty$, $i = 1(1)\infty$. According to Theorem 1 we have

$$\lim_{n\to\infty} p_n(A)y = \tilde{p}(A)y. \tag{3.17}$$

Then the equations (15), (16) and (17) give the assertion (12).

3. Since $\lim_{i\to\infty}[q(\mu_i)]^{-1} = 1$ there is a constant d_1 such that

$$|[q(\mu_i)]^{-1}| \leqslant d_1 \quad \text{for } i = 1(1)\infty.$$

Let us define

$$By = \sum_{i=1}^{\infty} [q(\mu_i)]^{-1}(y, \phi_i)\phi_i + Py.$$

A simple calculation gives

$$Bq(A) = q(A)B = I.$$

From this we can state the formula

$$[q(A)]^{-1} = B.$$

Remark. For the polynomials $h(t)$ and $q(t)$ the assumptions of Theorem 2, part 1 and part 3 may hold. Then the operator $h(A)[q(A)]^{-1}$ exists and we obtain the series expansion

$$h(A)[q(A)]^{-1}y = Py + \sum_{i=1}^{\infty} h(\mu_i)[q(\mu_i)]^{-1}(y, \phi_i)\phi_i.$$

As simple example for part 2 we choose the exponential function

$$\exp\alpha t = 1 + \sum_{n=1}^{\infty} \frac{1}{n!}\alpha^n t^n$$

with the parameter α. For the corresponding operator function

$$\exp\alpha A = I + \sum_{n=1}^{\infty} \frac{1}{n!}\alpha^n A^n$$

we have

$$[\exp\alpha A]y = Py + \sum_{i=1}^{\infty} \exp(\alpha\mu_i)(y, \phi_i)\phi_i.$$

After these preliminaries we formulate the general approximation principle.

THEOREM 3. *For the positive parameter* α $(0 < \alpha \leqslant \alpha_0)$ *we consider*
a family of polynomials $p(t, \alpha)$,
a family of power series $p(t, \alpha)$,
a family of fractional functions $p(t, \alpha) = h(t, \alpha)[q(t, \alpha)]^{-1}$ *with the assumptions of Theorem* 2.

Let

$$\lim_{\alpha \to 0} [1-\mu_i p(\mu_i, \alpha)] = 0 \quad \textit{for } i = 1(1)\infty. \tag{3.18}$$

Further we assume the existence of a constant $c > 0$ *with the property*

$$|1-\mu_i p(\mu_i, \alpha)| \leqslant c \quad \textit{for } i = 1(1)\infty,\ 0 < \alpha \leqslant \alpha_0. \tag{3.19}$$

If the equation (1.4) *has a solution* y *for a given* f *(compare* 2.2), *then we have*

$$\lim_{\alpha \to 0} p(A, \alpha) f = y. \tag{3.20}$$

Proof. Corresponding to Theorem 2 we state the formula

$$p(A, \alpha) f = Pf + \sum_{i=1}^{\infty} p(\mu_i, \alpha)(f, \phi_i)\phi_i.$$

The condition $f \perp N(A)$ gives $Pf = 0$.

The series expansion (2.6) for the normal solution y yields the equation

$$\begin{aligned} p(A, \alpha) f &= y - \sum_{i=1}^{\infty} [1-\mu_i p(\mu_i, \alpha)]\mu_i^{-1}(f, \phi_i)\phi_i \\ &= y - \sum_{i=1}^{\infty} [1-\mu_i p(\mu_i, \alpha)](y, \phi_i)\phi_i. \end{aligned} \tag{3.21}$$

Then from Theorem 1 using the conditions (3.18) and (3.19) follows the fundamental formula (3.20). ∎

Example 1. If we use the function $p(t, \alpha) = (\alpha + t)^{-1}$ for $\alpha > 0$, then we obtain the regularization method of Tikhonov and the convergence formula (1.7).

