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SL_2 ACTIONS AND COHOMOLOGY OF SCHUBERT VARIETIES

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One of the most useful aspects of a flag manifold G/B is that its cohomology ring $H^*(G/B; C)$ admits several different descriptions. The semisimple or Borel–Chevalley description says that $H^*(G/B; C)$ is the coinvariant algebra $A(h)/I^W$ associated to the Cartan subalgebra h of the Lie algebra g of G. The nilpotent or Kostant description says that $H^*(G/B; C)$ is the coordinate ring $A(N \cap h)$ of the scheme theoretic intersection of the variety of nilpotent elements N of g with h. On the other hand, there is the classical description of $H^*(G/B; C)$, which goes back to Schubert. It is based on the calculation of the homology from the partition G/B into cells (the so-called *Schubert cells*). The relation between the semisimple and the classical descriptions of $H^*(G/B; C)$ has been studied by Gel'fand et al. in [8], where the authors constructed a basis in the semisimple description dual to the Schubert cycles. However, the similar problem has not yet been studied for the nilpotent case.

In this note we discuss the semisimple, nilpotent and the classical descriptions of $H^*(G/B; C)$ as a special case of what happens when one has an action of SL_2 on a smooth complex projective variety X. We will also discuss the similar situation for the singular subvarieties of X. When Y is a Schubert subvariety of the algebraic homogeneous space G/P, we will show that the cohomology ring $H^*(Y; C)$ of Y admits a semisimple description. However, it is still an open problem whether $H^*(Y; C)$ also admits a nilpotent description(1). In the case of a Grassmann manifold $G_{k,n}$ these problems have been solved completely. It is proved in [5] that the Plücker coordinates form

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(1) D. H. Peterson has shown recently that $H^*(Y, C)$ does not admit a nilpotent description in general.

a basis for the nilpotent description of $H^*(G_{k,n}; C)$, which are dual to the Schubert cycles, and $H^*(Y; C)$ admits a nilpotent description(2).

This is basically an expository article, and the details can be found in [3], [5], [6], [7], [11], [12].

1. Vector fields and cohomology

Let V be a holomorphic vector field on a smooth complex projective variety X with isolated but nontrivial zero set Z, and let $i(V)$: $\Omega_X^p \to \Omega_X^{p-1}$ be the contraction operator associated to V. Here Ω_X^p (resp. $\mathcal{O}_X$) denotes the sheaf of germs of holomorphic p-forms (resp. functions) on X. The structure sheaf $\mathcal{O}_Z$ of Z is by definition $\mathcal{O}_X/i(V)\Omega_X^1$. We have the fundamental Koszul complex of sheaves:

$$0 \to \Omega_X^n \to \Omega_X^{n-1} \to \ldots \to \Omega_X^1 \to \mathcal{O}_X \to 0,$$

in which the differential is $i(V)$, and $n = \dim X$. It follows from the general facts on hypercohomology that there are two spectral sequences $\{'E_r\}$ and $\{''E_r\}$ abutting to $\operatorname{Ext}^*(X; \mathcal{O}_Z, \Omega_X^n)$, where $'E_1^{p,q} = H^q(X; \Omega_X^{n-p})$ and $''E_2^{p,q} = H^p(X, \operatorname{Ext}^q(\mathcal{O}_Z, \Omega_X^n))$ (see [15, p. 710]). The key fact proved in [11] is that the first spectral sequence degenerates at $'E_1$. Thus, as a consequence of the finiteness of Z and $H^0(X; \mathcal{O}_Z) \cong \operatorname{Ext}^n(X; \mathcal{O}_Z, \Omega_X^n)$, we obtain the following theorem of Carrell and Lieberman, which describes the computation of the cohomology ring $H^*(X; C)$ of X on the zeros Z of V.

THEOREM 1.1. *The ring $A(Z) = H^0(X; \mathcal{O}_Z)$ of regular functions on Z has an increasing filtration F_i with $F_iF_j \subseteq F_{i+j}$ so that $\sum_{p\geq 0} H^p(X; \Omega^p)$ and $\operatorname{Gr} A(Z) = \sum_{p\geq 0} F_p/F_{p-1}$ are isomorphic graded algebras. Moreover, $H^q(X; \Omega_X^p)$ vanishes if $p \neq q$, so the cohomology ring $H^*(X; C)$ is isomorphic to $\operatorname{Gr} A(Z)$.*

Proof. See [11], [12].

The main difficulty in realizing the cohomology ring of X on Z lies in computing the filtration F_p. However, in addition to V, if X admits an algebraic C^* action $(\lambda, x) \to \lambda \cdot x$ with the property that there exists an integer $k \neq 0$ such that for any $\lambda \in C^*$, $d\lambda \cdot V = \lambda^k V$, then $A(Z)$ is graded and the filtration F_p of $A(Z)$ is the canonical filtration associated to this grading. Hence $A(Z)$ becomes isomorphic to $H^*(X; C)$. This fact was proved in [6]. In this situation, we can actually describe the grading of $A(Z)$ explicitly. In fact, let us assume that V has exactly one zero x_0, (the general case is similar). Since x_0 is also a fixed point of the C^* action λ, C^* acts on the tangent space $T_{x_0}X$ of X at x_0 and consequently on the symmetric algebra $A = \operatorname{Sym}(T_{x_0}^* X)$ of the cotangent space $T_{x_0}^* X$ of X at x_0. The weight decomposition of this action makes A into a graded algebra. In the following theorem, A will be regarded as a graded algebra with this gradation.

(2) It has been proved in [8] that $H^*(Y; C)$ also admits a nilpotent description when Y is a Schubert subvariety of GL_n/P.

