TOPICS IN ALGEBRA
BANACH CENTER PUBLICATIONS, VOLUME 26, PART 2
PWN POLISH SCIENTIFIC PUBLISHERS
WARSAW 1990

SYMMETRIC FUNCTIONS AND THE CHERN CHARACTERS OF A HYPERSURFACE WITH SINGULARITIES

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The theory of Chern characteristic classes can be considered as the modern form of Schubert's enumerative theory. Moreover, the Schubert calculus and symmetric functions are closely connected. This enables mathematicians to use symmetric functions in the study of Chern classes (see [L1], [ST]).

We study here some new family of symmetric functions in two sets of variables $\{x_i\}$, $\{y_j\}$, indexed by partitions.

In Section 2, we give a basis of the linear space $\mathrm{CH}^{(k)}(x, y)$ that they generate (Lemma 2.2 and Theorem 2.3). In Section 3, we take the polynomials in $\mathrm{CH}^{(k)}(x, y)$ which depend only upon one set of variables and get inequalities for the coefficients of their expansion. In Section 4, we obtain equalities between some elements of $\mathrm{CH}^{(k)}(x, y)$.

Finally in Section 5, we relate these polynomials to the Chern characters of hypersurfaces with singularities, as defined by Wu, and thus obtain by purely algebraic methods equalities and inequalities for Chern characters of hypersurfaces (Theorem 5.1, 5.2, 5.3, and Corollary 5.4).

1. A family of symmetric polynomials

Recall that the total Chern class of a vector bundle $\boldsymbol{E}$ can be formally factorized into the product $C(\boldsymbol{E}) = \prod(1+x_i)$ (see [HI]). Using symmetric functions, A. Lascoux [L2] calculated the Chern classes of the exterior power $\Lambda^2 \boldsymbol{E}$ and of symmetric power $S^2 \boldsymbol{E}$ of a vector bundle $\boldsymbol{E}$, also the Chern classes of a tensor product $\boldsymbol{E} \otimes \boldsymbol{F}$ of vector bundles. H. S. Tai [T] noticed that a certain family of symmetric functions is related to the Chern classes of algebraic varieties with ample canonical bundle.

This paper is in final form and no version of it will be submitted for publication elsewhere.

In this section we use other symmetric polynomials than the above-mentioned authors; these polynomials are related to the Chern characteristic classes of algebraic hypersurfaces.

Let $e(x) = x_1 + x_2 + \ldots + x_n$ and $e(y) = y_1 + y_2 + \ldots + y_m$ be the first elementary symmetric polynomial in the variables $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_m$ respectively. We consider now the symmetric polynomials in both sets of variables x and y defined by

$$\mathrm{CH}_i(x, y) = \sum_{j=0}^{i} (-1)^j \binom{n+1-j}{i-j} e^{i-j}(y) e^j(x), \qquad i = 1, 2, \ldots, n, \tag{1.1}$$

where $\binom{n}{m} = \dfrac{n!}{m!\,(n-m)!}$ are the binomial coefficients.

For any positive integer k, $k \leqslant n$, let $\pi = (\pi_1, \pi_2, \ldots, \pi_j)$ be a partition of k, i.e., $\pi_1 \geqslant \pi_2 \geqslant \ldots \geqslant \pi_j \geqslant 0$ and $\pi_1 + \pi_2 + \ldots + \pi_j = k$. We define the symmetric polynomial corresponding to the partition π as the product

$$\mathrm{CH}_\pi(x, y) = \mathrm{CH}_{\pi_1}(x, y) \cdot \mathrm{CH}_{\pi_2}(x, y) \cdot \ldots \cdot \mathrm{CH}_{\pi_j}(x, y). \tag{1.2}$$

This polynomial can be expanded as follows

$$\begin{aligned} \mathrm{CH}_\pi(x, y) &= \prod_{i=1}^{j} \left\{ \sum_{t=0}^{\pi_i} (-1)^t \binom{n+1-t}{\pi_i - t} e^{\pi_i - t}(y)\, e^t(x) \right\} \\ &= \sum_{i=0}^{k} a_i(\pi) \cdot e^{k-i}(y) e^i(x). \end{aligned} \tag{1.3}$$

Let us consider the linear space, denoted by $\mathrm{CH}^{(k)}(x, y)$, spanned by the set

$$\{\mathrm{CH}_\pi(x, y) \mid \pi \text{ partition of } k\}.$$

From expansion (1.3), it is easy to see that every element in $\mathrm{CH}^{(k)}(x, y)$ can be expressed as a linear combination of $e^{k-i}(y) e^i(x)$, $i = 0, 1, \ldots, k$. We are interested in those elements in the linear space $\mathrm{CH}^{(k)}(x, y)$ which are independent of the variables $y_1, y_2, \ldots, y_m$ (that is such that $a_0 = a_1 = \ldots \ldots = a_{k-1} = 0$). We denote the subset of such elements by $\mathrm{E}^{(k)}(x)$.

