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THE RANK OF G_0 FOR POLYCYCLIC GROUP ALGEBRAS

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Introduction

In this short note, we use J. A. Moody's fundamental induction theorem for polycyclic group algebras kG (stated below) to determine the rank of the Grothendieck group $G_0(kG)$ of the category of finitely generated kG-modules: Assuming that k is a splitting field for all finite subgroups of G, for simplicity, the rank of $G_0(kG)$ turns out to be equal to the number of G-conjugacy classes of torsion elements of G whose order is not divisible by char k. For a somewhat more general formulation, without a priori assumption on k, we refer to § 1 below. This result directly generalizes the case of finite groups G. Indeed, besides Moody's theorem, our main ingredient are certain standard techniques and results from finite group representation theory which we quote from [CR]. We remark, however, that, in contrast with the finite case, $G_0(kG)$ for general polycyclic-by-finite groups G usually has nontrivial torsion. This aspect of the structure of $G_0(kG)$ needs further clarification. Certain techniques for dealing with this question have been developed in [LP] where one can also find complete computations of $G_0(kG)$, including torsion, in a number of explicit examples.

For the convenience of the reader, we state Moody's result that was referred to above, concentrating on the case of group algebras. (Moody's theorem holds, more generally, for crossed products of polycyclic-by-finite groups over right Noetherian rings; see [M].)

MOODY'S INDUCTION THEOREM. *Let kG be the group algebra of the polycyclic-by-finite group G over the field k. Then, denoting by $\mathcal{F}$ the set of all*

This paper is in final form and no version of it will be submitted for publication elsewhere.

finite subgroups of G, the map

$$\bigoplus_{H\in\mathscr{F}} \mathrm{Ind}_H^G\colon \bigoplus_{H\in\mathscr{F}} G_0(kH) \to G_0(kG)$$

is surjective. (Here, $\mathrm{Ind}_H^G(\cdot) = (\cdot)\otimes_{kH} kG$, as usual.)

All modules considered here will be right modules. The element of G_0 corresponding to the finitely generated module V will be written as $[V]$. Further notation will be introduced as we go along.

1. Statement of the main result

Let G be a polycyclic-by-finite group and let k be a field with $\operatorname{char} k = p \geqslant 0$. Put

$$G_{p'} = \{x \in G \mid x \text{ has finite order } o(x) \text{ with } p \nmid o(x)\}$$

$$(= \{x \in G \mid x \text{ has finite order}\} \text{ in case } p = 0).$$

The elements of $G_{p'}$ are usually called the *p-regular torsion elements* of G, or simply *torsion elements* in case $p = 0$. Fix a positive integer m with

$$p \nmid m \text{ and } o(x) \mid m \text{ for all } x \in G_{p'}.$$

(For example, m can be taken to be the p'-part of the index of any torsion-free normal subgroup of finite index in G.) The Galois group $\mathrm{Gal}(k(\mu_m)/k)$, where μ_m denotes the group of mth roots of unity in an algebraic closure of k, is isomorphic to a subgroup of the group of units $U(\mathbf{Z}/m\mathbf{Z})$ in the usual fashion. Following [CR], this subgroup will be denoted by $I_m(k)$, so

$$\mathrm{Gal}(k(\mu_m)/k) \cong I_m(k) \subseteq U(\mathbf{Z}/m\mathbf{Z}).$$

Note that G acts on $G_{p'}$ by conjugation, and $U(\mathbf{Z}/m\mathbf{Z})$ also acts via $x^{t+m\mathbf{Z}} = x^t$ ($t \in \mathbf{Z}$, $x \in G_{p'}$). Since these two operations commute, we obtain an operation of the cartesian product $G \times I_m(k)$ on $G_{p'}$. The set of orbits under this action, $G_{p'}/G \times I_m(k)$, is independent of the particular chosen m and will be denoted by $T(G, k)$. So

$$T(G, k) = G_{p'}/G \times I_m(k).$$

It is a well-known fact that $T(G, k)$ is finite. In fact, any polycyclic-by-finite group G has only finitely many G-conjugacy classes of torsion elements (see Lemma 1 below). In case k is a splitting field for all finite subgroups of G, the action of $I_m(k)$ on $G_{p'}$ is trivial and so $T(G, k)$ is the set of G-conjugacy classes in $G_{p'}$. To see this, note that the assumption on k implies that, for any $x \in G_{p'}$, k contains a primitive root of unity of order $o(x)$. Since the operation of $I_m(k)$ on $\langle x \rangle$ factors through $\mathrm{Gal}(k(\mu_{o(x)})/k) = \langle 1 \rangle$, we conclude that x is fixed under $I_m(k)$.

