TOPICS IN ALGEBRA
BANACH CENTER PUBLICATIONS, VOLUME 26, PART 2
PWN-POLISH SCIENTIFIC PUBLISHERS
WARSAW 1990

CONSTRUCTIVE GEOMETRIC INVARIANT THEORY FOR CERTAIN NONREDUCTIVE GROUPS

FRANK D. GROSSHANS

Department of Mathematical Sciences, West Chester University
West Chester, Pennsylvania, U.S.A.

The geometric invariant theory of reductive groups has been studied intensively for the past twenty years. In this paper, we shall extend central portions of that theory to certain nonreductive groups. The situation we shall study may be described as follows.

Let k be an algebraically closed field, let G be a connected semisimple algebraic group over k, and let H be a subgroup of G. Let X be an affine variety on which G operates regularly. Let $k[X]^H$ be the algebra of functions in $k[X]$ which are fixed by H. An orbit Hx in X is said to be *H-separated* if $Hx = \{y \in X\colon f(y) = f(x) \text{ for all } f \in k[X]^H\}$.

If $H = G$, the separated orbits may be described using a one-parameter subgroup criterion, originally due to Hilbert and developed extensively by Mumford. Separated orbits give rise to good quotient structures for both affine and projective varieties.

The class of subgroups of G to be studied here consists of the "codimension 2 condition" subgroups (§ 2). This class includes the maximal unipotent subgroups of G and, more generally, unipotent radicals of parabolic subgroups. Given such a subgroups H, there is an affine variety Z on which G operates regularly and a point $z \in Z$ such that (i) the stabilizer of z in G is H, (ii) the orbit Gz is open and dense in Z, and (iii) $\dim(Z - Gz) \leqslant \dim Z - 2$.

The key idea in this paper is the relationship between the following conditions on a point x in X: (C1) the orbit Hx is H-separated on X; (C2) the orbit $G(x, z)$ is G-separated on $X \times Z$. If either both conditions hold at x or neither condition holds at x, we shall write (C1) $\sim$ (C2) at x. In general, if (C2) is true at x, then (C1) also holds at x; however, if (C1) is true at x, then (C2) may or may not hold at x. Nevertheless, there are many instances where (C1) $\sim$ (C2)

This paper is in final form and no version of it will be submitted for publication elsewhere.

at x, for example when x is any point whose stabilizer in G is finite (Theorem 2, § 3) or when x is any point in a finite-dimensional vector space X having sufficiently large dimension (Theorem 2, § 5). If (C1) $\sim$ (C2) at x, then we may give a one-parameter subgroup criterion for the orbit Hx to be H-separated on X. Furthermore, when (C1) $\sim$ (C2) at all x in X, we obtain the desired theorems concerning quotient varieties (§ 4).

The one-parameter subgroup criterion can be simplified considerably if H is normalized by a maximal torus in G (§ 6). We conclude the paper with several examples chosen to illustrate our main theorems.

Our notation and terminology generally follows that in [9]. Some assumptions used throughout the paper are explained in the note concluding § 2. The author would like to thank J. Dixmier for his help in improving this paper's exposition and also for the proof of Lemma 1, § 5.

§ 1. Reductive groups

We begin with some terminology and notation which will be used throughout this paper. Let k be an algebraically closed field. Let Y be an algebraic variety over k and let $k[Y]$ be the algebra of regular functions on Y. Let L be an algebraic group over k which operates regularly on Y via a mapping $L\times Y\to Y$ denoted by $(g, y)\to gy$. Let $k[Y]^L=\{f\in k[Y]$: $f(gy)=f(y)$ for all $g\in L$ and $y\in Y\}$. Let $Ly=\{gy\colon g\in L\}$ be the *orbit* of y and $L_y=\{g\in G\colon gy=y\}$ denote the *stabilizer* of y.

The orbit Ly of a point $y\in Y$ is called *separated* if $Ly=\{y'\in Y$: $f(y)=f(y')$ for all $f\in k[Y]^L\}$. We note that a separated orbit is always closed.

Now, we shall state some important facts in the case $L=G=$ connected, reductive algebraic group and Y is an irreducible affine variety.

(RED1) The algebra $k[Y]^G$ is finitely generated over k [12; Theorem 3.4, p. 49].

(RED2) Let W_1 and W_2 be disjoint, closed, G-stable subsets of Y. There is an $f\in k[Y]^G$ such that $f(w_1)=1$ for all $w_1\in W_1$ and $f(w_2)=0$ for all $w_2\in W_2$ [12; Lemma 3.3, p. 49].

(RED3) Let $y\in Y$. The orbit Gy is G-separated on Y if and only if it is closed in Y and $\dim(Gy)\geqslant\dim(Gy')$ for all $y'\in Y$ [12; Proposition 3.8, p. 67].

(RED4) Let $y\in Y$. Let S be a closed, G-stable subset of Y which meets the closure of the orbit Gy. There is a one-parameter subgroup γ of G such that $\text{limit}_{a\to 0}\gamma(a)y$ exists and is contained in S [10; Theorem 1.4, p. 302].

(RED5) Let $y\in Y$. The orbit Gy is affine if and only if the stabilizer G_y is reductive. (This is a small extension of the main result in [15].)

§ 2. The codimension 2 condition

Let G be a connected semisimple algebraic group over the algebraically closed field k. Let $k[G]$ denote the algebra of regular functions on G. The group G operates on itself via right and left multiplication. These actions give rise to actions on $k[G]$, namely

$$(\varrho(g)f)(g_1) = f(g_1 g) \quad \text{and} \quad (\lambda(g)f)(g_1) = f(g^{-1} g_1)$$

for all g, $g_1 \in G$ and $f \in k[G]$.

For H a subgroup of G, we put $H' = k[G]^H = \{f \in k[G]\colon \varrho(h)f = f \text{ for all } h \in H\}$. For R a subset of $k[G]$, we put $R' = \{g \in G\colon \varrho(g)r = r \text{ for all } r \in R\}$. Then H' is a k-subalgebra of $k[G]$ and R' is an algebraic subgroup of G.

A subgroup H of G is said to satisfy the *codimension* 2 *condition* on G/H if there is a rational representation of G on a finite-dimensional vector space V and a $v \in V$ so that $H = G_v$ and, if Z is the clossure of the orbit Gv, then $\dim(Z - Gv) \leqslant \dim Z - 2$.

Applying standard normalization arguments (e.g., [1; Theorem, p. 78]) to the normalization of Z in $k(G/H)$, we see that V and v may be chosen so that Z is normal and Gv is isomorphic to G/H. In this case, we shall call Z the affine model for $k[G]^H$. (The equality $k[Z] = k[G]^H$ holds since $\dim(Z - Gv) \leqslant \dim Z - 2$.)

THEOREM [8; (1.2)]. *Let G be a connected semisimple algebraic group over k and let X be an affine variety on which G acts regularly. Let H be a subgroup of G which satisfies the codimension 2 condition on G/H. Let Z be the affine model for $k[G]^H$ and let z be any point in Z with $G_z = H$. Then $k[X]^H$ is isomorphic to $k[X \times Z]^G$ via the mapping $\iota^*\colon k[X \times Z]^G \to k[X]^H$ defined by $(\iota^* F)(x) = F(x, z)$.*

As a consequence of the isomorphism above and (RED1), it may be shown that $k[X]^H$ is finitely generated. Lists of subgroups H satisfying the codimension 2 condition may be found in [7], [13], and [17]. For example, unipotent radicals of parabolic subgroups of G satisfy the codimension 2 condition.