Example 2. The linear differential equation

$$y'(\tau) + Ay(\tau) = f$$

in the Hilbert space H with the initial condition $y(0) = y_0$ has the solution

$$y(\tau) = \Big[\int_0^{\tau} \exp(-A(\tau - s))\,ds\Big] f + \exp(-\tau A) y_0.$$

If we choose for the function

$$w(t, \tau) = [1 - \exp(-t\tau)]t^{-1}, \qquad t \geqslant 0,$$

the corresponding operator function, we obtain

$$y(\tau) = w(A, \tau) f + \exp(-\tau A) y_0.$$

For the function $w(t, \tau)$ we can state the relations

$$1-\mu_i w(\mu_i, \tau) = \exp(-\mu_i \tau),$$
$$|1-\mu_i w(\mu_i, \tau)| \leqslant 1,$$
$$\lim_{\tau\to\infty} [1-\mu_i w(\mu_i, \tau)] = 0.$$

We find that

$$\lim_{\tau\to\infty} \exp(-\tau A) y_0 = P y_0.$$

Then Theorem 3 yields the equation

$$\lim_{\tau\to\infty} y(\tau) = y + P y_0.$$

4. Iterative methods

In this section we consider general iterative methods of the form (1.6) with a starting element y_0.

THEOREM 4. 1. *Let $p(t)$ and $q(t)$ be polynomials with the properties*

$$p(0) = q(0) = 1, \quad q(t) > 0 \quad \textit{for } t > 0. \tag{4.1}$$

The polynomial $s(t)$ satisfies the equality

$$q(t) = p(t) + ts(t). \tag{4.2}$$

2. *Assume the existence of a value $\tau > 0$ such that*

$$|p(\tau\mu_i)[q(\tau\mu_i)]^{-1}| < 1 \tag{4.3}$$

for all eigenvalues μ_i of A.

If the normal solution y for the equation (1.4) *exists* (*compare* 2.2), *then we have for the sequence* (y_n) *of the iterative method* (1.6) *the convergence property*

$$\lim_{n\to\infty} y_n = y + P y_0. \tag{4.4}$$

Proof. The proposition (4.1) assures the existence of the operator $[q(\tau A)]^{-1}$. Now by (1.6) we obtain the iterative method

$$y_n = [q(\tau A)]^{-1} p(\tau A) y_{n-1} + [q(\tau A)]^{-1} s(\tau A)\tau f.$$

Further we consider

$$y_n = T y_{n-1} + Wf$$

with the operators

$$T = [q(\tau A)]^{-1} p(\tau A), \qquad W = [q(\tau A)]^{-1} \tau s(\tau A).$$

A simple calculation yields

$$(4.5) \qquad y_n = (I + \sum_{k=1}^{n-1} T^k) Wf + T^n y_0 = h_n(T) Wf + T^n y_0,$$

if we define the polynomial $h_n(t)$ by

$$h_n(t) = 1 + \sum_{k=1}^{n-1} t^k = (1-t^n)(1-t)^{-1}.$$

The first part $h_n(T)Wf$ of equation (4.5) is an approximate method with the operator function

$$h_n(T)W = h_n([q(\tau A)]^{-1} p(\tau A)) \tau [q(\tau A)]^{-1} s(\tau A).$$

This function is a fractional function. Further we use $Pf = 0$.

By Theorem 2 we can assert that

$$h_n(T)Wf = \sum_{i=1}^{\infty} \tau h_n([q(\tau\mu_i)]^{-1} p(\tau\mu_i)) [q(\tau\mu_i)]^{-1} s(\tau\mu_i)(f, \phi_i)\phi_i.$$