Our goal here is to prove the following

THEOREM. *Let G be a polycyclic-by-finite group and let k be a field. Then $G_0(kG)$ is a finitely generated abelian group with* $\operatorname{rank} G_0(kG) = |T(G, k)|$.

In particular, if all torsion in G is p-torsion ($G_{p'} = \{1\}$) then $G_0(kG)$ has rank 1. We remark that finite generation of $G_0(kG)$ is immediate from Moody's theorem and is included in the result for completeness only.

2. Finiteness for conjugacy classes

Finiteness of $T(G, k)$ is a consequence of the following technical lemma which also proves that G has only finitely many conjugacy classes of finite subgroups.

LEMMA 1. *Let G be a polycyclic-by-finite group. Then there exists a torsion-free normal subgroup N of finite index in G having the following property:*

If U and V are finite subgroups of G so that U is conjugate to a subgroup of V modulo N, say $U^xN \subseteq VN$ ($x \in G$), then there exists an element $y \in G$ with $U^y \subseteq V$ and $u^y \equiv u^x \bmod N$ for all $u \in U$.

In particular, if two finite subgroups of G, or two torsion elements of G, are conjugate modulo N then they are conjugate in G.

Proof. We argue by induction on the Hirsch number of G. The case when G is finite being clear we assume that G is infinite. Then G has an infinite torsion-free abelian normal subgroup A ([P, Lemma 10.2.9]). Put $B = A^n$ where $n > 0$ is chosen so that $|U|$ divides n for all finite subgroups U of G. (For example, n can be taken to be the index of any torsion-free normal subgroup of finite index in G.) Then B is an infinite normal subgroup of G, and so G/B has smaller Hirsch number than G. Thus, by induction, G/B has an appropriate normal subgroup N/B. We will show that N works for G. So assume that $U^xN \subseteq VN$ holds for finite subgroups U, V of G and $x \in G$. Then, for some $z \in G$, $U^zB \subseteq VB$ and $u^z \equiv u^x \bmod N$ for all $u \in U$. In particular, each $u \in U$ can be uniquely written in the form

$$u = zvz^{-1}\varphi(u) \quad \text{with } v \in V \text{ and } \varphi(u) \in B.$$

One easily checks the equality $\varphi(u_1u_2) = \varphi(u_1)^{u_2}\varphi(u_2)$ for u_1, $u_2 \in U$. Thus, putting $b = \prod_{u \in U} \varphi(u) \in B$, we see that $b = \prod_{u_1 \in U} \varphi(u_1u) = b^u\varphi(u)^{|U|}$. Letting $a \in A$ be such that $a^{|U|} = b^{-1}$ (note that $B \subseteq A^{|U|}$), we deduce that

$$\varphi(u) = a^{-1}a^u$$

holds for all $u \in U$. Thus $u = zvz^{-1}a^{-1}u^{-1}au$, or

$$u = azvz^{-1}a^{-1} = v^{y^{-1}} \quad \text{with } y = az \in G.$$

Therefore, $u^y = v \in V$ and so $U^y \subseteq V$. Moreover, $u^y = v \equiv u^z \bmod B$ and

$u^z \equiv u^x \mod N$ together imply that $u^y \equiv u^x \mod N$ holds for all $u \in U$. Thus N has the required property.

The assertion about conjugacy of finite subgroups of G modulo N is clear. Finally, if two torsion elements h and k are conjugate modulo N, say $h^x \equiv k \mod N$, then the foregoing yields an element $y \in G$ with $h^y = k^r$ for some integer r and $h^y \equiv h^x \mod N$. But then $k^r \equiv k \mod N$ and so $k^r = k$, since N is torsion-free. This proves the lemma.