THEOREM 1.2 (the nilpotent description). *There exists a C^*-invariant open affine neighborhood U of x_0 such that*

(i) *U is C^*-equivariantly isomorphic to* $\operatorname{Spec}(A)$, *and consequently the ring of regular functions $A(U)$ on U admits a graded algebra structure,*

(ii) *the ideal $I(Z)$ of the zero scheme Z of V is homogeneous in the graded algebra $A(U)$,*

(iii) *the graded algebra $A(U)/I(Z)$ $(= A(Z))$ is isomorphic to $H^*(X; C^*)$.*

Proof. See [6]. We note that U is the big cell $x_0^- = \{x \in X; \lim_{\lambda\to\infty} \lambda\cdot x = x_0\}$ of the negative Białynicki-Birula decomposition of X.

A natural way of obtaining such a C^* action on X is to consider those X which admit an action of the group of upper triangular matrices $\mathbf{B}$ in SL_2. For simplicity, we shall consider X with an SL_2 action. For the rest of the section, we fix an algebraic SL_2 action on X with the property that the vector field V generated by $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ has only isolated zeros Z. The C^* action $(\lambda, x) \to \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix} \cdot x$ on X, which is induced from SL_2, satisfies $d\lambda\cdot V = \lambda^2 V$ for any λ in C. Moreover, it has only isolated fixed points ([14]). Let Z_S be the fixed point scheme of this C^* action λ, and let V_S denote the vector field on X generated by $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. Since the zero scheme of V_S is Z_S, we obtain two descriptions of $H^*(X; C)$:

(a) *the semisimple description*: There exists an increasing filtration F_i of $A(Z_S)$ with $F_i F_j \subseteq F_{i+j}$ so that

$$\phi\colon \operatorname{Gr} A(Z_S) = \sum_{p \geq 0} F_p/F_{p-1} \xrightarrow{\sim} H^*(X; C).$$

(b) *the nilpotent description*: There exists a canonical grading on $A(Z)$ such that

$$\psi\colon A(Z) \xrightarrow{\sim} H^*(X; C).$$

Remark. By a theorem of Horrocks (see [14]) Z contains exactly one closed point x_0.

The third description of $H^*(X; C)$ is based on the calculation of the homology group from the partition of X into Białynicki-Birula cells. Let

$$Z_S = \{x_0, x_1, \ldots, x_r\}, \quad \text{and let} \quad x_i^+ = \{x \in X\colon \lim_{\lambda\to 0} \lambda\cdot x = x_i\},$$

$$x_i^- = \{x \in X\colon \lim_{\lambda\to\infty} \lambda\cdot x = x_i\} \quad \text{for } i = 0, 1, \ldots, r.$$

The following theorem is due to A. Białynicki-Birula ([9]).

THEOREM 1.3 (the classical description). $X = \bigcup_{i=0}^{r} x_i^+$ (resp. $\bigcup_{i=0}^{r} x_i^-$) *is a locally closed decomposition of* X, *and each* x_i^+ (*resp.* x_i^-) *is isomorphic to* A^{m_i} *for some* m_i. *Moreover, the homology classes* $[\overline{x_i^+}]$ (*resp.* $[\overline{x_i^-}]$) *of the Zariski closures* $\overline{x_i^+}$ (*resp.* $\overline{x_i^-}$) *of* x_i^+ (*resp.* x_i^-) *form a free basis of the homology algebra* $H_*(X; \mathbf{Z})$ *of* X *over* $\mathbf{Z}$.

Proof. See [9] and also [13].

One of the most interesting problems is to compare these various descriptions of $H^*(X; \mathbf{C})$. Namely, find a basis in the nilpotent (resp. semisimple) description of $H^*(X; \mathbf{C})$ which is dual to the Białynicki-Birula cycles $\{[\overline{x_i^+}]: i = 0, 1, \ldots, r\}$. In general this problem looks very difficult. A special case will be discussed in the next section.

The generalization of the semisimple and nilpotent description of $H^*(X; \mathbf{C})$ to the singular subvarieties of X have been studied in [7]. The authors obtained the following theorems as particular cases: Let $\mathbf{B}$ denote the group of upper triangular matrices in SL_2, Y a $\mathbf{B}$-invariant subvariety of X, and $i^*: H^*(X; \mathbf{C}) \to H^*(Y; \mathbf{C})$ the cohomology map of the inclusion $i: Y \subsetneq X$. In the semisimple case we have:

THEOREM 1.4. *The filtration* F_p *of* $A(Z_s)$ *induces a filtration on the coordinate ring* $A(Y \cap Z_s)$ *of the scheme theoretic intersection* $Y \cap Z_s$ *such that the associated graded algebra* $\mathrm{Gr}\, A(Y \cap Z_s)$ *admits a homomorphism into* $H^*(Y; \mathbf{C})$ *making the following diagram commute*:

$$\begin{array}{ccc} \phi\colon \mathrm{Gr}\, A(Z_s) & \xrightarrow{\sim} & H^*(X; \mathbf{C}) \\ \downarrow & & \downarrow i^* \\ \bar{\phi}\colon \mathrm{Gr}\, A(Y \cap Z_s) & \to & H^*(Y; \mathbf{C}). \end{array}$$

Moreover, if i^* *is surjective, then* $\bar{\phi}$ *is an isomorphism* (*i.e.*, $H^*(Y; \mathbf{C})$ *admits a semisimple description*).

In the nilpotent case $A(Z)$, however, the situation is different: Since the scheme theoretic intersection $Y \cap Z$ is a $\mathbf{B}$-invariant subscheme of X, there exists a canonical grading of the coordinate ring $A(Y \cap Z)$ of $Y \cap Z$ such that the natural map $A(Z) \to A(Y \cap Z)$ is a graded algebra homomorphism.

THEOREM 1.5. $A(Y \cap Z)$ *admits a homomorphism into* $H^*(Y; \mathbf{C})$ *making the following diagram commute*:

$$\begin{array}{ccc} \psi\colon A(Z) & \xrightarrow{\sim} & H^*(X; \mathbf{C}) \\ \downarrow & & \downarrow i^* \\ \bar{\psi}\colon A(Y \cap Z) & \to & H^*(Y; \mathbf{C}). \end{array}$$

If i^* is surjective, we know by Theorem 1.4 that $H^*(Y; \mathbf{C})$ admits a semisimple description. However, in the nilpotent case it is not known whether $H^*(Y; C)$ admits a nilpotent description, i.e., whether $\bar{\psi}$: $A(Y \cap Z) \to H^*(Y; C)$ is an isomorphism.