In order to describe $\mathrm{E}^{(k)}(x)$ we introduce an extra parameter t, and study the invariance under the change of variables $y_i \to y_i + t$, $i = 1, 2, \ldots, m$. First, for what concerns the derivatives of the $\mathrm{CH}_i(x, y)$, we have the following relation:

LEMMA 1.1 *The symmetric polynomials* $\mathrm{CH}_i(x, y)$ *satisfy*

$$\frac{d}{dt} \mathrm{CH}_i(x, y+t)|_{t=0} = m \cdot (n+2-i) \cdot \mathrm{CH}_{i-1}(x, y), \qquad i = 1, 2, \ldots, n. \tag{1.4}$$

Proof. It is clear that

$$\frac{d}{dt} e^{i-j}(y+t)|_{t=0} = m \cdot (i-j) \cdot e^{i-j-1}(y). \tag{1.5}$$

By Definition (1.1), we get

$$\begin{aligned}\frac{d}{dt}\mathrm{CH}_i(x, y+t)|_{t=0} &= \sum_{j=0}^{i}(-1)^j\binom{n+1-j}{i-j}e^j(x)\frac{d}{dt}e^{i-j}(y+t)|_{t=0}\\ &= \sum_{j=0}^{i-1}(-1)^j\binom{n+1-j}{i-j}\cdot e^j(x)\cdot m\cdot(i-j)\cdot e^{i-j-1}(y)\\ &= \sum_{j=0}^{i-1}(-1)^j\cdot m\cdot(n+2-i)\cdot\binom{n+1-j}{i-j-1}\cdot e^{i-j-1}(y)e^j(x)\\ &= m\cdot(n+2-i)\cdot\mathrm{CH}_{i-1}(x, y), \qquad i = 1, 2, .., n.\end{aligned}$$

The lemma is proved. ∎

Now, for the elements in $\mathrm{CH}^{(k)}(x, y)$ we have

THEOREM 1.2. *For any positive integer k, $1 \leqslant k \leqslant n$, the element $W(x, y)$ in $\mathrm{CH}^{(k)}(x, y)$ is independent of the variables $y_1, y_2, \ldots, y_m$, in other words, $W(x, y)$ belongs to $\mathbf{E}^{(k)}(x)$ if and only if $W(x, y)$ satisfies*

$$(1.6)\qquad (n+1)\frac{\partial W(x, y)}{\partial \mathrm{CH}_1(x, y)}+n\cdot\mathrm{CH}_1(x, y)\frac{\partial W(x, y)}{\partial \mathrm{CH}_2(x, y)}+\ldots +(n+2-k)\cdot\mathrm{CH}_{k-1}(x, y)\frac{\partial W(x, y)}{\partial \mathrm{CH}_k(x, y)} = 0.$$

In fact, the element $W(x, y)$ is independent of the variables $y_1, y_2, \ldots, y_m$ if and only if

$$(1.7)\qquad \frac{d}{dt}W(x, y+t)|_{t=0} = 0.$$

By the chain rule of differentiation we have

$$\frac{d}{dt}W(x, y+t)|_{t=0} = \sum_{i=1}^{k}\frac{\partial W(x, y)}{\partial \mathrm{CH}_i(x, y)}\cdot\frac{d}{dt}\mathrm{CH}_i(x, y+t)|_{t=0} = 0,$$

and this together with (1.4) implies (1.6). ∎

2. Dimension of the linear space $\mathrm{CH}^{(k)}(x, y)$

The linear space $\mathbf{CH}^{(k)}(x, y)$ is spanned by the set

$$\{\mathrm{CH}_\pi(x, y)|\pi \text{ partition of } k\}.$$

That is, the elements in $\mathbf{CH}^{(k)}(x, y)$ are linear combinations of the $\mathrm{CH}_\pi(x, y)$. But these polynomials are not linearly independent for $k \geqslant 4$ and we will determine a subfamily which is a basis of $\mathbf{CH}^{(k)}(x, y)$.

LEMMA 2.1. *For any positive integer k, $k \leqslant n$, let $\pi = (\pi_1, \pi_2, \ldots, \pi_j)$ be an arbitrary partition of k. If we expand the symmetric polynomial $\mathrm{CH}_\pi(x, y)$ corresponding to π as*

$$\mathrm{CH}_\pi(x, y) = \sum_{i=0}^{k} a_i(\pi) \cdot e^{k-i}(y) e^i(x), \tag{2.1}$$

then we have

$$a_1(\pi) = -\frac{k}{n+1} \cdot a_0(\pi). \tag{2.2}$$

Proof. From the expansion (1.3) we have

$$a_0(\pi) = \binom{n+1}{\pi_1}\binom{n+1}{\pi_2}\cdots\binom{n+1}{\pi_j}$$

and

$$-a_1(\pi) = \sum_{i=1}^{j} \binom{n+1}{\pi_1}\binom{n+1}{\pi_2}\cdots\overline{\binom{n+1}{\pi_i}}\cdots\binom{n+1}{\pi_j}\cdot\binom{n}{\pi_i - 1},$$

where the notation $\overline{\binom{n+1}{\pi_i}}$ stands for the absence of the factor enclosed. Hence we get

$$-a_1(\pi) = \sum_{i=1}^{j} a_0(\pi)\cdot\binom{n}{\pi_i - 1}\binom{n+1}{\pi_i}^{-1} = a_0(\pi)\cdot\sum_{i=1}^{j}\frac{\pi_i}{n+1} = \frac{k}{n+1}\cdot a_0(\pi),$$

as desired. ∎

For any positive integer k, we consider now the so-called hooked partitions of k, i.e., the partitions $\eta_i = (i, 1, \ldots, 1)$ ($k-i$ ones), $i = 1, 2, \ldots, k$, and denote the symmetric polynomials corresponding to the hooked partitions η_i by $\mathrm{CH}_{\eta_i}(x, y)$.