In the rest of this section, we will use the notation given in the Theorem, above. If g, g_1 are in G and f is in $k[G]$, then $\lambda(g)\varrho(g_1)f = \varrho(g_1)\lambda(g)f$. Therefore, if $f \in H'$ and $g \in G$, the function $\lambda(g)f$ is in H'. It follows that G acts on H' via λ. Next, let $N_G(H)$ be the normalizer of H in G. If $n \in N_G(H)$ and $f \in H'$, then $\varrho(n)f \in H'$. Hence, $N_G(H)$ acts on H' via ϱ.

The action of G on H' via λ gives rise to an action of G on Z, the affine model of $k[G]^H$. We shall denote this by $G \times Z \to Z$, $(g, z') \to gz'$. The action of $N_G(H)$ on H' via ϱ gives rise to an action of $N_G(H)$ on Z which we denote by $N_G(H) \times Z \to Z$, $(n, z') \to n \cdot z'$.

Now, let $\pi\colon G \to Z$ be defined by $\pi(g) = gz$. Let $\pi^*\colon k[Z] \to k[G]^H$ be the associated algebra homomorphism. Then $n \cdot gz = gn^{-1}z$ for all $g \in G$,

$n \in N_G(H)$. Indeed, let $f \in k[Z]$; then $f(gn^{-1}z) = (\pi^* f)(gn^{-1})$. Furthermore,

$$f(n \cdot gz) = \varrho(n^{-1})(\pi^* f)(g) = (\pi^* f)(gn^{-1}).$$

Let γ be a one-parameter subgroup of G so that $\text{limit}_{a \to 0} \gamma(a)z$ exists. Let z' be any other point in Z whose stabilizer in G is H. Then $z' = nz$ for some $n \in N_G(H)$. Therefore,

$$\gamma(a)z' = \gamma(a)nz = n^{-1} \cdot \gamma(a)z$$

so $\text{limit}_{a \to 0} \gamma(a)z'$ also exists. This allows us to define $\Gamma(G/H)$ to be the collection of all nontrivial one-parameter subgroups γ in G such that $\text{limit}_{a \to 0} \gamma(a)z'$ exists where z' is any point in Z whose stabilizer in G is H.

Notation. We shall use the notation and assumptions given in the Theorem, above, throughout this paper. Furthermore, $\Gamma(G/H)$ will denote the set described above. If T is any torus in G, we shall denote by $\Gamma(G/H; T)$ the set of all one-parameter subgroups γ in T such that $\gamma \in \Gamma(G/H)$.

§ 3. Separated orbits

In this section, we undertake the systematic study of the relationship between the following two conditions on a point x in X: (C1) the orbit Hx is H-separated on X; (C2) the orbit $G(x, z)$ is G-separated on $X \times Z$. If either both conditions hold at x or neither condition holds at x, we shall write (C1) $\sim$ (C2) at x. We shall see that condition (C2) always implies condition (C1) but that (C1) does not imply (C2), in general. The basic tools to be used are (RED4), § 1, and the Theorem, § 2. In particular, we shall use the isomorphism $\iota^*: k[X \times Z]^G \to k[X]^H$ again and again. Furthermore, we should observe that the proof of the Theorem, § 2, implies that $k[X]^H = k[X \times Gz]^G = k[X \times Z]^G$. Finally, we note that $(x', z) \in G(x, z)$ if and only if $x' \in Hx$ (since $H = G_z$).

LEMMA 1. *Let $x \in X$.*

(a) *The orbit Hx is H-separated on X if and only if the orbit $G(x, z)$ is G-separated on $X \times Gz$.*

(b) *If the orbit $G(x, z)$ is G-separated on $X \times Z$, then the orbit Hx is H-separated on X and H_x is a reductive algebraic group.*

Proof. Suppose that $G(x, z)$ is G-separated on $X \times Gz$. Let $x' \in X$, $x' \notin Hx$. Then $(x', z) \notin G(x, z)$. By assumption, there is an $F \in k[X \times Gz]^G$ such that $F(x', z) \neq F(x, z)$. Then

$$(\iota^* F)(x') = F(x', z) \neq F(x, z) = (\iota^* F)(x).$$

This proves that the orbit Hx is H-separated on X.

Next, suppose that the orbit Hx is H-separated on X. Let $(x', gz) \in X \times Gz$. $(x', gz) \notin G(x, z)$. Then $(g^{-1}x', z) \notin G(x, z)$ and, so, $g^{-1}x' \notin Hx$. By assumption, there is an $f \in k[X]^H$ so that $f(g^{-1}x') \neq f(x)$. Let $F \in k[X \times Gz]^G$ be chosen so that $\iota^* F = f$. Then

$$F(x', gz) = F(g^{-1}x', z) = f(g^{-1}x') \neq f(x) = F(x, z).$$

This completes the proof of (a).

If the orbit $G(x, z)$ is G-separated on $X \times Z$, then (by the definitions) it is G-separated on $X \times Gz$. Therefore, the orbit Hx is H-separated on X by (a). Furthermore, the orbit $G(x, z)$ is closed in $X \times Z$ (since it is G-separated) and, by (RED5), we may conclude that the stabilizer in G of (x, z) is reductive. But this stabilizer is $G_x \cap G_z = G_x \cap H = H_x$.

EXAMPLE. According to Lemma 1(b), condition (C2) always implies condition (C1). In this example, we shall use Lemma 1(b) to see that (C1) need not imply (C2).

Let us identify k^3 with 3×1 column matrices. Let $G = \mathrm{SL}_3(k)$ act on k^3 via left multiplication. Let $U = \{(a_{ij}) \in G\colon a_{ij} = 0$ for $i > j$ and $a_{ii} = 1$ for $i = 1, 2, 3\}$. Then U is a maximal unipotent subgroup of G, $\dim U = 3$, and the codimension 2 condition is true for G/U.

The orbits of U on k^3 are closed so U cannot have an orbit on k^3 of dimension 3. For if $\dim Uv = 3$, then $Uv = k^3$ and $0 \in Uv$. However, there are points $v \in k^3$ such that $\dim Uv = 2$, e.g., $v = (0, 0, 1)^T$. It follows (by standard arguments, e.g. [12; Lemma 3.7(c), p. 66]) that there is a nonempty open subset of k^3 consisting of points v with $\dim Uv = 2$.

There is a nonempty open subset of V consisting of points v so that Uv is separated [5; 2.4.2, p. 338]. Hence, there are points v in V so that Uv is U-separated on k^3 and $\dim Uv = 2$. Since U_v is a nontrivial unipotent group, the orbit $G(v, z)$ cannot be G-separated on $V \times Z$ (by Lemma 1(b)). This concludes the example.

Let A be any subset of k^n. We shall denote the Zariski-closure of A by $\mathrm{cl}(A)$.