For the eigenvalues we have the identity

$$q(\tau\mu_i) = p(\tau\mu_i) + \tau\mu_i s(\tau\mu_i).$$

For the expression in formula (3.18) we can establish the equality

$$(4.6) \qquad \begin{aligned} 1 - \mu_i \tau h_n([q(\tau\mu_i)]^{-1} p(\tau\mu_i)) [q(\tau\mu_i)]^{-1} s(\tau\mu_i) & \\ = 1 - \tau\mu_i \frac{1 - ([q(\tau\mu_i)]^{-1} p(\tau\mu_i))^n}{q(\tau\mu_i) - p(\tau\mu_i)} s(\tau\mu_i) & \\ = ([q(\tau\mu_i)]^{-1} p(\tau\mu_i))^n. & \end{aligned}$$

According to statement 2 we have

$$\lim_{n\to\infty} ([q(\tau\mu_i)]^{-1} p(\tau\mu_i))^n = 0,$$

$$|([q(\tau\mu_i)]^{-1} p(\tau\mu_i))^n| \leqslant 1$$

for every eigenvalue μ_i. Hence Theorem 3 gives for the normal solution y the equation

$$(4.7) \qquad \lim_{n\to\infty} h_n(T)Wf = y.$$

From

$$T^n y_0 = \sum_{i=1}^{\infty} (p(\tau\mu_i)[q(\tau\mu_i)]^{-1})^n (y_0, \phi_i)\phi_i + Py_0$$

we get

$$(4.8) \qquad \lim_{n\to\infty} T^n y_0 = Py_0.$$

The equations (4.7) and (4.8) demonstrate the assertion (4.4). ∎

Remark. For the special choice $y_0 = 0$ we can derive the error formula

$$y - y_n = \sum_{i=1}^{\infty} ([q(\tau\mu_i)]^{-1} p(\tau\mu_i))^n \frac{1}{\mu_i}(f, \varphi_i)\varphi_i.$$

EXAMPLE. 1. Let $p(t) = 1 - t$ and $q(t) = 1$. Then we have $s(t) = 1$ and the iterative method (1.6) becomes the Landweber iteration with the convergence condition (4.3)

$$\tau \max_i \mu_i = \tau \|A\| < 2.$$

2. $p(t) = 1 - 2\alpha t + \beta t^2$ and $q(t) = 1$. Then we have $s(t) = 2\alpha - \beta t$. For the constants may be valid $\alpha > 0$, $\beta > 0$, $\beta^2 < 2\alpha$. Then the convergence condition (4.3) takes the form

$$\beta\tau \max_i \mu_i = \beta\tau \|A\| < 2\alpha.$$

3. Let $p(t) = 1 - t + \frac{1}{2}t^2 - \frac{1}{6}t^3$ and $q(t) = 1$. It follows that $s(t) = 1 - \frac{1}{2}t + \frac{1}{6}t^2$. If the condition

$$\tau \max_i \mu_i = \tau \|A\| < 2.4$$

is satisfied, we get $|p(\tau\mu_i)| < 1$. (Compare P. Albrecht [2]. Chapter 7.3, Example 7.3.4).

4. Applying the Padé-approximation for the exponential function we can construct general iterative methods. For the used properties of Padé-approximation compare R. P. Grigorieff [6], Chapter 3.5. The diagonal Padé-approximation

$$\frac{P_{jj}(t)}{Q_{jj}(t)}, \quad j \geqslant 1,$$

contains the polynomials $P_{jj}(t)$ and $Q_{jj}(t)$ with the degree j and the properties $P_{jj}(0) = Q_{jj}(0) = 1$. We have $Q_{jj}(-t) > 0$ for $t \geqslant 0$, because $Q_{jj}(t)$ possesses only zeros with positive real part.

For $t > 0$ holds the inequality

$$|P_{jj}(-t)[Q_{jj}(-t)]^{-1}| < 1.$$

For the choice of $p(t) = P_{jj}(-t)$ and $q(t) = Q_{jj}(-t)$ the assumptions of Theorem 4 are valid. The parameter $\tau > 0$ can be chosen in a different way. The simplest method uses the linear polynomials

$$p(t) = P_{11}(-t) = 1 - \tfrac{1}{2}t, \qquad q(t) = Q_{11}(-t) = 1 + \tfrac{1}{2}t$$

and the polynomial $s(t) = 1$. The corresponding method becomes

$$(I + \tfrac{1}{2}\tau A)y_n = (I - \tfrac{1}{2}\tau A)y_{n-1} + \tau f, \qquad n = 1(1)\infty,$$

with any starting element y_0.