LEMMA 2.2. *The symmetric polynomials $\mathrm{CH}_{\eta_i}(x, y)$, $i = 1, 2, \ldots, k$, corresponding to hooked partitions of k are linearly independent.*

Proof. We use induction on k. When $k = 1$, the lemma is trivial. We asumme that the lemma is true for $k-1$. We have to prove that if

$$\sum_{i=1}^{k} \boldsymbol{b}_i \cdot \mathrm{CH}_{\eta_i}(x, y) = 0. \tag{2.3}$$

then $\boldsymbol{b}_i = 0$, $i = 1, 2, \ldots, k$.

For any hooked partition of k, if we expand the symmetric polynomial $\mathrm{CH}_{\eta_i}(x, y)$ as

$$\mathrm{CH}_{\eta_i}(x, y) = \sum_{i=1}^{k} \boldsymbol{a}_i(\eta_i) \cdot e^{k-i}(y) e^i(x),$$

then by definition, the coefficient $\boldsymbol{a}_k(\eta_i)$ is $(-1)^k$. Now (2.3) becomes

$$\sum_{i=1}^{k} \boldsymbol{b}_i \cdot \mathrm{CH}_{\eta_i}(x, y) = \sum_{i=1}^{k} \boldsymbol{a}_i(k) \cdot e^{k-i}(y) e^i(x) = 0.$$

The coefficient of $e^k(x)$ in the last summation is

$$\boldsymbol{a}_k(k) = \sum_{i=1}^{k} (-1)^k \cdot \boldsymbol{b}_i = 0. \tag{2.4}$$

It is trivial that summation (2.3) is independent of the variables $y_1, y_2, \ldots, y_m$, hence it satisfies condition (1.6). However, for the hooked partitions we have

$$\begin{aligned}
\frac{d}{dt}\mathrm{CH}_{\eta_i}(x, y+t)|_{t=0} &= \frac{d}{dt}\mathrm{CH}_i(x, y+t)\mathrm{CH}_1^{k-i}(x, y+t)|_{t=0} \\
&= m\cdot(n+2-i)\mathrm{CH}_{i-1}(x, y)\mathrm{CH}_1^{k-i}(x, y) \\
&\quad + m\cdot(k-i)(n+1)\mathrm{CH}_i(x, y)\mathrm{CH}_1^{k-i-1}(x, y) \\
&= m\cdot(n+2-i)\mathrm{CH}_{\eta_{i-1}(k-1)}(x, y) + m\cdot(k-i)(n+1)\mathrm{CH}_{\eta_i(k-1)}(x, y), \\
&\qquad\qquad i = 2, 3, \ldots, k.
\end{aligned} \tag{2.5}$$

So that condition (1.6) becomes

$$\begin{aligned}
\frac{d}{dt}\sum_{i=1}^{k} \boldsymbol{b}_i \cdot \mathrm{CH}_{\eta_i}(x, y+t)|_{t=0} &= \sum_{i=1}^{k} \boldsymbol{b}_i \cdot \frac{d}{dt}\mathrm{CH}_{\eta_i}(x, y+t)|_{t=0} \\
&= m\cdot k\cdot(n+1)\cdot \boldsymbol{b}_1 \cdot \mathrm{CH}_{\eta_1(k-1)}(x, y) + m\cdot n\cdot \boldsymbol{b}_2 \cdot \mathrm{CH}_{\eta_1(k-1)}(x, y) \\
&\quad + m\cdot \sum_{i=1}^{k-1} \{(k-i)(n+1)\cdot \boldsymbol{b}_i + (n+1-i)\cdot \boldsymbol{b}_{i+1}\}\cdot \mathrm{CH}_{\eta_i(k-1)}(x, y).
\end{aligned}$$

By the induction hypothesis, the $k-1$ symmetric polynomials $\mathrm{CH}_{\eta_i(k-1)}(x, y)$ are linearly independent, thus we obtain

$$\begin{aligned}
&k(n+1)\cdot \boldsymbol{b}_1 + n\cdot \boldsymbol{b}_2 = 0, \\
&(k-i)(n+1)\cdot \boldsymbol{b}_i + (n+1-i)\cdot \boldsymbol{b}_{i+1} = 0, \qquad i = 2, 3, \ldots, k-1.
\end{aligned} \tag{2.6}$$

The solutions of this system are

$$\begin{aligned}
&\boldsymbol{b}_1 = (-1)^{k-1}(n+1)^{-(k-1)}\cdot\frac{n}{k}\cdot\binom{n-1}{k-2}\cdot \boldsymbol{b}_k, \\
&\boldsymbol{b}_i = (-1)^{k-i}(n+1)^{-(k-i)}\binom{n+1-i}{k-i}\cdot \boldsymbol{b}_k, \qquad i = 2, 3, \ldots, k-1.
\end{aligned} \tag{2.7}$$

And from (2.4) we have

$$\sum_{i=1}^{k} \boldsymbol{b}_i = \boldsymbol{B}\cdot \boldsymbol{b}_k = 0, \tag{2.8}$$

where

$$B = 1+\sum_{i=2}^{k-1}(-1)^{k-i}(n+1)^{-k+i}\binom{n+1-i}{k-i}+(-1)^{k-1}(n+1)^{-k+1}\cdot\frac{n}{k}\cdot\binom{n-1}{k-2}.$$

But the constant B is not equal to zero (see Lemma 3.1 in the next section), hence equation (2.8) leads to $\boldsymbol{b}_k = 0$, and therefore $\boldsymbol{b}_i = 0$, $i = 1, 2, \ldots, k-1$, as desired. ■

Now we establish the main result of this section.