LEMMA 2. *Let $x \in X$. Suppose that the orbit Hx is H-separated on X.*

(a) *The orbit $G(x, z)$ is G-separated on $X \times Z$ if and only if it is closed in $X \times Z$.*

(b) *The closure of the orbit $G(x, z)$ in $X \times Z$ is a union:*

$$G(x, z) \cup [\mathrm{cl}\, G(x, z) \cap (X \times (Z - Gz))].$$

Proof. We noted earlier (§ 1) that separated orbits are always closed. So, let us assume that the orbit $G(x, z)$ is closed in $X \times Z$ and show that it is separated in $X \times Z$. Let $(x', z') \in X \times Z$ but $(x', z') \notin G(x, z)$. If $(x', z') \in X \times Gz$, then (by Lemma 1(a)) there is an $F \in k[X \times Gz]^G = k[X \times Z]^G$ such that

$F(x', z') \neq F(x, z)$. So, we may assume that $(x', z') \in X \times (Z - Gz) = X_1$. The set X_1 is G-stable, closed, and $X_1 \cap G(x, z) = \emptyset$. We apply (RED2) to find $F \in k[X \times Z]^G$ such that $F(x', z') = 1$, $F(x, z) = 0$.

To prove (b), let us assume that $(x', gz) \in \operatorname{cl} G(x, z)$. Then the points (x', gz) and (x, z) cannot be separated by any function in $k[X \times Z]^G$. Since Hx is H-separated on X by assumption, the orbit $G(x, z)$ is G-separated on $X \times Gz$ by Lemma 1(a). Hence, $(x', gz) \in G(x, z)$.

THEOREM 1. *Let $x \in X$ be a point such that the orbit Hx is H-separated on X but the orbit $G(x, z)$ is not G-separated on $X \times Z$.*

(a) *There is a one-parameter subgroup $\gamma \in \Gamma(G/H)$ such that* $\operatorname{limit}_{a \to 0} \gamma(a) x$ *exists.*

(b) *Let $\gamma \in \Gamma(G/H)$ be any one-parameter subgroup such that* $\operatorname{limit}_{a \to 0} \gamma(a) x$ *exists. Let* $x' = \operatorname{limit}_{a \to 0} \gamma(a) x$. *Let x'' be any point in X so that* $\operatorname{limit}_{a \to 0} \gamma(a) x'' = x'$. *Then $x'' \in Hx$.*

Proof. According to Lemma 2(a), the orbit $G(x, z)$ is not closed in $X \times Z$. Then statement (a) follows from (RED4). To prove (b), let $\operatorname{limit}_{a \to 0} \gamma(a)(x, z) = (x', z')$. Let F be any element in $k[X \times Z]^G$ and let $f = \iota^* F \in k[X]^H$. Then,

$$\begin{aligned} f(x'') = F(x'', z) &= F(\operatorname{limit}_{a \to 0} \gamma(a)(x'', z)) \\ &= F(x', z') = F(\operatorname{limit}_{a \to 0} \gamma(a)(x, z)) = F(x, z) = f(x). \end{aligned}$$

Since x'' and x cannot be separated by any $f \in k[X]^H$, we see that $x'' \in Hx$.

THEOREM 2. *Let $x \in X$ satisfy* $\dim(Gx) = \dim G$. *Then the following conditions are equivalent.*

(a) *The orbit Hx is not H-separated on X.*
(b) *The orbit $G(x, z)$ is not G-separated on $X \times Z$.*
(c) *There is a $\gamma \in \Gamma(G/H)$ such that* $\operatorname{limit}_{a \to 0} \gamma(a) x$ *exists.*

Proof. The implication (a) $\to$ (b) follows from Lemma 1(b). Now, suppose that the orbit $G(x, z)$ is not G-separated on $X \times Z$. The stabilizer of (x, z) in G is finite since G_x is finite, by assumption. We apply (RED3) to see that the orbit $G(x, z)$ cannot be closed in $X \times Z$. Next, we apply (RED4) to obtain the one-parameter subgroup γ in (c).

Finally, suppose that statement (c) holds and that $\operatorname{limit}_{a \to 0} \gamma(a)(x, z) = (x', z')$. Then, $\operatorname{limit}_{a \to 0} \gamma(a)(x', z)$ is also equal to (x', z') so we cannot separate (x, z) and (x', z) by any element in $k[X \times Z]^G$. This means that we cannot separate x and x' by any element in $k[X]^H$. But $x' \notin Hx$. Indeed, if $x' = hx$ for some $h \in H$, then $G_{x'} = hG_x h^{-1}$. But then each $\gamma(a)$ is in $hG_x h^{-1}$, contradicting the assumption that G_x is finite.

THEOREM 3. *Let H be a unipotent subgroup of G. Let $x \in X$. Suppose that* (C1) $\sim$ (C2) *at x. Then the following conditions are equivalent.*

(a) *The orbit* Hx *is not* H*-separated on* X.

(b) *There is a one-parameter subgroup* $\gamma \in \Gamma(G/H)$ *such that* $\operatorname{limit}_{a \to 0} \gamma(a)x$ *exists.*

Proof. Suppose that statement (a) holds; by Lemma 1(b), the orbit $G(x, z)$ is not G-separated on $X \times Z$. If $\dim G(x, z) < \dim G$, then the stabilizer in G of (x, z) is a nontrivial unipotent group (H_x, in fact). Hence, $G(x, z)$ cannot be closed in $X \times Z$ by (RED5). If $\dim G(x, z) = \dim G$, then the orbit $G(x, z)$ cannot be closed in $X \times Z$ by (RED3). Thus, in any case, the orbit $G(x, z)$ is not closed in $X \times Z$ and we obtain statement (b) from (RED4).

Next, assume that statement (b) holds and that $\operatorname{limit}_{a \to 0} \gamma(a)(x, z) = (x', z')$. Then each $\gamma(a)$ is in the stabilizer in G of (x', z'). But the stabilizer in G of each point in $X \times Gz$ is conjugate to a subgroup of H and, so, must be unipotent. This shows that $(x', z') \notin X \times Gz$. Hence, $G(x, z)$ is not closed in $X \times Z$ and cannot be G-separated in $X \times Z$.

§ 4. Quotients

We begin by recalling some definitions and theorems in [5]. Let Y be an algebraic variety over k and let L be an algebraic group over k which operates regularly on Y. A *geometric quotient* of Y by L is a pair (W, ϕ) where W is an algebraic variety over k and ϕ: $Y \to W$ is a morphism such that (i) ϕ is open, constant on L-orbits, and defines a bijection of the set Y/L onto W; (ii) if W' is an open subset of W, then the morphism of $k[W']$ into $k[\phi^{-1}(W')]^L$ defined by ϕ is bijective.

Next, suppose that Y is quasi-affine. Let $\Omega_2(Y, L)$ be the interior of the union of all L-separated orbits on Y. Then the variety $\Omega_2(Y, L)/L$ exists, is quasi-affine, and is open in the affine soheme $\operatorname{Spec} k[Y]^L$ [5; (2.2.3)].

THEOREM 1. *Let* X *be an affine variety on which* G *acts regularly. Suppose that* (C1) $\sim$ (C2) *at each point* x *in* X. *Let* $X(H)$ *be the set of all* H*-separated orbits in* X. *Then* $X(H)$ *is open in* X, *the quotient* $X(H)/H$ *exists, is quasi-affine and open in* $\operatorname{Spec} k[X]^H$.

Proof. Let W consist of all the G-separated orbits in $X \times Z$. Then W is open in $X \times Z$ [12; Proposition 3.8, p. 67]. Hence, $X(H)$ is open in X since $X(H) \times \{z\} = W \cap (X \times \{z\})$. The rest of the theorem follows from the result just cited.

Before considering projective varieties, several observations on Theorem 1 may be in order. First, if H is unipotent, then $X(H)$ is open and dense in X [5; (2.4.2)]. Second, whether H is unipotent or not, the algebra $k[X]^H$ is finitely generated since H is always assumed to satisfy the codimension 2 condition (§ 2). A small modification of the proof for Theorem 1, along with an application of Theorem 2, § 3, gives the following result.