3. Finite groups and characters

Let H be a finite group and let k be a field with $\operatorname{char} k = p \geqslant 0$. Fix a p'-integer m dividing the order of each $x \in H_{p'}$, as in § 1, and put $k_1 = k_0(\mu_m)$, where k_0 is the prime subfield of k and μ_m is the group of mth roots of unity in an algebraic closure of k_0. In case $\operatorname{char} k = p > 0$, we fix a *p-modular system* (K, R, k_1), that is,

R is a discrete valuation ring with maximal ideal $\mathfrak{p}$ and $\operatorname{char} R = 0$,
$k_1 \cong R/\mathfrak{p}$, and
$K = \operatorname{Fract}(R)$ is the field of fractions of R.

(Such a p-modular system exists, e.g. by [CR, Proposition (16.21)].) In case $\operatorname{char} k = 0$, we simply take $K = k_1$. Finally, with $T(H, k)$ being defined as in § 1, we let $K^{T(H,k)}$ denote the K-vector space with basis $T(H, k)$ or, equivalently, the K-space of functions $\chi\colon T(H, k) \to K$. The following lemma is a standard result in finite group representation theory ([CR, Theorems (21.5) and (21.25) and their proofs]).

LEMMA 2. *Let H be a finite group and let k be a field. Let K be defined as above, so* $\operatorname{char} K = 0$. *Then we have an isomorphism of K-vector spaces*

$$(\operatorname{ch}_H)^{\sim}\colon\ G_0(kH) \otimes_{\mathbf{Z}} K \xrightarrow{\cong} K^{T(H,k)}.$$

The isomorphism is given by Brauer characters ([CR, Definition (21.26)]) in case $\operatorname{char} k > 0$, and by ordinary characters if $\operatorname{char} k = 0$.

4. Colimits over the Frobenius category $\mathscr{F} = \mathscr{F}(G)$

Let G be a polycyclic-by-finite group. The *Frobenius category* $\mathscr{F} = \mathscr{F}(G)$ of G is defined to have objects the set of all finite subgroups of G, and morphisms are inclusions of G-conjugates: If $H^x \subseteq E$ $(H, E \in \operatorname{ob}\mathscr{F},\ x \in G)$, then we have a morphism $H \to E$ given by $h \mapsto h^x$ $(h \in H)$. As usual, we will write $H \in \mathscr{F}$ instead of $H \in \operatorname{ob}\mathscr{F}$ in the following.

Now let k be a given ground field, $\operatorname{char} k = p \geqslant 0$. Fix a positive integer m as in § 1 and construct the field K from m and k as in § 3, so $\operatorname{char} K = 0$. For each $H \in \mathscr{F}$, we define the orbit set $T(H, k)$ as in § 1 and § 3: $T(H, k) = H_{p'}/H \times I_m(k)$.

Assume that $H \to E$ is a morphism in $\mathscr{F}$, given by the inclusion $H^x \subseteq E$ for $H, E \in \mathscr{F}$ and $x \in G$. Then we get a commutative diagram

$$\begin{tikzcd} T(G,k) \arrow[r, equal, "\mathrm{id}"] & T(G,k) \\ T(H,k) \arrow[u] \arrow[r] & T(E,k) \arrow[u] \end{tikzcd} \qquad \begin{tikzcd} h^{G\times I_m(k)} \arrow[r, equal] & (h^x)^{G\times I_m(k)} \\ h^{H\times I_m(k)} \arrow[u, mapsto] \arrow[r, mapsto] & (h^x)^{E\times I_m(k)} \arrow[u, mapsto] \end{tikzcd}$$

for $h \in H_{p'}$. By K-linear extension we obtain a corresponding commutative diagram of K-vector spaces

$$\begin{tikzcd} & K^{T(G,k)} & \\ K^{T(H,k)} \arrow[ur] \arrow[rr, "\tau_{H,E,x}"'] & & K^{T(E,k)} \arrow[ul] \end{tikzcd}$$

Similarly, induction of modules yields a commutative diagram

$$\begin{tikzcd} & G_0(kG) & \\ G_0(kH) \arrow[ur, "\mathrm{Ind}_H^G"] \arrow[rr, "\gamma_{H,E,x}"'] & & G_0(kE) \arrow[ul, "\mathrm{Ind}_E^G"'] \end{tikzcd} \qquad \begin{tikzcd} & {[V \otimes_{kH} kG]} & \\ {[V]} \arrow[ur] \arrow[rr, mapsto] & & {[V \otimes_{kH} kE] = [(\mathrm{Ind}_H^{E^{x^{-1}}}(V))^x]} \arrow[ul] \end{tikzcd}$$

Here, in $V \otimes_{kH} kE$, we view kE as a left kH-module via the map $kH \to kE$ given by $\alpha \mapsto \alpha^x$ ($\alpha \in kH$), and $(\cdot)^x$ denotes the x-conjugate of the module in question.