Remark. Clearly, one also has all the results above for an action of $\mathbf{B}$ on X with the property that V_s has only isolated zeros.

DEFINITION. The plus decomposition $X = \bigcup_{i=0}^{r} x_i^+$ of X is said to satisfy the frontier condition if each Zariski closure $\overline{x_i^+}$ is a union of certain x_j^+.

THEOREM 1.6. *If each x_i^+ intersects each x_j^- transversally, then the plus decomposition of X satisfies the frontier condition.*

Proof. See [10].

THEOREM 1.7. *If the plus decomposition of X satisfies the frontier condition, then for any Bialynicki-Birula subvariety $Y = \overline{x_j^+}$ of X we have*

(i) *$H^*(Y; C)$ admits a semisimple description.*

(ii) *The graded algebra isomorphism ψ: $A(Z) \to H^*(X; C)$ induces a surjective map $\bar{\psi}$: $A(Y \cap Z) \to H^*(Y; C)$.*

Proof. By [14], the plus decomposition of X is $\mathbf{B}$-invariant. Thus, each $Y = \overline{x_j^+}$ is a $\mathbf{B}$-invariant subvariety of X. The rest follows from Theorems 1.4 and 1.5, because the frontier condition implies that i^*: $H^*(X; C) \to H^*(Y; C)$ is surjective.

2. Cohomology of Schubert varieties

In this section we apply the results of the previous section to the algebraic homogeneous space G/P. We give the relations between the semisimple, nilpotent, Borel–Chevalley, and Kostant descriptions of $H^*(G/B; C)$. Moreover, we compute the cohomology rings of Schubert varieties in G/P.

Let G be a complex semisimple linear algebraic group, B a fixed Borel subgroup of G, H a maximal torus of G in B, P a parabolic subgroup of G containing B, W the Weyl group of (H, G), g and h the Lie algebras of G and H, respectively. We denote by Δ the set of roots of H in G, Δ_+ the set of positive roots in Δ associated to B, $\Sigma = \{\alpha_1, \ldots, \alpha_l\}$ the set of simple roots in Δ_+, $\{e_\beta \in g: \beta \in \Delta\}$ the set of root vectors such that $\{[e_\beta, e_{-\beta}]: \beta \in \Delta\}$ is dual to Δ (i.e., $[e_\beta, e_{-\beta}]$ is the co-root associated to $\beta \in \Delta$). The integer obtained from the canonical perfect pairing between one parameter subgroups μ: $C^* \to H$ and characters χ: $H \to C^*$ is denoted by $\langle \mu, \chi \rangle$. The height $\sum_1^l m_i$ of any $\beta = \sum_1^l m_i \alpha_i$ in Δ is denoted by $h(\beta)$.

For any given regular nilpotent element n in g, by the Jacobson–Morosov Lemma ([16]) there exists an sl_2-triple $\{n, f, s\}$ with the property $[n, f] = s$, $[s, f] = -2f$, and $[s, n] = 2n$. This gives an algebraic SL_2 action on G/B (resp. G/P) so that the vector field V generated by $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ has exactly one zero $x_0 = B$ (resp. P). Conversely, each such an SL_2 action on G/B (resp. G/P) is obtained in this way. We will take, without loss of generality, the principal regular nilpotent element $n = \sum_1^l e_{\alpha_i}$ throughout the rest of the paper. In this case $s \in h$ is a regular semisimple, while $f \in b_u^-$ is a regular nilpotent element of g. Here b_u^- is the Lie algebra of the nilpotent radical B_u^- of the Borel subgroup B^- of G opposite to B. Let μ: $C^* \to H$ be the one parameter subgroup of H associated to s (i.e., $d\mu(1) = s$). It is clear that the vector field V_s (resp. V) generated by $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ (resp. $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$) is induced from the C^* (resp. C) action μ (resp. $\exp(tn)$).

We start with the semisimple description of the cohomology ring of the flag space G/B. Since $s \in h$ is regular, the zeros Z_s of V_s is the fixed point scheme of H on G/B. Thus, Z_s may be viewed as the orbit under W of $x_0 = B$. This implies that $A(Z_s)$ is precisely the ring of complex valued functions on W. We will consider $A(Z_s)$ as a W-module by setting $w \cdot f(v) = f(vw)$ for all $w, v \in W$ and $f \in A(Z_s)$. To identify the filtration of $A(Z_s)$, we consider instead the finite reduced affine scheme $W \cdot s \subset h$. Its coordinate ring $A(W \cdot s)$ is a W-module with a W-invariant filtration $F_i \subset F_{i+1}$ such that $F_i F_j \subset F_{i+j}$. Indeed, F_i consists of the restrictions to $W \cdot s$ of polynomials on h of degree at most i. The key fact proved in [3] is that $A(W \cdot s)$ and $A(Z_s)$ are isomorphic as filtered W-algebras. In fact, the explicit isomorphism

$$\theta\colon A(W \cdot s) \to A(Z_s) \quad \text{is given by} \quad \theta(f)(w) = (w \cdot f)(s) = f(w^{-1} \cdot s)$$
$$\text{for all } f \in A(W \cdot s) \text{ and all } w \in W.$$

NOTATION. Throughout the rest of the paper we will take $A(W \cdot s)$ for $A(Z_s)$.