THEOREM 2. *Let X be an affine variety on which G acts regularly. Let $X'(H)$ be the set of all points x in X so that* (i) $\dim Gx = \dim G$ *and* (ii) *the orbit Hx is H-separated on X. Then $X'(H)$ is open in X, the quotient $X'(H)/H$ exists, is quasi-affine and open in* $\operatorname{Spec} k[X]^H$.

Next, we consider projective varieties. We shall follow closely the ideas and terminology in [12; pp. 73–77]. Let V be a finite-dimensional vector space on which G operates linearly. Let $P(V)$ be the corresponding projective space; if $v \in V$ and $v \neq 0$, then we shall denote its image in $P(V)$ by $[v]$. Let X^* be a projective variety in $P(V)$ which is stable with respect to the action of G, i.e., $g[v] \in X^*$ for all $g \in G$, $[v] \in X^*$. Let X be the affine variety in V lying over X^*. The algebra $k[X]$ is a graded k-algebra. If f is any nonzero homogeneous element in $k[X]$, we put $X_f^* = \{[v] \in X^* \colon f(v) \neq 0\}$. If $\deg f \geqslant 1$, then (it is known that) X_f^* is affine and $k[X_f^*] = (k[X]_f)_0$, the algebra of elements having degree 0 in the graded k-algebra $k[X]_f$.

THEOREM 3. *Suppose that* (C1) $\sim$ (C2) *at each point $x \in X$. Let $X^*(H)$ consist of all those points $[x]$ in X^* such that the orbit Hx is H-separated on X. Then $X^*(H)$ is open in X^*, the quotient $X^*(H)/H$ exists and is quasi-projective.*

Proof. Let $X(H)$ be the set of all $x \in X$ such that the orbit Hx is H-separated on X. Then $X(H)$ is open in X by Theorem 1. Furthermore, if $c \in k^*$ and $x \in X(H)$, then $cx \in X(H)$. For let $y \in X$ and suppose that $F(y) = F(cx)$ for every homogeneous polynomial F in $k[X]^H$. If $\deg F = d$, then $F(y) = F(cx) = c^d F(x)$ and $F(c^{-1}y) = F(x)$. Since $x \in X(H)$, $c^{-1}y \in Hx$ and $y \in H(cx)$. Hence, $X^*(H)$ is open in X^*. We shall construct the quotient variety $X^*(H)/H$ in several steps.

(1) Let $R = k[X]^H$. The algebra R is finitely generated over k and is a graded subalgebra of $k[X]$. Let X_R^* denote the union of all those X_f^* where f is any nonzero homogeneous elements in R. There is a projective variety Y and a morphism $\phi\colon X_R^* \to Y$ such that the following conditions hold [12; p. 76]:

(a) Y is covered by affine open sets Y_f, one for each homogeneous element of R having degree $\geqslant 1$ and $k[Y_f]$ is isomorphic to $(R_f)_0$;

(b) $\phi^{-1}(Y_f) = X_f^*$ and $\phi\colon X_f^* \to Y_f$ is the morphism of affine varieties corresponding to the inclusion of $(R_f)_0 = (k[X]_f^H)_0$ in $(k[X]_f)_0$.

Let f be a nonzero homogeneous element in $k[X]^H$; we put $X^*(H)_f = X^*(H) \cap X_f^*$. We may assume that $X^*(H)$ is the union of all the $X^*(H)_f$. (For if $x \in X(H)$ and $f(x) = 0$ for all homogeneous $f \in k[X]^H$, then x cannot be separated from 0, so, $x = 0$. But if 0 is H-separated, then H fixes every point in X and the Theorem is immediate.) We postpone the proof of the next step to the end of this section.

(2) Let f be a nonzero homogeneous polynomial in $k[X]^H$ and let $[x] \in X^*(H)_f$. Then the orbit $H[x]$ is H-separated in (the affine variety) X_f^*.

(3) The restriction of ϕ to $X^*(H)_f$, namely, $\phi\colon X^*(H)_f \to \phi(X^*(H)_f) \subset Y_f$ is a quotient mapping by (2) and the remarks preceding Theorem 1. Hence, $\phi(X^*(H))$ is open in Y. The proof that ϕ now defines a quotient on $X^*(H)$ is straightforward and we omit the details.

We now prove (2). We begin with two lemmas, always keeping the terminology introduced just before Theorem 3.

LEMMA 1. *Let x, x' be in X. Suppose that there is a positive integer d such that $f(x)^d = f(x')^d$ for all homogeneous polynomials f in $k[X]$. Then there is a $c \in k^*$ such that $x' = cx$.*

Proof. We may assume that $x \neq 0$. We may also assume that the characteristic of k does not divide d. (For if $d = p^m q$ with $(p, q) = 1$ and $p = \operatorname{char} k$, then we may replace d by q.) Let μ be any coordinate function on V such that $\mu(x) \neq 0$. Then, there is a dth root of unity, c, such that $\mu(x') = c\mu(x)$. Let η be any other coordinate function on V such that $\eta(x) \neq 0$. If $a \in k$, then

$$(\mu(x) + a\eta(x))^d = (\mu(x') + a\eta(x'))^d.$$

Comparing coefficients of a, we see that $\mu(x)^{d-1}\eta(x) = \mu(x')^{d-1}\eta(x')$. We multiply each side of this equality by $\mu(x)$ and use the equation $\mu(x)^d = \mu(x')^d$ to see that $\mu(x')\eta(x) = \mu(x)\eta(x')$. Hence, $\eta(x') = c\eta(x)$. Since η was chosen arbitrarily, $x' = cx$.

Let $S = k[X \times Z] = k[X] \otimes k[Z]$. Let $k[X]_d$ denote the set of all polynomials in $k[X]$ which are homogeneous of degree d. We put $S_d = k[X]_d \otimes k[Z]$. Then S is a graded k-algebra with $S_0 = k[Z]$. Furthermore, the action of G on S preserves degree and the mapping ι^* gives an isomorphism between $(S_d)^G$ and $(k[X]_d)^H$.

In what follows, we let $f \in (k[X]_d)^H$ and choose $F \in (S_d)^G$ so that $\iota^* F = f$. We shall denote the localization of S at F by S_F. Also, let $(S_F)_0$ be the algebra consisting of all elements having degree 0 in the graded k-algebra S_F. We note that G acts on $(S_F)_0$. Let W be the affine variety corresponding to $(S_F)_0$. Let

$$(X \times Z)_F = \{(x', z') \in X \times Z\colon\ F(x', z') \neq 0\}.$$

Let $\pi\colon (X \times Z)_F \to W$ be the map corresponding to the inclusion of $(S_F)_0$ in S_F.

Remark. Let x, x' be elements in X with $f(x) = f(x') \neq 0$. Suppose that there is a g in G such that $\pi(x', z) = g\pi(x, z)$. Then g is in H and there is a c in k^* such that $c' = cgx$. Indeed, if $s \in k[Z]$, then $s \in k[W]$ and $s(x', z) = s(gx, gz)$. Hence, $z = gz$ and g is in H. In general, if s is in $k[X]_e$, then s^d/F^e is in $k[W]$. Therefore, $s^d(x') = s^d(gx)$ and $x' = cgx$ by Lemma 1.