THEOREM 2.3. *For any given positive integer k, $k \leqslant n$, the dimension of the linear space* $\mathbf{CH}^{(k)}(x, y)$ *is k.*

Proof. From expansion (1.3) we know that every element in the linear space $\mathbf{CH}^{(k)}(x, y)$ is a linear combination of the $e^{k-i}(y)e^{i}(x)$, $i = 0, 1, \ldots, k$. Therefore the dimension of $\mathbf{CH}^{(k)}(x, y)$ is less than or equal to $k+1$. Taking into account the relation between the coefficients of $e^k(y)$ and $e^{k-1}(y)e(x)$ given by Lemma 2.1, we see that in fact the dimension of $\mathbf{CH}^{(k)}(x, y)$ is at most k.

On the other hand, Lemma 2.2 giving k linearly independent elements, we conclude that the dimension of $\mathbf{CH}^{(k)}(x, y)$ is exactly k. ■

3. Positivity properties

Let $s_\lambda(x)$ denote the Schur function in the variables $x_1, x_2, \ldots, x_n$, corresponding to the partition λ of n. We have on the space of symmetric functions a scalar product (see [M]) for which

$$\langle e^n(x), s_\lambda(x)\rangle = X^\lambda(1^n), \tag{3.1}$$

where $X^\lambda(1^n)$ is the number of standard tableaux of shape λ, hence is a positive integer; it is also the dimension of the irreducible representation of the symmetric group corresponding to the partition λ (see [J-K], [SH4]).

Since the elements of $\mathbf{E}^k(x)$, i.e., the elements in $\mathbf{CH}^{(k)}(x, y)$ independent of variables $y_1, y_2, \ldots, y_m$, have the following form

$$\sum_\pi c(\pi)\cdot \mathrm{CH}_\pi(x, y) = \boldsymbol{C}\cdot e^k(x), \tag{3.2}$$

where $\boldsymbol{C}$ is a constant, we can rewrite (3.1) as

$$\langle e^{n-k}(x)\cdot\sum_\pi c(\pi)\mathrm{CH}_\pi(x, y), s_\lambda(x)\rangle = \boldsymbol{C}\cdot\langle e^n(x), s_\lambda(x)\rangle, \tag{3.3}$$

where $s_\lambda(x)$ is any Schur function corresponding to a partition λ of n. Thanks to the positivity of the number $X^\lambda(1^n)$, we introduce

DEFINITION. For every element of $\mathbf{E}^k(x)$

$$W(x) = \sum_\pi c(\pi)\cdot \mathrm{CH}_\pi(x, y) = \boldsymbol{C}\cdot e^k(x),$$

we say it is *positive* (or *negative*) if the constant C is positive (or negative respectively).

To study the positivity of elements of $\mathbf{E}^k(x)$, we prove first

LEMMA 3.1. *For any positive integer k, $2 \leqslant k \leqslant n$, we set*

$$
\begin{aligned}
c_1(k) &= (-1)^{k-1}\cdot\frac{n}{k}\cdot\binom{n-1}{k-2},\\
c_i(k) &= (-1)^{k-i}(n+1)^{i-1}\binom{n+1-i}{k-i}, \qquad i = 2, 3, \ldots, k.
\end{aligned}
\tag{3.4}
$$

Then the sum

$$C_k = c_1(k)+c_2(k)+\ldots+c_k(k)$$

is positive.

Proof. We write

$$C_k = \sum_{j=0}^{[\frac{k}{2}]} \{c_{k-2j}(k)+c_{k-2j-1}(k)\},$$

where $[k/2]$ means the integer part of $k/2$ and where $c_0(k) = c_{-1}(k) = 0$. Using the equation

$$\binom{n+1-(k-2j-1)}{2j+1} = \frac{n+1-(k-2j-1)}{2j+1}\cdot\binom{n+1-(k-2j)}{2j},$$

we have

$$
\begin{aligned}
&c_{k-2j}+c_{k-2j-1}\\
&\quad = (n+1)^{k-2j-1}\binom{n+1-(k-2j)}{2j} - (n+1)^{k-2j-2}\binom{n+1-(k-2j-1)}{2j+1}\\
&\quad = (n+1)^{k-2j-2}\cdot\left\{n+1-\frac{n+1-(k-2j-1)}{2j+1}\right\}\cdot\binom{n+1-(k-2j)}{2j}.
\end{aligned}
$$

It is clear that every term $c_{k-2j}(k)+c_{k-2j-1}(k)$, $j = 0, 1, \ldots, [k/2]$, is positive. The lemma is proved. ∎

Now, we obtain the positivity of some combination of the symmetric polynomials corresponding to the hooked partitions of k.