The geometric significance of the identification of $A(W \cdot s)$ with $A(Z_s)$ is the fact that if $[d\chi]$ is the function on $W \cdot s$ defined by the restriction $d\chi | W \cdot s$, then we have

$$\phi([d\chi]) = c_1(L_\chi^*), \quad \text{the Chern class of the dual of the line bundle } L_\chi \text{ associated to } \chi \text{ ([3])}.$$

One can put these in a familiar context by recalling the Borel–Chevalley isomorphism β: $A(h)/I^W \to H^*(G/B; C)$ between coinvariant algebra of h and the cohomology of G/B induced by setting $\beta(d\chi) = c_1(L_\chi)$. Here $A(h)$ is the coordinate ring of h, and I^W is the homogeneous ideal generated by the W-invariant functions f on h such that $f(0) = 0$. Note that W acts on $A(h)$ according to the rule $(w \cdot f)(\mathrm{h}) = f(w^{-1} \cdot \mathrm{h})$ for $f \in A(h)$, $w \in W$ and $\mathrm{h} \in h$. The natural map π: $A(h) \to A(W \cdot s)$ is by definition degree preserving, so it induces a surjective homomorphism $\mathrm{Gr}\,\pi$: $A(h) \to \mathrm{Gr}\,A(W \cdot s)$. The following theorem which is proved in [3] summarizes the connection.

THEOREM 2.1. *The kernel of* $\mathrm{Gr}\,\pi$ *is* I^W. *Moreover, we have the following commutative diagram of W-equivariant isomorphisms*:

$$\begin{tikzcd} A(h)/I^W \arrow[r, "\beta"] \arrow[d, "\overline{\mathrm{Gr}\,\pi}"'] & H^*(G/B;\mathbf{C}) \\ \mathrm{Gr}A(W\cdot s) \arrow[ur, "\tilde{\phi}"'] & \end{tikzcd}$$

where $\tilde{\phi}$ *denotes* ϕ *twisted by the sign representation.*

THEOREM 2.2. *The plus decomposition* $G/B = \bigcup (wB)^+$, $w \in W$, *of* G/B *determined by the* $\mathbf{C}^*$ *action* μ *is the Bruhat decomposition* $G/B = \bigcup BwB$, $w \in W$, *of* G/B. *Therefore, the cycle classes* $[X_w]$ *of the Schubert varieties* $X_w = \overline{BwB}$ *form a basis of* $H_*(G/B;\mathbf{C})$.

Proof. It follows from Theorem 1.3 and [1].

Recall that there exists a partial ordering $\leqslant$ on W with the property that $v \leqslant w$ if and only if $BvB \subseteq X_w = \overline{BwB}$ ([8]). This immediately implies that $X_w = \bigcup_{v \leqslant w} BvB$, and therefore the Bruhat decomposition satisfies the frontier condition.

THEOREM 2.3. *Let* e *be the identity element of* W *and* $X_w = \overline{BwB}$ *the Schubert subvariety of* G/B *associated to* w *in* W. *Then* $H^*(X_w;\mathbf{C})$ *is the graded ring associated to the degree filtration of the coordinate ring* $A([e,w]\cdot s)$ *of the subvariety* $[e,w]\cdot s = \{v\cdot s \colon v \leqslant w\}$ *of* $W\cdot s$. *Moreover, there is a commutative diagram of algebra homomorphisms*

$$\begin{tikzcd} A(h)/I^W \arrow[r, "\sim"] \arrow[d] & \mathrm{Gr}\,A(W\cdot s) \arrow[r, "\phi", "\sim"'] \arrow[d] & H^*(G/B;\mathbf{C}) \arrow[d, "i^*"] \\ A(h)/\mathrm{gr}\,I([e,w]\cdot s) \arrow[r, "\sim"] & \mathrm{Gr}\,A([e,w]\cdot s) \arrow[r, "\bar{\phi}", "\sim"'] & H^*(X_w;\mathbf{C}), \end{tikzcd}$$

where $\mathrm{gr}\,I([e,w]\cdot s)$ *is the ideal generated by the leading terms of the functions* $f \in A(h)$ *vanishing on* $[e,w]\cdot s$.

Proof. It follows from Theorem 1.7, the discussions above, and the fact that $X_w \cap Z_s \cong [e,w]\cdot s$. For more details see [7].

The situation is similar for the space G/P. In fact, let W_P denote the Weyl group of (H, P). The following is proved in [3].

THEOREM 2.4. *The ring of* W_P*-invariant elements* $A(W\cdot s)^{W_P}$ *of* $A(W\cdot s)$ *has an increasing filtration such that* $\mathrm{Gr}\,A(W\cdot s)^{W_P} \xrightarrow{\sim} H^*(G/P;\mathbf{C})$, *and moreover this isomorphism is compatible with the inclusions* $A(W\cdot s)^{W_P} \subset A(W\cdot s)$ *and* $H^*(G/P;\mathbf{C}) \hookrightarrow H^*(G/B;\mathbf{C})$.

Since by definition $A(W\cdot s)^{W_P}$ is $A(W_P/W\cdot s)$ we obtain the following corollary (see [3], [7]).

COROLLARY. *The following diagram of algebra homomorphism commutes:*

$$\begin{array}{ccccc} (A(h)/I^W)^{W_P} & \xrightarrow{\sim} & \operatorname{Gr} A(W_P\backslash W\cdot s) & \overset{\Phi}{\xrightarrow{\sim}} & H^*(G/P; C) \\ \downarrow & & \downarrow & & \downarrow \\ A(h)/I^W & \xrightarrow{\sim} & \operatorname{Gr} A(W\cdot s) & \xrightarrow{\sim} & H^*(G/B; C), \end{array}$$

where $(A(h)/I^W)^{W_P}$ is the ring of W_P-invariant elements of $A(h)/I^W$.

Let $\overline{[e, w]}\cdot s$ denote the image of $[e, w]\cdot s$ under the natural map $W\cdot s \to W_P\backslash W\cdot s$, and let $\operatorname{Gr} A(\overline{[e, w]}\cdot s)$ be the graded ring associated to the filtration induced from $A(W_P\backslash W\cdot s)$.