The assignments $H \mapsto K^{T(H,k)}$ and $H \mapsto G_0(kH)$ ($H \in \mathscr{F}$) together with the above maps yield two functors from $\mathscr{F}$ into the category of abelian groups $\mathscr{A}b$. The collection of Brauer or ordinary (if $\operatorname{char} k = 0$) character maps ch_H: $G_0(kH) \to K^{T(H,k)}$ ($H \in \mathscr{F}$) defines a natural transformation between these two functors. The required commutativity of the diagram

$$\begin{tikzcd} G_0(kH) \arrow[r, "\gamma_{H,E,x}"] \arrow[d, "\mathrm{ch}_H"'] & G_0(kE) \arrow[d, "\mathrm{ch}_E"] \\ K^{T(H,k)} \arrow[r, "\tau_{H,E,x}"'] & K^{T(E,k)} \end{tikzcd}$$

for $H^x \subseteq E (H, E \in \mathscr{F}$ and $x \in G)$ follows from [CR, Lemma (21.28) and (10.3)].

The colimits of the above two functors $\mathscr{F} \to \mathscr{A}b$ will be denoted by

$$\varinjlim_{H \in \mathscr{F}} K^{T(H,k)} \quad \text{and} \quad \varinjlim_{H \in \mathscr{F}} G_0(kH).$$

We explain the construction and universal property of $\varinjlim_{H \in \mathscr{F}} G_0(kH)$, the case of $\varinjlim_{H \in \mathscr{F}} K^{T(H,k)}$ being entirely analogous. To construct $\varinjlim_{H \in \mathscr{F}} G_0(kH)$, one forms the direct sum $\bigoplus_{H \in \mathscr{F}} G_0(kH)$ and factors out the subgroup that is generated by the elements $\sigma_H(\alpha_H) - \sigma_E \gamma_{H,E,x}(\alpha_H)$ ($\alpha_H \in G_0(kH)$, $H^x \subseteq E$), where σ_H: $G_0(kH)$

$\to \bigoplus_{H\in\mathscr{F}} G_0(kH)$ is the canonical embedding. Letting $\psi_H\colon\ G_0(kH) \to \varinjlim_{H\in\mathscr{F}} G_0(kH)$ denote the map σ_H followed by the canonical projection of $\bigoplus_{H\in\mathscr{F}} G_0(kH)$ onto $\varinjlim_{H\in\mathscr{F}} G_0(kH)$, it is clear that $\varinjlim_{H\in\mathscr{F}} G_0(kH)$ is characterized by the following *universal property*: The maps $\psi_H\colon\ G_0(kH) \to \varinjlim_{H\in\mathscr{F}} G_0(kH)$ satisfy $\psi_H = \psi_E \circ \gamma_{H,E,x}$ whenever $H^x \subseteq E$ $(H,\ E\in\mathscr{F},\ x\in G)$. Moreover, if $f_H\colon\ G_0(kH) \to A$ (some abelian group) is another collection of maps satisfying the corresponding equalities $f_H = f_E \circ \gamma_{H,E,x}$, then there exists a unique homomorphism $f\colon\ \varinjlim_{H\in\mathscr{F}} G_0(kH) \to A$ with $f_H = f \circ \psi_H$ for all $H \in \mathscr{F}$.

We note some immediate consequences of the above description. The construction of $\varinjlim_{H\in\mathscr{F}} G_0(kH)$ makes it clear that this group is generated by the subgroups $\psi_H(G_0(kH))$, where H runs through some representative set of G-conjugacy classes of maximal members of $\mathscr{F}$. Since such a representative set is finite, by Lemma 1, and since all $G_0(kH)$ are finitely generated, it follows that $\varinjlim_{H\in\mathscr{F}} G_0(kH)$ is a finitely generated abelian group.