THEOREM 3.2. *For the k symmetric polynomials $\mathrm{CH}_{\eta_i}(x, y)$ corresponding to the hooked partitions of k, if we write*

$$W_k(x) = \sum_{i=1}^{k} c_i(k)\cdot \mathrm{CH}_{\eta_i}(x, y), \tag{3.5}$$

where the coefficients $c_i(k)$, $i = 1, 2, \ldots, k$, are given by (3.4), *then $(-1)^k\cdot W_k(x)$ is positive.*

Proof. In the proof of Lemma 2.2, the solutions of system (2.6) are given by formula (2.7). If we set $b_k = (n+1)^{k-1} = c_k(k)$, then $b_i = c_i(k)$, $i = 1, 2, \ldots, k$, that is, the $c_i(k)$ given by (3.4) are also solutions of system (2.6). According to the proof of Lemma 2.2, $W(x)$ is independent of the variables $y_1, y_2, \ldots, y_m$. Therefore we have

$$W_k(x) = \sum_{i=1}^{k} c_i(k)\cdot \mathrm{CH}_{\eta_i}(x, y) = (-1)^k \cdot \sum_{i=1}^{k} c_i(k)\cdot e^k(x) = (-1)^k \cdot C_k \cdot e^k(x).$$

The theorem follows from Lemma 3.1. ■

EXAMPLE 3.1. For $k = 2$, we have $c_2 = (n+1)$, $c_1 = -n/2$ and

$$W_2(x) = (n+1)\cdot \mathrm{CH}_2(x, y) - \frac{n}{2}\cdot \mathrm{CH}_1^2(x, y) = \frac{n+2}{2}\cdot e^2(x).$$

EXAMPLE 3.2. For $k = 3$, we have

$$c_3 = (n+1)^2, \quad c_2 = -(n+1)(n-1), \quad c_1 = \frac{n}{3}\cdot(n-1)$$

and

$$W_3(x) = (n+1)^2\cdot \mathrm{CH}_3(x, y) - (n^2-1)\cdot \mathrm{CH}_2(x, y)\mathrm{CH}_1(x, y) + \frac{n}{3}(n-1)\mathrm{CH}_1^3(x, y)$$

$$= -\frac{1}{3}(n+2)(n+3)\cdot e^3(x).$$

EXAMPLE 3.3. For $k = 4$, we have

$$c_4 = (n+1)^3, \quad c_3 = -(n+1)^2(n-2), \quad c_2 = (n+1)\binom{n-1}{2}, \quad c_1 = -\frac{n}{4}\cdot\binom{n-1}{2}$$

and

$$W_4(x) = (n+1)^3\cdot \mathrm{CH}_4(x, y) - (n+1)^2(n-2)\cdot \mathrm{CH}_3(x, y)\mathrm{CH}_1(x, y)$$

$$+ (n+1)\binom{n-1}{2}\cdot \mathrm{CH}_2(x, y)\mathrm{CH}_1^2(x, y) - \frac{n}{4}\cdot\binom{n-1}{2}\cdot \mathrm{CH}_1^4(x, y)$$

$$= \frac{1}{8}(n+2)(3n^2+13n+16)\cdot e^4(x).$$

EXAMPLE 3.4. For $k = 5$, we have

$$c_5 = (n+1)^4, \quad c_4 = -(n+1)^3(n-3), \quad c_3 = (n+1)^2\binom{n-2}{2},$$

$$c_2 = -(n+1)\binom{n-1}{3}, \quad c_1 = \frac{n}{5}\cdot\binom{n-1}{3}$$

and

$$W_5(x) = (n+1^4\cdot \mathrm{CH}_5(x, y) - (n+1)^3(n-3)\cdot \mathrm{CH}_4(x, y)\mathrm{CH}_1(x, y)$$
$$+(n+1)^2\binom{n-2}{2}\cdot \mathrm{CH}_3(x, y)\mathrm{CH}_1^2(x, y)$$
$$-(n+1)\binom{n-1}{3}\cdot \mathrm{CH}_2(x, y)\mathrm{CH}_1^3(x, y)+\frac{n}{5}\cdot\binom{n-1}{3}\cdot \mathrm{CH}_1^5(x, y)$$
$$= -\frac{1}{30}(n+2)(n+3)(11n^2+39n+40)\cdot e^5(x).$$

Notice that when $k \geqslant 4$, there are partitions $\pi = (\pi_1, \pi_2, \ldots, \pi_j)$ which are without parts equal to one, i.e., $\pi_1 \geqslant \pi_2 \geqslant \ldots \geqslant \pi_j \geqslant 2$. For example partition (2,2) of 4 is such a partition. For this kind of partitions we have

COROLLARY 3.3. *For any partition $(\pi_1, \pi_2, \ldots, \pi_j)$ of k without parts equal to one, the symmetric polynomial*

$$(-1)^k W_\pi(x) = (-1)^k W_{\pi_1}(x) W_{\pi_2}(x)\cdot \ldots \cdot W_{\pi_j}(x) = C_{\pi_1}\cdot C_{\pi_2}\cdot \ldots \cdot C_{\pi_j}\cdot e^k(x)$$

is positive.