THEOREM 2.5. (i) *The cycle classes $[X_{\bar{w}}]$ of the Schubert subvarieties $X_{\bar{w}} = \overline{BwP}$, $\bar{w}\in W/W_P$, of G/P form a basis of $H_*(G/P; C)$.*

(ii) *For any Schubert subvariety $X_{\bar{w}} = \overline{BwP}$ of G/P, $H^*(X_{\bar{w}}; C)$ is isomorphic with $\operatorname{Gr} A(\overline{[e, w]}\cdot s)$. The restriction map $H^*(G/P; C) \to H^*(X_{\bar{w}}; C)$ corresponds to the natural map $\operatorname{Gr} A(W_P\backslash W\cdot s) \to \operatorname{Gr} A(\overline{[e, w]}\cdot s)$.*

Proof. (i) follows from the fact that the plus decomposition of G/P determined by the C^* action μ is the Bruhat decomposition $G/P = \bigcup B\bar{w}P$ of G/P ([2]). (ii) follows from Theorem 1.7 and the previous discussions.

This theorem, in particular, says that the cohomology rings of Schubert varieties in G/P admit semisimple descriptions. The situation for the nilpotent description, however, is different as we shall discuss now. Let θ be an arbitrary subset of Σ, Δ_θ the subset of Δ_+ consisting of linear combinations of θ, and P_θ the parabolic subgroup of G corresponding to θ. We shall take $P = P_\theta$ without loss of generality, and keep the notation as before.

PROPOSITION 2.1. *There exists an H-invariant open affine neighborhood U of x_0 in G/B (resp. G/P) together with a natural holomorphic local coordinate system $z_{-\alpha}$ at x_0 such that $A(U)$ is isomorphic to $C[z_{-\alpha}\colon \alpha\in\Delta_+]$ (resp. $C[z_{-\alpha}\colon \alpha\in\Delta_+\backslash\Delta_\theta]$), where the grading is determined by taking degree $z_{-\alpha} = h(\alpha)$ for $\alpha\in\Delta_+$.*

Proof. By Theorem 1.2, $U = x_0^- = B^- B$ (resp. $B^- P$), which is also H-invariant. For $\alpha\in\Delta$, let $P_\alpha\colon g\to g_\alpha \cong Ce_\alpha$ be the projection map. The canonical isomorphism $\exp\colon b_u^- \to B_u^-$ defines the following coordinate functions $z_{-\alpha}$ on $B^- B = B_u^- B$ (resp. $B^- P$): For $\alpha\in\Delta_+$, $z_{-\alpha}(\exp(x)B) = P_{-\alpha}(x)$, $x\in b_u^-$. Since the tangent action of the C^* action $\lambda\to\lambda\cdot x = \mu(\lambda)x$ (the left multiplication on G/B (resp. G/P)) induced from $\mu\colon C^*\to H$ is equivalent to the adjoint action and $2h(\beta) = \langle\mu, \beta\rangle$ for any β in Δ, we have $d\lambda\cdot e_{-\alpha} = \lambda^{-2h(\alpha)}e_{-\alpha}$. This shows that degree $z_{-\alpha} = h(\alpha)$, and therefore we have the claim.

COROLLARY. *The ideal $I(Z)$ of the zero scheme Z of V is homogeneous in $C[z_{-\alpha}\colon \alpha\in\Delta_+]$ (resp. $C[z_{-\alpha}\colon \alpha\in\Delta_+\backslash\Delta_\theta]$), and the graded algebra*

$R = C[z_{-\alpha}\colon \alpha\in\Delta_+]/I(Z)$ *(resp.* $R_\theta = C[z_{-\alpha}\colon \alpha\in\Delta_+\setminus\Delta_\theta]/I(Z)$*) is the nilpotent description* $A(Z)$ *of the cohomology ring of* G/B *(resp.* G/P*).*

We shall now give the algebraic relation between the semisimple and nilpotent descriptions of $H^*(G/P; C)$. Consider the grading on the coordinate ring $A(b_u^-)$ of b_u^- induced from the adjoint action of $\mu\colon C^*\to H$ (i.e., for $\lambda\in C^*$ and $f\in A(b_u^-)$, $(\lambda\cdot f) = f(Ad\mu(\lambda^{-1})(x))$ for $x\in b_u^-$). It is clear that $C[z_{-\alpha}\colon \alpha\in\Delta_+]\cong A(b_u^-)$ as a graded algebra. Let b^- be the Lie algebra of B^-, and let $\tilde{\alpha}\colon A(h)\to A(b_u^-)$ be the comorphism of the linear map $\varrho\colon b_u^- \xrightarrow{\text{adn}} b^- \xrightarrow{\text{proj}} h$. Since $A(h)$ is generated by the characters $d\chi$ of h and $\lambda\cdot\tilde{\alpha}(d\chi) = \lambda^2\tilde{\alpha}(d\chi)$, $\tilde{\alpha}$ is a graded algebra homomorphism.

THEOREM 2.6. *The algebra homomorphism* $\tilde{\alpha}\colon A(h)\to A(b_u^-)$ *induces a graded algebra isomorphism* $\alpha\colon A(h)/I^W\to A(b_u^-)/I(Z)$ *giving the following commutative diagram of isomorphisms*:

$$\begin{array}{ccc} A(h)/I^W & \xrightarrow{\alpha} & A(b_u^-)/I(Z) \\ {\scriptstyle \overline{\mathrm{Gr}\,\pi}}\downarrow & & \downarrow{\scriptstyle \psi} \\ \mathrm{Gr}\,A(W\cdot s) & \xrightarrow{\phi} & H^*(G/B;\ C) \end{array}$$

Proof. See [6].

By a theorem of Kraft, Kostant description $A(N\cap h)$ of $H^*(G/B; C)$ can also be added to the diagram above as follows: The restriction map res: $A(g)\to A(h)$ induces an isomorphism $\overline{\mathrm{res}}$ of graded algebras

$$\overline{\mathrm{res}}\colon A(N\cap h)\to \mathrm{Gr}\,A(W\cdot s)$$

such that the diagram

$$\begin{tikzcd} A(N\cap h) \arrow[r, "k"] \arrow[dr, "\overline{\mathrm{res}}"'] & A(h)/I^W \arrow[d, "\overline{\mathrm{Gr}\,\pi}"] \\ & \mathrm{Gr}A(W\cdot s) \end{tikzcd}$$

commutes, where k is the usual map and $A(N\cap h)$ is the coordinate ring of the scheme theoretic intersection of the nilpotent cone N in g with h (see [6], [17]).