LEMMA 2. *Let (x, z) be a point in $(X \times Z)_F$ such that the orbit $G(x, z)$ is G-separated on $X \times Z$. Then $G\pi(x, z)$ is G-separated in W.*

Proof. According to (RED3), it suffices to show that $G\pi(x, z)$ is closed and of maximal dimension in W. Let γ be a one-parameter subgroup in G so that $\text{limit}_{a\to 0}\gamma(a)\pi(x, z)$ exists. If s is any element in S_e, then s^d/F^e is in $k[W]$ and $\text{limit}_{a\to 0}(s^d/F^e)(\gamma(a)\pi(x, z))$ must exist. Since F is in S^G, we see that $\text{limit}_{a\to 0}s^d(\gamma(a)(x, z))$ must exist and, so, $\text{limit}_{a\to 0}\gamma(a)(x, z)$ exists in $X \times Z$. But $G(x, z)$ is closed and, so, there is a g in G with $\text{limit}_{a\to 0}\gamma(a)(x, z) = g(x, z)$. Then $\text{limit}_{a\to 0}\gamma(a)\pi(x, z) = g\pi(x, z)$. We may now apply (RED4) to conclude that the orbit $G\pi(x, z)$ is closed.

To prove that $G\pi(x, z)$ has maximal dimension, we shall show that the stabilizers in G of (x, z) and $\pi(x, z)$ have the same dimension. Let $g \in G$ satisfy $g\pi(x, z) = \pi(x, z)$. According to the remark above, g is in H and there is a c in k^* with $x = cgx$. But $\{c \in k^*\colon$ there is an $h \in H$ with $hx = cx\}$ must be finite. For otherwise, $(0, z)$ is in the closure of the (closed) orbit $G(x, z)$.

Now we prove (2). Let $[x']$ be a point in X_f^* which cannot be separated from $[x]$ by an polynomial in $k[X_f^*]^H$. We may assume that $f(x) = f(x')$. If $\pi(x', z) \in G\pi(x, z)$, then $[x'] \in H[x]$ by the remark above. Otherwise, by Lemma 2, there is a G-invariant polynomial s/F^e which separates $\pi(x', z)$ and $\pi(x, z)$. Then $\iota^* s/f^e$ separates $[x]$ and $[x']$.

THEOREM 4. *Let $X_0^*(H)$ consist of all those points $[x]$ in X^* such that the orbit Hx is H-separated on X and $G_x = \{e\}$. Then $X_0^*(H)$ is open in X^*, the quotient $X_0^*(H)/H$ exists and is quasi-projective.*

§ 5. Dimension arguments

Assumptions and notation

As always, G will denote a connected semisimple algebraic group over k. If γ is a non-trivial one-parameter subgroup of G, we define $P(\gamma)$ to be $\{g \in G\colon \text{limit}_{a\to 0}\gamma(a)g\gamma(a)^{-1}$ exists in $G\}$. Then, $P(\gamma)$ is a parabolic subgroup of G [11; p. 55]. Let $T = T(\gamma)$ be any maximal torus in $P(\gamma)$ containing γ. Let $B(\gamma) = TU(\gamma)$ be any Borel subgroup in $P(\gamma)$. Let $\Phi(G, T)$ be the set of roots of G relative to T. For each $\alpha \in \Phi(G, T)$, let U_α denote the corresponding one-dimensional unipotent subgroup of G and let $\varepsilon_\alpha\colon \mathbf{G}_a \to U_\alpha$ be the corresponding isomorphism [9; Theorem, p. 161]. The group $P(\gamma)$ corresponds to those roots $\alpha \in \Phi(G, T)$ such that $\langle\alpha, \gamma\rangle \geqslant 0$; its unipotent radical corresponds to the collection of all those roots α such that $\langle\alpha, \gamma\rangle > 0$. Let $W(G, T)$ be the Weyl group of G relative to T. If $s \in N_G(T)$, we shall denote by w_s the element $sT \in W(G, T)$.

LEMMA 1. *Let χ be a nonzero character of T such that $\langle\chi, \gamma\rangle \geqslant 0$ (resp. $\langle\chi, \gamma\rangle \leqslant 0$). There is an element $w_s \in W(G, T)$ such that $\langle w_s\chi, \gamma\rangle < 0$ (resp. $\langle w_s\chi, \gamma\rangle > 0$).*

Proof. We may assume that G is simple. Let $X(T)$ be the character group of T and let $V = X(T) \otimes \mathbf{R}$. Since G is simple, $W(G, T)$ acts irreducibly on V [2; Corollaire, p. 146]. Let $A_+ = \{v \in V\colon \langle v, \gamma \rangle \geqslant 0\}$, let $A_- = \{v \in V\colon \langle v, \gamma \rangle \leqslant 0\}$, and let $A_0 = A_+ \cap A_-$. Suppose for instance, that $\chi \in A_+$. Let $\chi' = \sum_w w\chi$, where the sum is over all $w \in W(G, T)$. Since χ' is fixed by each w in $W(G, T)$, we have $\chi' = 0$. So, if $w\chi$ is in A_+ for all $w \in W(G, T)$, we see that $w\chi \in A_0$ for all $w \in W(G, T)$. Then, $\sum_w \mathbf{R} w\chi$ is a $W(G, T)$-invariant, nontrivial, proper subspace of V. This contradicts our assumption that G is simple.

LEMMA 2 [14; Lemma, p. 245]. *Let V be any finite-dimensional vector space on which G acts via a rational representation. Let χ be a weight of T on V, let $v \in V$, $v \neq 0$, satisfy $t \cdot v = \chi(t)v$ for all $t \in T$, and let $\alpha \in \Phi(G, T)$ be a root such that $U_\alpha \subset G_v$. Then,*

(a) $(\chi, \alpha) \geqslant 0$;

(b) $(\chi, \alpha) = 0$ *if and only if* $U_{-\alpha} \subset G_v$.

LEMMA 3. *Let V be any finite-dimensional vector space on which G acts via a rational representation. Suppose that $\{v \in V\colon G_v$ contains an infinite normal subgroup of $G\} = \{0\}$. Let T be (any) maximal torus in G. Let P be any parabolic subgroup of G which contains T and let P_u be the unipotent radical of P. If v is any vector in V with $G_v \supset TP_u$, then $v = 0$.*

Proof. We may assume that G is simply connected and is a direct product of simple algebraic groups, say, $G = G_1 \times \ldots \times G_r$. Let $P = RP_u$ be a Levi decomposition of P where R is a reductive group. Let U_R be any maximal unipotent subgroup of R which is normalized by T; let $U = U_R P_u$. Let $\{\alpha_1, \ldots, \alpha_m, \ldots, \alpha_n\}$ be a simple root system in $\Phi(G, T)$ corresponding to U; we shall assume that $\{\alpha_1, \ldots, \alpha_m\}$ is a simple root system in $\Phi(R, T)$ corresponding to U_R.