EXAMPLE 3.5. For partition (2,2) of 4, we have

$$W_2^2(x) = \left\{(n+1)\cdot \mathrm{CH}_2(x, y) - \frac{n}{2}\cdot \mathrm{CH}_1^2(x, y)\right\}^2 = \left(\frac{n+2}{2}\right)^2\cdot e^4(x)$$

and it is positive.

4. Relations in $\mathbf{CH}^{(k)}(x, y)$

For any positive integer k, the dimension of the linear space $\mathbf{CH}^{(k)}(x, y)$ is k. If the number of partitions of k is greater than the dimension of $\mathbf{CH}^{(k)}(x, y)$, then any $k+1$ symmetric polynomials $\mathrm{CH}_\pi(x, y)$ must satisfy at least one relation.

Recall that the number of partitions of k is greater than k when $k \geqslant 4$. Therefore, there exist some relations between the symmetric polynomials $\mathrm{CH}_\pi(x, y)$ when $k \geqslant 4$.

THEOREM 4.1. *When $k \geqslant 4$, for any partition $(\pi_1, \pi_2, \ldots, \pi_j)$ of k which is without parts equal to one, the symmetric polynomials $W_{\pi_1}(x), W_{\pi_2}(x), \ldots$ $\ldots, W_{\pi_j}(x)$ and the k symmetric polynomials corresponding to hooked partitions of k satisfy the following relation*

$$C_k\cdot W_{\pi_1}(x)\cdot W_{\pi_2}(x)\cdot \ldots \cdot W_{\pi_j}(x) - C_{\pi_1}\cdot C_{\pi_2}\cdot \ldots \cdot C_{\pi_j}\cdot W_k(x) = 0. \tag{4.1}$$

We call *non-hooked partition* a partition of k which is not a hooked partition, and write it $\pi = (\pi_1, \pi_2, \ldots, \pi_j, 1, \ldots, 1)$ (i ones), $i \geqslant 0$, $j \geqslant 2$, $\pi_1 \geqslant \pi_2 \geqslant \ldots \geqslant \pi_j \geqslant 2$, $\pi_1+\pi_2+\ldots+\pi_j = k-i$. When $i = 0$, the non-hooked partition is a partition without parts equal to one. When $i \geqslant 0$, then $(\pi_1, \pi_2, \ldots, \pi_j)$ is a partition of $k-i$ without parts equal to one, we have

THEOREM 4.2. *For any non-hooked partition of* k, $\pi = (\pi_1, \pi_2, \ldots, \pi_j, 1^i)$, *the symmetric polynomial* $\mathrm{CH}_\pi(x, y)$ *and the* $k-i$ *symmetric polynomials corresponding to the hooked partitions* η_i, $i = 1, 2, \ldots, k-i$, *satisfy the relation*

$$(4.2)\quad C_{\pi_1}\cdot C_{\pi_2}\cdot\ldots\cdot C_{\pi_j}\cdot W_{k-i}(x)\cdot \mathrm{CH}_1^i(x, y) - C_{k-i}\cdot W_{\pi_1}(x)\cdot W_{\pi_2}(x)\cdot\ldots\cdot W_{\pi_j}(x)\cdot \mathrm{CH}_1^i(x, y) = 0.$$

EXAMPLE 4.1. For $k = 4$, the non-hooked partition (2, 2) is without parts equal to one, and the hooked partitions are (4), (3, 1), (2, 1, 1), (1, 1, 1, 1). From Examples 3.3 and 3.5 we have

$$W_4(x) = \tfrac{1}{8}(n+2)(3n^2+13n+16)\cdot e^4(x),$$

and

$$W_2^2(x) = \left(\frac{n+2}{2}\right)^2\cdot e^4(x),$$

therefore we get

$$(4.3)\qquad 2(n+2)\cdot W_4(x)-(3n^2+13n+16)\cdot W_2^2(x) = 0.$$

By the definition of $W_2(x)$ and $W_4(x)$, equation (4.3) becomes

$$(4.4)\quad (3n^2+13n+16)\mathrm{CH}_2^2(x, y)+2(n+1)(n+2)\mathrm{CH}_4(x, y) -2(n^2-4)\mathrm{CH}_3(x, y)\mathrm{CH}_1(x, y)+4(n+1)^2\mathrm{CH}_2(x, y)\mathrm{CH}_1^2(x, y) -n(n+1)\mathrm{CH}_1^4(x, y) = 0.$$

EXAMPLE 4.2. For partition (2, 2, 1) of 5, we have

$$(4.5)\qquad \{2(n+2)\cdot W_4(x)-(3n^2+13n+16)\cdot W_2^2(x)\}\cdot \mathrm{CH}_1(x, y) = 0.$$