Remark. Under the isomorphism $\psi\colon A(b_u^-)/I(Z)\xrightarrow{\sim} H^*(G/B; C)$ it can be shown that $\psi(z_{-\alpha_i}) = c_1(L_{w_i})$, where $\{w_i\colon i = 1, \ldots, l\}$ is the set of fundamental dominant weights associated to Σ. It would be interesting to compute $\psi(z_{-\alpha})$ in $H^*(G/B; C)$ for any $\alpha\in\Delta_+$. One of the main differences between the semisimple and nilpotent descriptions of $H^*(G/B; C)$ is that in the nilpotent case the values of ψ at the generators $z_{-\alpha}$, $\alpha\in\Delta_+$, of $A(b_u^-)$ are not known in general.

COROLLARY. *When the isomorphism* α: $A(h)/I^W \to A(b_u^-)/I(Z)$ *is restricted to the subalgebra* $(A(h)/I^W)^{W_P}$ *one obtains the following commutative diagram*:

$$\begin{array}{ccccc} A(h)/I^W & \xrightarrow{\alpha} & R & \xrightarrow{\psi} & H^*(G/B;C) \\ \uparrow & & \uparrow & & \uparrow \\ (A(h)/I^W)^{W_P} & \xrightarrow{\sim} & R_\theta & \xrightarrow{\sim} & H^*(G/P;C) \end{array}$$

Let $X_{\bar{w}} = \overline{BwP}$, $\bar{w} \in W/W_P$, be the Schubert subvariety of G/P associated to $\bar{w}$. By Theorem 1.7 we know that the isomoprhisms ψ: $A(Z) \xrightarrow{\sim} H^*(G/P;C)$ induces a surjective map $\bar{\psi}$: $A(X_{\bar{w}} \cap Z) \to H^*(X_{\bar{w}};C)$. It is an open problem whether $\bar{\psi}$ is an isomorphism. It has been conjectured in [6] that $\bar{\psi}$ is an isomorphism, i.e., the cohomology rings of Schubert varieties admit nilpotent descriptions. The conjecture has been verified recently for the Grassmann manifold ([5]) as we shall discuss in the next section.

3. The relations of Plücker coordinates to Schubert calculus

For the completeness of the article we start with the relation between the semisimple and classical descriptions of $H^*(G/P;C)$. We then discuss the connection between the nilpotent and classical descriptions of the cohomology ring of the Grassmann manifold $G_{k,n}$. Finally we prove that the cohomology ring of any Schubert subvariety of $G_{k,n}$ admits a nilpotent description. We keep the notation of Section 2.

We start recalling the operators A_w: $A(h) \to A(h)$, $w \in W$ ([8]). For each $\alpha \in \Delta$, the element $f - \sigma_\alpha \cdot f$ is divisible by α, where σ_α is the reflection corresponding to α. Thus A_α: $A(h) \to A(h)$, $A_\alpha f = f - \sigma_\alpha \cdot f/\alpha$, is a well-defined linear operator on $A(h)$. Let $\alpha_1, \ldots, \alpha_k \in \Sigma$, and let $w = \sigma_{\alpha_1} \ldots \sigma_{\alpha_k}$ be any element of W. Then we have

(i) if the length $l(w)$ of w is less than k, then $A_{\alpha_1} \ldots A_{\alpha_k} = 0$,

(ii) if $l(w) = k$, then the operator $A_{\alpha_1} \ldots A_{\alpha_k}$ depends only on w and not on representation of w in the form $w = \sigma_{\alpha_1} \ldots \sigma_{\alpha_k}$. In this case we put $A_w = A_{\alpha_1} \ldots A_{\alpha_k}$.

We note that the operator A_w: $A(h) \to A(h)$ preserves the ideal I^W, and thus induces an operator $\bar{A}_w$: $A(h)/I^W \to A(h)/I^W$ of homogeneous degree $-l(w)$. Let w_0 be the unique element of W of maximal length, and $P_{w_0} = (1/|W|) \prod_{\alpha \in \Delta_+} \alpha \pmod{I^W}$, where $|W|$ is the order of W. For each $w \in W$, let $P_w = \bar{A}_{w^{-1}w_0}(P_{w_0})$. The following is proved in [8]:

THEOREM 3.1. (i) $\{P_w: w \in W\}$ *is a free* $\mathbf{Z}$-*basis of* $A(h)/I^W$.

(ii) *Let* Y_w *be the cycle class of the Schubert variety* $X_{w_0 w} = \overline{Bw_0 wB}$ *in* $H_*(G/B;C)$. *Then under the Poincaré duality map* $\mathscr{P}$: $H_*(G/B;C) \to H^*(G/B;C)$, $\mathscr{P}(Y_w) = \beta(P_w)$, *where* β *is the Borel–Chevalley isomorphism given in Theorem* 2.1.

The situation for G/P, $P = P_\theta$, is similar. In fact, let $W_p^\perp$ denote the set of $w \in W$ such that $w\theta \subset \Delta_+$, and $A(h)^{W_P}$ the ring of W_P-invariant elements of $A(h)$. The following theorem is also proved in [8]; however, by using the result of Section 2 and the Gysin homomorphism of the natural map π: $G/B \to G/P$, it can also easily be obtained from the theorem above. For more details see [4, p. 260].