Let W be the subspace of V spanned by all vectors uv where $u \in U$. If $\dim W = 1$, then TU fixes v; hence, G fixes v (since G/TU is complete) and $v = 0$ by assumption. So we may assume that $\dim W > 1$. We note that each $uv = u_R v$ for some $u_R \in U_R$ since $P_u \subset G_v$. The vector space W is stable with respect to TU and the weights of T on W have the form $e_1\alpha_1 + \ldots + e_m\alpha_m$ where each e_i is a nonnegative integer [9; Proposition, p. 165]. There are nonzero T weight vectors $w \in W$ fixed by U. Let w_0 be such a vector and suppose (to simplify the notation) that w_0 has T-weight $\omega = e_1\alpha_1 + \ldots + e_r\alpha_r$ where $e_i > 0$. Since U fixes w_0, we have $(\omega, \alpha) \geqslant 0$ for all roots α of U by Lemma 2a. If there is a simple root β among $\{\alpha_{r+1}, \ldots, \alpha_n\}$ connected to one of the $\alpha_1, \ldots, \alpha_r$, then $(\omega, \beta) < 0$. We may conclude that $\{\beta \in \Phi(G, T)\colon (\omega, \beta) = 0\}$ contains all the roots of a simple factor, say G_1, of G. Then G_1 fixes w_0 (by Lemma 2b) which contradicts our assumption.

THEOREM 1. *Let V be any finite-dimensional vector space on which G acts via a rational representation. Suppose that $\{v \in V\colon G_v$ contains an infinite normal* *subgroup of* $G\} = \{0\}$. *Let* γ *be a nontrivial one-parameter subgroup in* G *and let* $V(\gamma) = \{v \in V\colon \operatorname{limit}_{a \to 0} \gamma(a) v = 0\}$. *Then*

$$\dim V \leqslant \dim V(\gamma)\,[1 + \operatorname{Card} W(G, T)].$$

Proof. Let χ be a nonzero weight of T on V and let $V_\chi = \{v \in V\colon t \cdot v = \chi(t) v$ for all $t \in T\}$. Since $sV(\chi) = V(w_s(\chi))$ for all $w_s \in W(G, T)$, we may apply Lemma 1 to conclude that there is a $w_s \in W(G, T)$ such that $sV(\chi) \subset V(\gamma)$. It follows that

$$\dim\Big(\sum_{\chi \neq 0} V(\chi)\Big) \leqslant \operatorname{Card} W(G, T) \cdot \dim V(\gamma).$$

The proof of Theorem 1 now follows from

LEMMA 4. *If* $V_0 = \{v \in V\colon t \cdot v = v$ *for all* $t \in T\}$, *then* $\dim V_0 \leqslant \dim V(\gamma)$.

Proof. Let $\alpha_1, \ldots, \alpha_r$ be all the roots in the unipotent radical $P(\gamma)_u$ of $P(\gamma)$. We recall that each $\langle \alpha_i, \gamma \rangle > 0$. Let $u = \varepsilon_{\alpha_1}(1) \ldots \varepsilon_{\alpha_r}(1)$. Let $v \in V_0$, $v \neq 0$. We claim that (i) $uv \neq v$ and (ii) $uv - v \in V(\gamma)$. To prove (i), we first note that $G_v \supset T$. Then $G_v \cap P(\gamma)_u$ is directly spanned by those U_α it contains [9; Proposition, p. 170]. If u is in $G_v \cap P(\gamma)_u$, then so is each U_α where $\alpha = \alpha_1, \ldots, \alpha_r$. Then $P(\gamma)_u \subset G_v$ and, by Lemma 3, $v = 0$ which is a contradiction. To prove (ii), we note that $uv - v$ is a sum of weight vectors whose weights have the form $m_1 \alpha_1 + \ldots + m_r \alpha_r$ where each m_i is a nonnegative integer [9; Proposition, p. 165]. Since each $\langle \alpha_i, \gamma \rangle > 0$, we see that $uv - v \in V(\gamma)$. This proves (ii). Therefore, the linear transformation $(u - I)$ gives an isomorphism of V_0 into $V(\gamma)$ and $\dim V_0 \leqslant \dim V(\gamma)$.

THEOREM 2. *Let* V *be any finite-dimensional vector space on which* G *acts via a rational representation. Suppose that* $\{v \in V\colon G_v$ *contains an infinite normal subgroup of* $G\} = \{0\}$. *Suppose that*

$$\dim V > \dim H\,[\operatorname{Card} W(G, T) + 1].$$

Then (C1) $\sim$ (C2) *for all* $v \in V$.

Proof. We need only show that condition (C1) implies condition (C2) (Lemma 1(b), § 3). Let $v \in V$ be a point such that the orbit Hv is H-separated on V but the orbit $G(v, z)$ is not G-separated on $V \times Z$. According to Theorem 1, § 3, there is a one-parameter subgroup γ in $\Gamma(G/H)$ such that $v + V(\gamma) \subset Hv$. Hence, $\dim H \geqslant \dim V(\gamma)$. The desired contradiction is now obtained by applying Theorem 1.

§ 6. Groups normalized by a maximal torus

Assumptions and notation

In this section, we shall always assume that H is a unipotent subgroup of G. Let $N_G(H)$ be the normalizer of H in G. We shall assume throughout this section that $N_G(H)$ is a parabolic subgroup of G. Let T be any maximal torus in $N_G(H)$. Let $\Phi(G, T)$ be the set of roots of G relative to T. For each root $\alpha \in \Phi(G, T)$, let U_α denote the corresponding one-dimensional unipotent subgroup of G. The group H is connected and directly spanned by those U_α which are contained in H [9; Proposition, p. 170]. Let $\Phi(H, T)$ be the set of all roots $\alpha \in \Phi(G, T)$ such that $U_\alpha \subset H$. Since T normalizes H, the group T acts on $H' = k[G]^H$ via right multiplication, ϱ. If χ is any character of T, we put $H'_\chi = \{f \in H' : \varrho(t)f = \chi(t)f \text{ for all } t \in T\}$. Let $X(G/H; T)$ be the set of all characters χ of T such that $H'_\chi \neq \{0\}$. Finally, we recall that $\Gamma(G/H; T)$ consists of all one-parameter subgroups γ of T such that $\gamma \in \Gamma(G/H)$.

LEMMA 1. *Let $\chi \in X(T)$. Then $\chi \in X(G/H; T)$ if and only if $(\chi, \alpha) \geqslant 0$ for all $\alpha \in \Phi(H, T)$.*

Proof. If $H'_\chi = \{0\}$, then $(\chi, \alpha) \geqslant 0$ for all $\alpha \in \Phi(H, T)$ by Lemma 2a, § 5. Now, suppose that $(\chi, \alpha) \geqslant 0$ for all $\alpha \in \Phi(H, T)$. Let $X(T)$ be ordered so that χ is a dominant weight. Let V be the irreducible G-module with highest weight χ and let v be a highest weight vector in V. Let $\{v = v_1, \ldots, v_r\}$ be a basis of V and let $\{\mu_1, \ldots, \mu_r\}$ be the corresponding dual basis of V^*. Since $(\chi, \alpha) \geqslant 0$ for all $\alpha \in \Phi(H, T)$, we have $H \subset G_v$. We define $f \in k[G]$ by $f(g) = \mu_1(gv)$. Then (it is easy to check that) $f \neq 0$ and $f \in H'_\chi$. This completes the proof.

LEMMA 2. *Let γ be a nontrivial one-parameter subgroup of T. Then $\gamma \in \Gamma(G/H; T)$ if and only if $\langle \chi, \gamma \rangle \geqslant 0$ for all $\chi \in X(G/H; T)$.*

Proof. We note that $\text{limit}_{a \to 0}\gamma(a)z$ exists in Z if and only if $\text{limit}_{a \to 0} f(\gamma(a)z)$ exists for all $f \in k[Z]$. Let $\pi: G \to Z$ be defined by $\pi(g) = gz$. Let $\pi^*: k[Z] \to k[G]^H$ be the corresponding isomorphism.