EXAMPLE 4.3. For partition (3, 2) of 5, from Examples 3.1, 3.2 and 3.4 we have

$$(4.6)\qquad 5(n+2)\cdot W_5(x)-(11n^2+39n+40)\cdot W_2(x)\cdot W_3(x) = 0.$$

By definition, (4.6) leads to

$$(4.7)\quad (11n^2+39n+40)(n+1)\cdot \mathrm{CH}_3(x, y)\mathrm{CH}_2(x, y) -(11n^2+39n+40)(n-1)\cdot \mathrm{CH}_2^2(x, y)\mathrm{CH}_1(x, y) -5(n+2)(n+1)^2\cdot \mathrm{CH}_5(x, y) +5(n+1)(n+2)(n-3)\cdot \mathrm{CH}_4(x, y)\mathrm{CH}_1(x, y) -2(4n^3+6n^2+5n+15)\cdot \mathrm{CH}_3(x, y)\mathrm{CH}_1^2(x, y) +10(n-1)(n+1)^2\cdot \mathrm{CH}_2(x, y)\mathrm{CH}_1^3(x, y)-2n(n^2-1)\cdot \mathrm{CH}_1^5(x, y) = 0.$$

5. The Chern characters of a hypersurface with singularities

Early in 1965, Wu Wen-tsün [W1], [W2] introduced the notion of Chern characteristic classes for algebraic varieties with arbitrary singularities. Wu used algebraic equivalence, and not only birational equivalence, which makes the classes concretely computable. Recently, Wu proved some inequalities for the Chern classes of 2 or 3-dimensional algebraic hypersurface with singularities in [W3]. Following the idea given in [W3] the present author obtained in [SH1], [SH2], [SH3] a series of inequalities for the Chern classes of n-dimensional algebraic hypersurface with singularities, and discovered some equalities for the Chern classes of an algebraic hypersurface with singularities when its dimension $n \geqslant 4$. These inequalities and equalities were obtained by using combinatorial method and solving some systems of equations.

In this section we use the Chern characteristic classes of a hypersurface with singularities defined by Wu (see [W1], [W2], [W3]).

We employ the notation used in [W3]. Let V_n be a hypersurface with singularities in an $(n+1)$-dimensional complex projective space. We denote the group of r-dimensional algebraic equivalence classes of V_n by $\mathrm{ALG}_r(V_n)$. According to [W3], we denote the Ehresmann classes of the hypersurface V_n corresponding to the Ehresmann symbols $[1|(0, 1, \ldots, n)]$ and $[0|(0, 1, \ldots, n-1), n+1]$ by $p(V_n)$ and $q(V_n)$ respectively. The multiplication $p^i q^j(V_n) \in \mathrm{ALG}_{n-i-j}(V_n)$ of the Ehresmann classes is well-defined in the intersection ring of V_n. Then the n Chern classes $\mathrm{CH}_i(V_n)$ belonging to $\mathrm{ALG}_{n-i}(V_n)$ of the hypersurface V_n are defined as

$$\mathrm{CH}_i(V_n) = \sum_{j=0}^{i} (-1)^j \binom{n+1-j}{i-j} (p^{i-j} q^j)(V_n), \qquad i = 1, 2, \ldots, n. \tag{5.1}$$

For a partition $\pi = (\pi_1, \pi_2, \ldots, \pi_j)$ of a positive integer k, the Chern class of the hypersurface V_n corresponding to the partition π is defined as

$$\mathrm{CH}_\pi(V_n) = \mathrm{CH}_{\pi_1}(V_n) \cdot \mathrm{CH}_{\pi_2}(V_n) \cdot \ldots \cdot \mathrm{CH}_{\pi_j}(V_n). \tag{5.2}$$

Now the connection between the Chern classes of V_n and the symmetric polynomials $\mathrm{CH}_i(x, y)$ defined by formula (1.1) is very clear. Moreover, the multiplication of Ehresmann classes $p(V_n)$ and $q(V_n)$ in the intersection ring of V_n coincides with the multiplication of the symmetric polynomials $e(y)$ and $e(x)$. The properties of symmetric polynomials that we obtained in the preceding sections can now be translated in terms of Chern classes of hypersurfaces with singularities.

Furthermore, according to [W3], for a partition π of k, the product

$$\mathrm{CH}_\pi(V_n) \cdot q^{n-k}(V_n) \in \mathrm{ALG}_0(V_n) \tag{5.3}$$

is an integer, and is a projective character in the sense of Severi and is called the Chern character of V_n corresponding to the partition π. Moreover, the

Chern character defined by Wu is just the usual Chern number when the hypersurface V_n is smooth.

On the other hand, one of the fundamental facts in intersection theory is that for a hypersurface V_n in $(n+1)$-dimensional complex projective space the intersection number $q^n(V_n)$ is necessarily nonnegative (this is similar to the positivity of the number $X^\lambda(1^n)$).

Therefore, we have the following conclusions for Chern classes or Chern characters of hypersurfaces with singularities.