The method with quadratic polynomials uses

$$p(t) = P_{22}(-t) = 1 - \tfrac{1}{2}t + \tfrac{1}{12}t^2,$$
$$q(t) = Q_{22}(-t) = 1 + \tfrac{1}{2}t + \tfrac{1}{12}t^2,$$
$$s(t) = 1.$$

For the subdiagonal Padé-approximation

$$\frac{P_{j+1,j}(t)}{Q_{j+1,j}(t)}, \quad j \geqslant 1,$$

we have

$$P_{j+1,j}(0) = Q_{j+1,j}(0) = 1,$$
$$|P_{j+1,j}(-t)[Q_{j+1,j}(-t)]^{-1}| < 1 \quad \text{for } t > 0.$$

In the simplest case holds

$$p(t) = P_{2,1}(-t) = 1,$$
$$q(t) = Q_{2,1}(-t) = 1 + t > 0 \quad \text{for } t > 0,$$
$$s(t) = 1.$$

The corresponding iterative method is the method (1.9) with the parameter $\tau = \alpha^{-1}$.

5. Numerical experiments

We consider semidegenerate kernels of polynom type. The semidegenerate kernel

$$k(x, s) = \begin{cases} \sum_{i=0}^{n} \sum_{k=0}^{n} a_{ik} x^i s^k, & -1 \leqslant s \leqslant x \leqslant 1, \\ \sum_{i=0}^{n} \sum_{k=0}^{n} a_{ik} s^i x^k, & -1 \leqslant x \leqslant s \leqslant 1, \end{cases}$$

yields a symmetric compact operator A. If the eigenvalues are positive then A is positive definite.

The functions y_n of the iterative method and the starting function y_0 are represented in form of T-polynomials

$$y_n(x) = \tfrac{1}{2} c_{n0} + \sum_{k=0}^{m(n)} c_{nk} T_k(x).$$

$T_n(x) n = 0(1)\infty$ are the Tchebyshev polynomials. The numerical algorithms are procedures for the systems of the coefficients $(c_{n0}, c_{n1}, \ldots, c_{n,m(n)})$. For such T-polynomials is worked out an algebra and procedures (compare S. Paszkowski [15]).

The kernel function

$$k(x,s)=\begin{cases}\frac{1}{8}(1+s-t+st), & -1\leqslant s\leqslant t\leqslant 1,\\ \frac{1}{8}(1+t-s+st), & -1\leqslant t\leqslant s\leqslant 1,\end{cases} \tag{5.1}$$

possesses the positive eigenvalues $\mu_i=(i\pi)^{-2}$. It is produced by substitution from the kernel function $\min(s,t)(1-\max(s,t))$ for $0\leqslant s,t\leqslant 1$. For the function

$$g(x)=\tfrac{61}{1920}-\tfrac{5}{128}x^2+\tfrac{1}{128}x^4-\tfrac{1}{1920}x^6$$

on the right-hand side of the equation (1.4) with the kernel function (5.1) we have the solution

$$y(x)=\tfrac{1}{16}x^4-\tfrac{3}{8}x^2+\tfrac{5}{16}.$$

The convergence condition for the Landweber iterative method is satisfied for the value $\tau=4$, since $\tau<2\pi^2$.

If we carry out 500 iterative steps and use series expansions with 26 coefficients, we get for the approximate solution the following error quantities:

maximal value of the defect	$0.104\cdot 10^{-7}$,
minimal value of the defect	$0.573\cdot 10^{-10}$,
maximal value of the error	$-0.208\cdot 10^{-4}$,
minimal value of the error	$0.170\cdot 10^{-7}$.

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