THEOREM 3.2. (i) *$\{P_\sigma \bmod (I^W \cap A(h)^{W_P})$: $\sigma \in W_P^\perp\}$ is a free $\mathbf{Z}$-basis of $(A(h)/I^W)^{W_P}$.*

(ii) *For each $\sigma \in W_P^\perp$, let Y_σ be the cycle class of the Schubert variety $X_{\overline{w_0\sigma}} = \overline{Bw_0\sigma P}$ in $H_*(G/P; C)$. Then under the Poincaré duality map $\mathscr{P}$, $\mathscr{P}(Y_\sigma) = \beta|(P_\sigma \bmod (I^W \cap A(h)^{W_P}))$, where $\beta|$ is the restriction of β to $(A(h)/I^W)^{W_P}$.*

Remark. It is very desirable to find an explicit basis of the nilpotent description $A(Z)$ of $H^*(G/P, C)$ which is dual to the Schubert cycles. By the theorems above and the corollary of Theorem 2.6 this amounts to saying to compute $\alpha(P_w)$ in $R = A(\mathfrak{h}_u^-)/I(Z)$.

In the rest of the section, we discuss some of the problems mentioned before for the full flag manifolds and the Grassmannians. We would like to note the results so far obtained are also valid for any reductive linear algebraic group G over C ([3], [5]).

Let $G = \mathrm{GL}_n$, B the group of upper triangular matrices in G, and H the group of diagonal matrices in B. Then $\Delta = \{\alpha_{ij} = x_i - x_j$: $i \neq j$, $1 \leqslant i, j \leqslant n\}$, $\Delta_+ = \{\alpha_{ij}$: $1 \leqslant i < j \leqslant n\}$, $\Sigma \in \{\alpha_{i\,i+1}$: $1 \leqslant i \leqslant n-1\}$, $W = S_n$ the symmetric group in $1, 2, \ldots, n$ and $n = \sum_{i=1}^{n-1} e_{i\,i+1}$ where e_{ij} is the $n \times n$ matrix having 1 in the (i, j)th entry and zero everywhere else. Let $\theta = \{\alpha_{i\,i+1}$: $1 \leqslant i \leqslant k\}$, and $P = P_\theta$. The algebraic homogeneous space G/B is the full flag manifold F_n and G/P is the Grassmann manifold $G_{k,n}$ of k-planes in C^n. It follows from Proposition 2.1 that $A(U)$ is isomorphic to $C[z_{ij}$: $1 \leqslant j \leqslant i \leqslant n]$ (resp. $C[z_{k+i\,j}$: $1 \leqslant i \leqslant n-k$, $1 \leqslant j \leqslant k]$) for the space F_n (resp. $G_{k,n}$), where $z_{ij}(x) = x_{ij}$ for $x \in G$ and the grading is determined by degree $z_{pq} = p - q$. In the rest of the paper we take $z_{ij} = 0$ if either $i > n$ or $j < 1$, and $z_{ii} = 1$ for $1 \leqslant i \leqslant n$. The following theorem has been stated in [5]. We take this opportunity to give its complete proof.

THEOREM 3.3. (i) *The graded algebra $A(Z)$ is isomorphic to $C[z_{ij}$: $1 < j \leqslant i \leqslant n]/I(Z)$, where $I(Z)$ is the homogeneous ideal generated by*

$$a_{ij}(z) = z_{i+1\,j} - z_{i\,j-1} + z_{ij}(z_{j\,j-1} - z_{j+1\,j}).$$

(ii) *Let $x_1 = z_{21}$, $x_2 = z_{32} - z_{21}, \ldots, x_j = z_{j+1\,j} - z_{j\,j-1}, \ldots, x_n = -z_{n\,n-1}$, and let $h_m(y_1, \ldots, y_s)$ be the m-th complete symmetric homogeneous function in $y_1, \ldots, y_s$. For any i, j the identity*

$$z_{ij} = h_{i-j}(x_1, \ldots, x_j)$$

holds in $A(Z)$. *In particular*, $A(Z) = C[x_1, \dots, x_n]/(\sigma_1, \dots, \sigma_n)$ *where* σ_i *is the i-th elementary symmetric function in* $x_1, \dots, x_n$.

(iii) *Under the isomorphism* $\psi\colon A(Z) \to H^*(F_n; C)$, $\psi(z_{ij}) = c_{i-j}(Q_j)$, *the* $(i-j)$*th Chern class of the universal quotient bundle* Q_j *of rank* $n-j$ *on* F_n.

Proof. To prove (i), it is enough to show that $i(V)(dz_{ij}) = V(z_{ij}) = a_{ij}(z)$. For this we need to compute the local expression of V in the local coordinates z_{ij}. Let $M = (z_{ij})$ be the $n \times n$ lower triangular unipotent matrix having z_{ij} as its entries. The change of the local coordinates z_{ij} by the action of $\exp(tn)$ around x_0 is given by the holomorphic functions $z_{ij}(t)$, $1 \leqslant j < i < n$, which satisfy the following matrix identity: $(*)$ $\exp(tn)MB_t = [z_{ij}(t)]$ for some $n \times n$ B_t in B, where $[z_{ij}(t)]$ is an $n \times n$ lower triangular unipotent matrix. We note that

$$\exp(tn)M = W(f_1, \dots, f_n) = \begin{bmatrix} f_1 & \dots f_n \\ f_1' & \dots f_n' \\ \vdots & \vdots \\ f_1^{(n-1)} & \dots f_n^{(n-1)} \end{bmatrix}$$

is a Wrońskian matrix, where $f_k = \sum_{i=0}^{n-k} \frac{t^{k+i-1}}{(k+i-1)!} z_{k+ik}$, $1 \leqslant k \leqslant n$. Let $|W(g_1, \dots g_l)|$ denote the determinant of the Wrońskian matrix $W(g_1, \dots, g_l)$, and $W(g_1, \dots, \hat{g}_i, \dots, g_l) = W(g_1, \dots, g_{i-1}, g_{i+1}, \dots, g_l)$, $W(\hat{g}_1) = 1$. By using standard formulas involving derivatives of determinants, one can check that the following matrix B_t, defined by

$$(B_t)_{ij} = \begin{cases} (-1^{i+j}|W(f_1, \dots, \hat{f}_i, \dots, f_j)|/|W(f_1, \dots, f_j)| & \text{if } i \leqslant j, \\ 0 & \text{otherwise,} \end{cases}$$

satisfies the identity $(*)$. From this, one obtains

$$z_{ij}(t) = \begin{vmatrix} f_1 & \dots f_j \\ \vdots & \vdots \\ f_1^{(j-2)} & \dots f_j^{(j-2)} \\ f_1^{(i-1)} & \dots f_j^{(i-1)} \end{vmatrix} \Big/ |W(f_1, \dots, f_j)|,$$