Suppose, first, that $\text{limit}_{a \to 0}\gamma(a)z$ exists. Let $\chi \in X(G/H; T)$ and let $f \in k[Z]$ be a nonzero function so that $\pi^* f \in H'_\chi$. For each $g \in G$, $t \in T$, we have

$$\chi(t)(\pi^* f)(g) = \varrho(t)(\pi^* f)(g) = f(gtz).$$

In particular, if $t = \gamma(a)$ we have

$$(+) \qquad a^d f(gz) = (g^{-1} f)(\gamma(a)z)$$

where $d = \langle \chi, \gamma \rangle$. Let g be chosen so that $f(gz) \neq 0$. By assumption, $\text{limit}_{a \to 0}(g^{-1} f)(\gamma(a)z)$ exists. Hence, we use equation (+) to conclude that $\langle \chi, \gamma \rangle \geqslant 0$.

Now, suppose that $\langle \chi, \gamma \rangle \geqslant 0$ for all $\chi \in X(G/H; T)$. Let f be a nonzero function in $k[Z]$ such that $\pi^* f \in H'_\chi$. Then

$$f(\gamma(a)z) = (\pi^* f)(\gamma(a)) = (\pi^* f)(e\gamma(a)) = \varrho(\gamma(a))(\pi^* f)(e) = a^d f(z)$$

where $d = \langle \chi, \gamma \rangle$. Hence, $\text{limit}_{a \to 0} f(\gamma(a)z)$ exists. It follows that $\text{limit}_{a \to 0} f(\gamma(a)z)$ exists for every function f in $k[Z]$.

Note. The algebra H' is finitely generated over k. We may suppose that $H' = k[f_1, \ldots, f_m]$ where $\varrho(t)f_i = \chi_i(t)f_i$ for some character χ_i in $X(T)$. It follows that $\chi \in X(G/H;\, T)$ if and only if there are nonnegative integers $e_1, \ldots, e_m$ such that $\chi = e_1\chi_1 + \ldots + e_m\chi_m$. Furthermore, $\langle \chi, \gamma \rangle \geqslant 0$ for all $\chi \in X(G/H;\, T)$ if and only if each $\langle \chi_i, \gamma \rangle \geqslant 0$.

LEMMA 3. *Let γ be a one-parameter subgroup of G. Let $P(\gamma) = \{g \in G$: $\operatorname{limit}_{a\to 0}\gamma(a)g\gamma(a)^{-1}$ exists in $G\}$. Then $P(\gamma)$ is a parabolic subgroup of G. Let X be any affine variety on which G acts regularly. Let $x \in X$ and suppose that $\operatorname{limit}_{a\to 0}\gamma(a)x$ exists. If $p \in P(\gamma)$, then $\operatorname{limit}_{a\to 0}\gamma(a)px$ exists.*

Proof. We observed in § 5 that $P(\gamma)$ is parabolic. Now,

$$\operatorname{limit}_{a\to 0}\gamma(a)px = \operatorname{limit}_{a\to 0}(\gamma(a)p\gamma(a)^{-1})(\gamma(a)x) = p'x'$$

where $p' = \operatorname{limit}_{a\to 0}\gamma(a)p\gamma(a)^{-1}$ is in $P(\gamma)$ and $x' = \operatorname{limit}_{a\to 0}\gamma(a)x$ is in X.

LEMMA 4. *Let $n \in N_G(H)$. Let z' be any point in Z whose stabilizer in G is H. Let $\gamma \in \Gamma(G/H)$. Then $\operatorname{limit}_{a\to 0}\gamma(a)nz'$ exists.*

Proof. This was shown at the end of § 2.

LEMMA 5. *Let X be any affine variety on which G acts regularly. Let $x \in X$. The following statements are equivalent.*

(a) *There is a $\gamma \in \Gamma(G/H)$ such that $\operatorname{limit}_{a\to 0}\gamma(a)x$ exists.*

(b) *There is a $\gamma \in \Gamma(G/H;\, T)$ and an $n \in N_G(H)$ such that $\operatorname{limit}_{a\to 0}\gamma(a)nx$ exists.*

Proof. Suppose that (a) holds. The intersection of two parabolic subgroups of G always contains a maximal torus in G [18]. Let T_1 be a maximal torus in $P(\gamma) \cap N_G(H)$ and let $p \in P(\gamma)$ be chosen so that $\gamma_1 = p^{-1}\gamma p \subset T_1$. Now γ_1 is in $\Gamma(G/H)$ since $\operatorname{limit}_{a\to 0}\gamma_1(a)z = p^{-1}\operatorname{limit}_{a\to 0}\gamma(a)pz$ exists by Lemma 3. Similarly, $\operatorname{limit}_{a\to 0}\gamma_1(a)x$ exists. Let $\operatorname{limit}_{a\to 0}\gamma_1(a)(x, z) = (x_1, z_1)$. Let $n \in N_G(H)$ be chosen so that $\gamma_2 = n\gamma_1 n^{-1} \subset T$. Then, $\operatorname{limit}_{a\to 0}\gamma_2(a)(nx, nz) = (nx_1, nz_1)$. We apply Lemma 4 to see that $\operatorname{limit}_{a\to 0}\gamma_2(a)(nx, z)$ exists. This proves (b).

If statement (b) holds, then $\operatorname{limit}_{a\to 0}\gamma(a)(nx, nz)$ exists by Lemma 4. The one-parameter subgroup $\gamma_1 = n^{-1}\gamma n$ has the desired properties.

We now restate Theorems 1, 2, 3 in § 3 in light of the lemmas above. First, though, let $x \in X$ and let $n \in N_G(H)$. The orbit Hx is H-separated on X if and only if the orbit $H(nx)$ is, since $N_G(H)$ sends $k[X]^H$ to itself.

THEOREM 1. *Let H be unipotent. Let $N_G(H)$ be a parabolic subgroup of G and let T be a maximal torus in $N_G(H)$. Let x be a point in X such that the orbit Hx is H-separated on X but the orbit $G(x, z)$ is not G-separated on $X \times Z$. There is a nontrivial one-parameter subgroup γ in T and an $n \in N_G(H)$ such that* (i) *$\operatorname{limit}_{a\to 0}\gamma(a)nx$ exists and* (ii) *$\langle \chi, \gamma \rangle \geqslant 0$ for all $\chi \in X(G/H;\, T)$. Furthermore,* *let* $x' = \operatorname{limit}_{a\to 0}\gamma(a)nx$. *If* x'' *is any point in* X *such that* $\operatorname{limit}_{a\to 0}\gamma(a)x'' = x'$, *then* $x'' \in Hnx$.

THEOREM 2. *Let* H *be unipotent. Let* $N_G(H)$ *be a parabolic subgroup of* G *and let* T *be a maximal torus in* $N_G(H)$. *Let* x *be a point in* X *such that* $\dim(Gx) = \dim G$. *Then the following statements are equivalent.*

(a) *The orbit* Hx *is not* H*-separated on* X.

(b) *There is a nontrivial one-parameter subgroup* γ *in* T *and an* $n \in N_G(H)$ *such that* (i) $\operatorname{limit}_{a\to 0}\gamma(a)nx$ *exists and* (ii) $\langle\chi, \gamma\rangle \geqslant 0$ *for all* $\chi \in X(G/H;\, T)$.