The universal property, applied with $A = G_0(kG)$ and $f_H = \mathrm{Ind}_H^G$, yields a homomorphism

$$\mathrm{Ind}_{\mathscr{F}}^G\colon\ \varinjlim_{H\in\mathscr{F}} G_0(kH) \to G_0(kG)$$

extending the induction maps Ind_H^G $(H\in\mathscr{F})$. Similarly, the natural transformation given by the character maps ch_H $(H\in\mathscr{F})$ yields a homomorphism

$$\varinjlim_{H\in\mathscr{F}} \mathrm{ch}_H\colon\ \varinjlim_{H\in F} G_0(kH) \to \varinjlim_{H\in\mathscr{F}} K^{T(H,k)}.$$

LEMMA 3 (notations as above). (a) $\mathrm{Ind}_{\mathscr{F}}^G\colon\ \varinjlim_{H\in\mathscr{F}} G_0(kH) \to G_0(kG)$ *is surjective. In particular*, $G_0(kG)$ *is a finitely generated abelian group.*

(b) $\varinjlim_{H\in\mathscr{F}} K^{T(H,k)} \cong K^{T(G,k)}$. *Under this isomorphism, the canonical map* $K^{T(H,k)} \to \varinjlim K^{T(H,k)}$ *becomes the K-linear map sending the orbit* $h^{H\times I_m(k)}$ *to* $h^{G\times I_m(k)}$ $(h\in H)$.

(c) *The map* $\varinjlim_{H\in\mathscr{F}} \mathrm{ch}_H$ *together with the isomorphism in* (b) *yield a K-linear isomorphism* $(\mathrm{ch}_{\mathscr{F}})^{\sim}\colon\ \big(\varinjlim_{H\in\mathscr{F}} G_0(kH)\big)\otimes_{\mathbf{Z}} K \xrightarrow{\cong} K^{T(G,k)}$.

Proof. (a) is a restatement of Moody's induction theorem: $G_0(kG)$ is generated by the images of the maps Ind_H^G $(H\in\mathscr{F})$.

(b) We have already noted above that the maps $K^{T(H,k)} \to K^{T(G,k)}$ sending $h^{H\times I_m(k)}$ to $h^{G\times I_m(k)}$ $(h\in H)$ satisfy the required commutativity conditions, and so we get a homomorphism $f\colon\ L := \varinjlim K^{T(H,k)} \to K^{T(G,k)}$ extending these maps. Letting $\varphi_H\colon\ K^{T(H,k)} \to L$ denote the canonical map, with $\varphi_H = \varphi_E \circ \tau_{H,E,x}$

whenever $H^x \subseteq E$ (above notation), we can define a back-map $g\colon K^{T(G,k)} \to L$ as follows. For a given $x \in G_{p'}$ pick any $H \in \mathscr{F}$ with $x^y \in H$ for some $y \in G$ and send the orbit $x^{G \times I_m(k)}$ to $\varphi_H((x^y)^{H \times I_m(k)}) = \varphi_{\langle x \rangle}(x^{\langle x \rangle \times I_m(k)}) \in L$. One checks that this is the required inverse for f.

(c) Composition of $\varinjlim_{H \in \mathscr{F}} \mathrm{ch}_H\colon \varinjlim_{H \in \mathscr{F}} G_0(kH) \to \varinjlim_{H \in \mathscr{F}} K^{T(H,k)}$ with the isomorphism in (b) gives a homomorphism

$$\mathrm{ch}_{\mathscr{F}}\colon \varinjlim_{H \in \mathscr{F}} G_0(kH) \to K^{T(G,k)}.$$

Tensoring this with K we obtain a linear map

$$(\mathrm{ch}_{\mathscr{F}})^{\sim} = \mathrm{ch}_{\mathscr{F}} \otimes_{\mathbf{Z}} \mathrm{id}_K\colon \big(\varinjlim_{H \in \mathscr{F}} G_0(kH)\big) \otimes_{\mathbf{Z}} K \to K^{T(G,k)}.$$