$1 \leqslant j < i \leqslant n$. By using, again, the formulas involving derivatives of determinants, one obtains

$$V(z_{ij}) = \frac{d}{dt}(z_{ij}(t))\Big|_{t=0} = z_{i+1\,j} - z_{ij-1} + z_{ij}(z_{jj-1} - z_{j+1\,j}).$$

For (ii), let $S = \{(i, j)\colon 1 \leqslant j < i \leqslant n\}$. Consider the partial order $\leqslant$ on S defined by: for (i, j) and (k, l) in S, $(i, j) \leqslant (k, l)$ if $i \leqslant k$, $j \leqslant l$. We prove the identity $z_{ij} = h_{i-j}(x_1, \ldots, x_j)$ in $A(Z)$ by induction on $(i, j) \in S$ relative to $\leqslant$. For the minimal element (2, 1) in S, we have, by definion, $z_{21} = h_1(x_1) = x_1$. For a given $(i+1, j) \in S$, by using the defining relations $a_{ij}(z) = 0$ in $A(Z)$ one obtains $z_{i+1\,j} = z_{i\,j-1} + x_j z_{ij}$, where $x_j = z_{j+1\,j} - z_{j\,j-1}$. Thus, by the induction hypothesis we get $z_{i+1\,j} = h_{i+1-j}(x_1, \ldots, x_{j-1}) + x_j h_{i-j}(x_1, \ldots, x_j)$. Since

$$h_{i+1-j}(x_1, \ldots, x_{j-1}) + x_j h_{i-j}(x_1, \ldots, x_j) = h_{i+1-j}(x_1, \ldots, x_j),$$

we get $z_{i+1\,j} = h_{i+1-j}(x_1, \ldots, x_j)$ in $A(Z)$, which gives the claim. This, in particular, implies that $z_{n+i\,n} = h_i(x_1, \ldots, x_n) = 0$ in $A(Z)$ for $i = 1, \ldots, n$. Thus, the ideal $(h_1(x_1, \ldots, x_n), \ldots, h_n(x_1, \ldots, x_n)) = (\sigma_1, \ldots, \sigma_n)$ lies in $I(Z)$. By comparing the dimensions we get $A(Z) = \mathbf{C}[x_1, \ldots, x_n]/I(Z)$ (this is the identification given by α in Theorem 2.6). Part (iii) follows from part (ii), Theorems 2.1, 2.6, and the well-known formula for $c_k(Q_j)$ in $H^*(F_n; \mathbf{C})$.

We shall now give the explicit description of the isomorphism $\psi\colon A(Z) \to H^*(G_{k,n}; \mathbf{C})$ by providing the representatives of Schubert cycles in $A(Z) = \mathbf{C}[z_{k+i\,j}\colon 1 \leqslant i \leqslant n-k,\ 1 \leqslant j \leqslant k]/I(Z)$.

For any permutation $\tau = (a_1, \ldots, a_n)$ in $W = S_n$, let $\tau(e)$ be the $n \times n$ permutation matrix obtained from the identity matrix e by permuting the rows relative to $(a_1, \ldots, a_n)$. Let $S = \{(i) = (i_1, \ldots, i_k)\colon 1 \leqslant i_1 < \ldots < i_k \leqslant n\}$. For any (i) in S there exists a unique permutation $(i_1, \ldots, i_k, i_{k+1}, \ldots, i_n)$ with the property $i_{k+1} < \ldots < i_n$. We denote this permutation by $\sigma(i)$. For $(i) = (i_1, \ldots, i_k)$ in S, let $X_{(i)} = \overline{B\sigma(i)(e)P}$ be the Schubert subvariety of $G_{k,n}$ associated to $1 \leqslant i_1 < \ldots < i_k \leqslant n$, let $\Omega(i_1, \ldots, i_k)$ be the Poincaré dual of the cycle class of the Schubert variety $X_{(n-i_k+1, \ldots, n-i_1+1)}$ in $H^*(G_{k,n}; \mathbf{C})$.

THEOREM 3.4. *For any* $1 \leqslant i_1 < i_2 < \ldots < i_k \leqslant n$, *we have* $\psi(P_{(i_1, \ldots, i_k)} \bmod I(Z)) = \Omega(i_1, \ldots, i_k)$, *where* $P_{(i_1, \ldots, i_k)}$ *is the Plücker coordinate of* $G_{k,n}$ *associated to* $1 \leqslant i_1 < \ldots < i_k \leqslant n$.

Proof. See [5]. It basically follows from Theorem 3.3 and the Schubert calculus.

COROLLARY. *For any Schubert subvariety* $X_{(i)}$ *of* $G_{k,n}$, *the graded algebra isomorphism* $\psi\colon A(Z) \to H^*(G_{k,n}; \mathbf{C})$ *induces an isomorphism* $\bar{\psi}\colon A(X_{(i)} \cap Z) \xrightarrow{\sim} H^*(X_{(i)}; \mathbf{C})$ *which commutes with the natural maps* $A(Z) \to A(X_{(i)} \cap Z)$ *and* $H^*(G_{k,n}; \mathbf{C}) \to H^*(X_{(i)}; \mathbf{C})$.

Proof. It follows from the theorem above and the fact that the ideal $I(X_{(i)})$ of $X_{(i)}$ at x_0 is generated by the Plücker coordinates $P_{(j)}$ with $(j) \nleqslant (i)$. For more details see [5].

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