THEOREM 5.1. *For any positive integer k, $k \leqslant n$, the k Chern characters corresponding to hooked partitions of k*

$$\mathrm{CH}_{\eta_i}(V_n)\cdot q^{n-k}(V_n), \qquad i = 1, 2, \ldots, k,$$

satisfy the inequality

$$(-1)^k \cdot \sum_{i=1}^{k} c_i(k)\cdot \mathrm{CH}_{\eta_i}(V_n)\cdot q^{n-k}(V_n) \geqslant 0, \tag{5.4}$$

where the $c_i(k)$ are

$$\begin{aligned} c_1(k) &= (-1)^{k-1}\cdot\frac{n}{k}\cdot\binom{n-1}{k-2}, \\ c_i(k) &= (-1)^{k-i}(n+1)^{i-1}\binom{n+1-i}{k-i}, \qquad i = 2, 3, \ldots, k. \end{aligned} \tag{5.5}$$

EXAMPLE 5.1. For $k = 2$, we have

$$\left\{(n+1)\cdot \mathrm{CH}_2(V_n) - \frac{n}{2}\cdot \mathrm{CH}_{1,1}(V_n)\right\}\cdot q^{n-2}(V_n) \geqslant 0,$$

and when $n = 2$, it becomes

$$3\cdot \mathrm{CH}_2(V_n) - \mathrm{CH}_{1,1}(V_n) \geqslant 0.$$

EXAMPLE 5.2. For $k = 3$, we have

$$-\left\{(n+1)^2\cdot \mathrm{CH}_3(V_n) - (n+1)(n-1)\cdot \mathrm{CH}_{2,1}(V_n) + \frac{n}{3}(n-1)\cdot \mathrm{CH}_{1,1,1}(V_n)\right\} \times q^{n-3}(V_n) \geqslant 0,$$

and when $n = 3$, it becomes

$$-8\cdot \mathrm{CH}_3(V_n) + 4\cdot \mathrm{CH}_{2,1}(V_n) - \mathrm{CH}_{1,1,1}(V_n) \geqslant 0.$$

EXAMPLE 5.3. For $k = 4$, we have

$$\left\{(n+1)^3\cdot \mathrm{CH}_4(V_n) - (n+1)^2(n-2)\cdot \mathrm{CH}_{3,1}(V_n) + (n+1)\binom{n-1}{2}\cdot \mathrm{CH}_{2,1,1}(V_n) - \frac{n}{4}\binom{n-1}{2}\cdot \mathrm{CH}_{1,1,1,1}(V_n)\right\}\cdot q^{n-4}(V_n) \geqslant 0.$$

EXAMPLE 5.4. For $k = 5$, we have

$$-\left\{(n+1)^4 \cdot \mathrm{CH}_5(V_n) - (n+1)^3(n-3) \cdot \mathrm{CH}_{4,1}(V_n) + (n+1)^2\binom{n-2}{2} \cdot \mathrm{CH}_{3,1,1}(V_n)\right.$$

$$\left. -(n+1)\binom{n-1}{3} \cdot \mathrm{CH}_{2,1,1,1}(V_n) + \frac{n}{5}\binom{n-1}{3} \cdot \mathrm{CH}_{1,1,1,1,1}(V_n)\right\} \cdot q^{n-5}(V_n) \geqslant 0.$$

THEOREM 5.2. *For any partition $\pi = (\pi_1, \pi_2, \ldots, \pi_j)$ of k without parts equal to one, if we write*

$$W_r(V_n) = \sum_{i=1}^{r} c_i(r) \cdot \mathrm{CH}_{\eta_i(r)}(V_n), \tag{5.6}$$

where the $c_i(r)$ are given by (5.5), then the following inequality is valid:

$$(-1)^k \cdot W_{\pi_1}(V_n) \cdot W_{\pi_2}(V_n) \cdot \ldots \cdot W_{\pi_j}(V_n) \cdot q^{n-k}(V_n) \geqslant 0. \tag{5.7}$$

EXAMPLE 5.5. For partition (2, 2) of 4, we have

$$\left\{(n+1) \cdot \mathrm{CH}_2(V_n) - \frac{n}{2} \cdot \mathrm{CH}_{1,1}(V_n)\right\}^2 \cdot q^{n-4}(V_n) \geqslant 0.$$

EXAMPLE 5.6. For partition (3, 2) of 5, we have

$$-\left\{(n+1)^2 \cdot \mathrm{CH}_3(V_n) - (n^2-1) \cdot \mathrm{CH}_{2,1}(V_n) + \frac{n}{3}(n-1) \cdot \mathrm{CH}_{1,1,1}(V_n)\right\}$$

$$\times \left\{(n+1) \cdot \mathrm{CH}_2(V_n) - \frac{n}{2} \cdot \mathrm{CH}_{1,1}(V_n)\right\} \cdot q^{n-5}(V_n) \geqslant 0.$$

THEOREM 5.3. *Let $\mathrm{CH}^{(k)}(V_n)$ be the linear space spanned by*

$$\{\mathrm{CH}_\pi(V_n) \mid \pi \text{ partition of } k\}.$$

Then the dimension of $\mathrm{CH}^{(k)}(V_n)$ is k.

COROLLARY 5.4. *When $4 \leqslant k \leqslant n$, any system of $k+1$ Chern classes, corresponding to partitions of k, of a hypersurface V_n must satisfy at least one relation.*

EXAMPLE 5.7. For $k = 4$, we have

$$(3n^2+13n+16) \cdot \mathrm{CH}_{2,2}(V_n) + 2(n+1)(n+2)\mathrm{CH}_4(V_n)$$

$$-2(n^2-4) \cdot \mathrm{CH}_{3,1}(V_n)$$

$$+4(n+1)^2 \cdot \mathrm{CH}_{2,1,1}(V_n) - n(n+1) \cdot \mathrm{CH}_{1,1,1,1}(V_n) = 0.$$

More examples are provided in [SH2].

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