Since the functor $(\cdot) \otimes_{\mathbf{Z}} K$ preserves colimits ([MacL, p. 115]), $(\mathrm{ch}_{\mathscr{F}})^{\sim}$ is identical with the composite map

$$\big(\varinjlim_{H \in \mathscr{F}} G_0(kH)\big) \otimes_{\mathbf{Z}} K \xrightarrow{\cong} \varinjlim_{H \in \mathscr{F}} (G_0(kH) \otimes_{\mathbf{Z}} K) \xrightarrow{\delta} \varinjlim_{H \in \mathscr{F}} K^{T(H,k)} \xrightarrow{\cong} K^{T(G,k)},$$

where $\delta := \varinjlim_{H \in \mathscr{F}} (\mathrm{ch}_H)^{\sim}$. Finally, Lemma 2 implies that δ is an isomorphism, and hence so is $(\mathrm{ch}_{\mathscr{F}})^{\sim}$. This completes the proof.

5. Proof of the main result

As before, let G be a polycyclic-by-finite group and let k be a field of characteristic $p \geqslant 0$. We already know, by Lemma 3(a), that $G_0(kG)$ is finitely generated. It remains to establish the equality rank $G_0(kG) = |T(G, k)|$. This will be a consequence of the following more precise result. Here, the field K is chosen as in § 3 and § 4.

LEMMA 4. *The map* $\mathrm{Ind}_{\mathscr{F}}^{G}\colon \varinjlim_{H \in \mathscr{F}} G_0(kH) \twoheadrightarrow G_0(kG)$ *gives rise to a K-linear isomorphism*

$$(\mathrm{Ind}_{\mathscr{F}}^{G})^{\sim} = \mathrm{Ind}_{\mathscr{F}}^{G} \otimes_{\mathbf{Z}} \mathrm{id}_K\colon \big(\varinjlim_{H \in \mathscr{F}} G_0(kH)\big) \otimes_{\mathbf{Z}} K \xrightarrow{\cong} G_0(kG) \otimes_{\mathbf{Z}} K.$$

Consequently, $G_0(kG) \otimes_{\mathbf{Z}} K \cong K^{T(G,k)}$.

Proof. In view of Lemma 3, it suffices to show that $(\mathrm{Ind}_{\mathscr{F}}^{G})^{\sim}$ is injective. For this, we may enlarge K if necessary. Choose N as in Lemma 1 and let $^{-}\colon G \to G/N$ denote the canonical map. We may assume that K is a splitting field for $\bar{G}$ as well. So $(\mathrm{ch}_{\bar{G}})^{\sim}\colon G_0(k\bar{G}) \otimes_{\mathbf{Z}} K \xrightarrow{\cong} K^{T(\bar{G},k)}$ via Brauer or ordinary characters, by Lemma 2.

Since kN has finite global dimension (N is torsion-free), we have a homomorphism

$$\pi = \pi_{G,N}\colon\ G_0(kG) \to G_0(k\bar{G}),$$
$$[V] \mapsto \sum_{i\geq 0} (-1)^i[\mathrm{Tor}_i^{kG}(V, k\bar{G})] = \sum_{i\geq 0} (-1)^i[H_i(N, V)]$$

(see [B, p. 454]). For example, if $V|_{kN}$ is projective then $\pi([V]) = [V/V(\omega N)]$, where ωN is the augmentation ideal of kN. One easily checks that, for any subgroup H of G, the following diagram commutes:

$$\begin{tikzcd} G_0(kH) \arrow[r, "\pi_{H,N\cap H}"] \arrow[d, "\mathrm{Ind}_H^G"'] & G_0(k\bar{H}) \arrow[d, "\mathrm{Ind}_{\bar{H}}^{\bar{G}}"] \\ G_0(kG) \arrow[r, "\pi_{G,N}"'] & G_0(k\bar{G}) \end{tikzcd}$$

For $H \in \mathscr{F}$ one has $N \cap H = \langle 1 \rangle$, and $\pi_{H,N\cap H}$ is the obvious isomorphism $G_0(kH) \xrightarrow{\cong} G_0(k\bar{H})$ coming from the isomorphism $H \cong \bar{H}$. The maps $G_0(kH) \xrightarrow{\cong} G_0(k\bar{H}) \xrightarrow{\mathrm{Ind}_{\bar{H}}^{\bar{G}}} G_0(k\bar{G})$ $(H \in \mathscr{F})$ give rise to a map

$$\mathrm{Ind}_{\mathscr{F}}^{\bar{G}}\colon\ L := \varinjlim_{H\in\mathscr{F}} G_0(kH) \to G_0(k\bar{G}).$$

In view of the above commutative diagram, $\mathrm{Ind}_{\mathscr{F}}^{\bar{G}}$ factors as follows:

$$\begin{tikzcd} L = \varinjlim_{H\in\mathscr{F}} G_0(kH) \arrow[rr, "\mathrm{Ind}_{\mathscr{F}}^{\bar{G}}"] \arrow[dr, "\mathrm{Ind}_{\mathscr{F}}^{G}"'] & & G_0(k\bar{G}) \\ & G_0(kG) \arrow[ur, "\pi"'] & \end{tikzcd}$$

Thus, in order to show that $(\mathrm{Ind}_{\mathscr{F}}^{G})^{\sim}$ is injective, it suffices to prove the injectivity of $(\mathrm{Ind}_{\mathscr{F}}^{\bar{G}})^{\sim} = \mathrm{Ind}_{\mathscr{F}}^{\bar{G}} \otimes_{\mathbf{Z}} \mathrm{id}_K\colon L \otimes_{\mathbf{Z}} K \to G_0(k\bar{G}) \otimes_{\mathbf{Z}} K$. This is a consequence of the following commutative diagram:

$$\begin{tikzcd} K^{T(G,k)} \arrow[r] & K^{T(\bar{G},k)} \\ L \otimes K \arrow[u, "\cong", "(\mathrm{ch}_{\mathscr{F}})^{\sim}"'] \arrow[r, "(\mathrm{Ind}_{\mathscr{F}}^{\bar{G}})^{\sim}"'] & G_0(k\bar{G}) \otimes K \arrow[u, "(\mathrm{ch}_{\bar{G}})^{\sim}"'] \end{tikzcd}$$

Here, the top horizontal arrow sends the $G \times I_m(k)$-orbit of $x \in G_{p'}$ to the $\bar{G} \times I_m(k)$-orbit of $\bar{x} \in \bar{G}$. By Lemma 1, this map is injective, and hence so is $(\mathrm{Ind}_{\mathscr{F}}^{\bar{G}})^{\sim}$. This proves the lemma, and hence the theorem.

We remark that, if N is chosen as in Lemma 1, then $\varinjlim_{H\in\mathscr{F}} G_0(kH)$ is isomorphic with $\varinjlim_{\bar{H}\in\bar{\mathscr{F}}} G_0(k\bar{H})$, where $\bar{\mathscr{F}}$ is the full subcategory of $\mathscr{F}(G/N)$ having objects $\{\bar{H} = HN/N \mid H \in \mathscr{F}\}$. This follows from Lemma 1. The above argument also proves this fact modulo torsion.

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References

[B] H. Bass, *Algebraic K-Theory*, Benjamin, New York 1968.

[CR] C. W. Curtis and I. Reiner, *Methods of Representation Theory*, Vol. 1, Wiley-Interscience, New York 1981.

[LP] M. Lorenz and D. S. Passman, *The structure of G_0 for certain polycyclic group algebras and related algebras*, Contemp. Math., to appear.

[MacL] S. MacLane, *Categories for the Working Mathematician*, Springer, New York 1971.

[M] J. A. Moody, *A Brauer induction theorem for G_0 of certain infinite groups*, J. Algebra, to appear.

[P] D. S. Passman, *The Algebraic Structure of Group Rings*, Wiley-Interscience, New York 1977.

Added in proof (April 10, 1990). The equation $\mathrm{ch}_E \circ \gamma_{H,E,x} = \tau_{H,E,x} \circ \mathrm{ch}_H$ which is claimed to hold on p. 49 requires a somewhat modified definition of ch_X ($X \in \mathscr{F}$). Namely, one must include a factor $|C_X(x)|^{-1}$ in the definition of ch_X. The rest works as before. We take this opportunity to announce that a forthcoming article *Colimits of functors and Grothendieck groups of polycyclic group algebras* (with K. A. Brown) contains a proof of our main result in a somewhat more general setting, as well as further related results on $G_0(kG)$.

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