THEOREM 3. *Let* H *be unipotent. Let* $N_G(H)$ *be a parabolic subgroup of* G *and let* T *be a maximal torus in* $N_G(H)$. *Let* $x \in X$ *and suppose that* (C1) $\sim$ (C2) *at* x. *Then, the following statements are equivalent.*

(a) *The orbit* Hx *is not* H*-separated on* X.

(b) *There is a nontrivial one-parameter subgroup* γ *in* T *and an* $n \in N_G(H)$ *such that* (i) $\operatorname{limit}_{a\to 0}\gamma(a)nx$ *exists and* (ii) $\langle\chi, \gamma\rangle \geqslant 0$ *for all* χ *in* $X(G/H;\, T)$.

THEOREM 4. *Let* $B = TU$ *be a Borel subgroup of* G. *Let* $\{\omega_1, \ldots, \omega_r\}$ *be a corresponding system of fundamental dominant weights of* T. *Let* V *be a finite-dimensional vector space on which* G *operates via a rational representation. Suppose that* $H = U$ *and that* (C1) $\sim$ (C2) *at a point* v *in* V. *Then the following conditions are equivalent.*

(a) *The orbit* Uv *is not* U*-separated on* V.

(b) *There is a nontrivial one-parameter subgroups* γ *in* T *and an element* u *in* U *such that* (i) $\operatorname{limit}_{a\to 0}\gamma(a)uv$ *exists and* (ii) $\langle\omega_1, \gamma\rangle \geqslant 0$ *for each* $i = 1, \ldots, r$.

Proof. This follows immediately from Theorem 3.

THEOREM 5. *Let* $B = TU$ *be a Borel subgroup of* G. *Let* $\{\omega_1, \ldots, \omega_r\}$ *be a corresponding system of fundamental dominant weights of* T. *Let* V *be a finite-dimensional vector space on which* G *operates via a rational representation. Let* v *be a point in* V *such that the orbit* Uv *is separated on* V *but the orbit* $G(u, v)$ *is not* G*-separated on* $V \times Z$. *There is a nontrivial one-parameter subgroup* γ *in* T *and an element* $u \in U$ *such that* (i) $\gamma(a)uv = uv$ *for all* $a \in k^*$ *and* (ii) $\langle\omega_i, \gamma\rangle \geqslant 0$ *for each* $i = 1, \ldots, r$. *Furthermore,* $Uv \supset uv + V(\gamma)$ *where* $V(\gamma) = \{v \in V\colon \operatorname{limit}_{a\to 0}\gamma(a)v = 0\}$.

Proof. According to Theorem 1, there is a nontrivial one-parameter subgroup γ in T and an element u_1 in U such that (i) $\operatorname{limit}_{a\to 0}\gamma(a)u_1 v$ exists, say is v', and (ii) $\langle\gamma, \omega_i\rangle \geqslant 0$ for each $i = 1, \ldots, r$. Furthermore, $U(u_1 v) \supset v' + V(\gamma)$. In particular, $v' \in Uv$. If we choose $u \in U$ so that $uv = v'$, we obtain the theorem.

§ 7. Binary forms

Throughout this section, we shall assume that $\operatorname{char} k = 0$. Let us fix a positive integer d and let V_d be the vector space consisting of all binary forms of degree d in the variables X_0 and X_1. The vector space V_d has a basis $\binom{d}{i} X_0^{d-i} X_1^i$ for

$i = 0, 1, \ldots, d$. Let $G = \mathrm{SL}_2(k)$, let $H = U = \{(a_{ij}) \in \mathrm{SL}_2(k)\colon a_{21} = 0,\ a_{11} = a_{22} = 1\}$, and let $T = \{(a_{ij}) \in \mathrm{SL}_2(k)\colon a_{21} = a_{12} = 0\}$. We shall denote elements $t = (a_{ij})$ in T by $t = \mathrm{diag}(a_{11}, a_{22})$.

The natural representation (by left multiplication) of G on the vector space k^2, consisting of all 2×1 column matrices, gives rise to an action of G on $k[X_0, X_1]$. In particular, we have

$$(a_{ij}) \cdot X_0 = a_{22} X_0 - a_{12} X_1 \quad \text{and} \quad (a_{ij}) \cdot X_1 = -a_{21} X_0 + a_{11} X_1.$$

This, in turn, gives a representation of G on V_d.

The invariants of G acting on $V_d \times k^2$ were called "covariants" in the nineteenth century. M. Roberts (1861) proved that $k[V_d]^U = k[V_d \times k^2]^G$ which is the Theorem of § 2 in this setting. Complete lists of covariants are known for $1 \leqslant d \leqslant 8$. (For $1 \leqslant d \leqslant 6$, a good exposition can be found in the text by Grace and Young. The cases $d = 7, 8$ were studied by von Gall. A modern account of $1 \leqslant d \leqslant 4$ may be found in [16].)

THEOREM 1. *Let $f \in V_d$. Then* (C1) $\sim$ (C2) *at f.*

Proof. Let $f = a_0 X_0^d + a_1 \binom{d}{1} X_0^{d-1} X_1 + \ldots + a_d \binom{d}{d} X_1^d$. If $d = 1$, the only U-invariants are polynomials in a_0. If $d = 2$, the only U-invariants are polynomials in a_0 and $4a_0 a_2 - a_1^2$. In both cases, the theorem follows at once. If $d \geqslant 3$, we may apply Theorem 2, § 5.

THEOREM 2. *Let $d \geqslant 1$ and let $f \in V_d$. The orbit Uf is U-separated on V_d if and only if the multiplicity of* $(1, 0)$ *as a root of f is* $< d/2$.

Proof. This follows from Theorem 4, § 6, since $\gamma(a) = \mathrm{diag}(a, a^{-1})$.

The result above was discovered recently by A. Cerezo [4] via a direct computation of invariants. It may be interesting to compare this theorem with the known facts on G, namely: the orbit Gf is G-separated in V_d if and only if the multiplicity of every root of f is $< d/2$ [11].

§ 8. Plane cubics: H = maximal unipotent subgroup

Let V be the vector space consisting of all polynomials f having the form

$$f = a_{30} X_1^3 + a_{21} X_1^2 X_2 + a_{12} X_1 X_2^2 + a_{03} X_2^3 + a_{20} X_0 X_1^2 + a_{11} X_0 X_1 X_2 + a_{02} X_0 X_2^2 + a_{10} X_0^2 X_1 + a_{01} X_0^2 X_2 + a_{00} X_0^3.$$

The representation by left multiplication of $G = \mathrm{SL}_3(k)$ on the vector space k^3, consisting of all 3×1 column matrices, gives rise to an action on $k[X_0, X_1, X_2]$. This, in turn, gives a representation of G on V.

Let $H = U = \{(a_{ij}) \in G\colon a_{ij} = 0 \text{ for } i > j \text{ and } a_{ii} = 1 \text{ for } i = 1, 2, 3\}$. Then U is a maximal unipotent subgroup of G. Let $T = \{(a_{ij}) \in G\colon a_{ij} = 0 \text{ for } i \neq j\}$. Then T is a maximal torus in G which normalizes U. We shall denote elements

$t = (a_{ij})$ in T by $t = \operatorname{diag}(a_{11}, a_{22}, a_{33})$. We note that $t \cdot X_0 = a_{11}^{-1} X_0$, $t \cdot X_1 = a_{22}^{-1} X_1$, and $t \cdot X_2 = a_{33}^{-1} X_2$.

Let γ be a one-parameter subgroup of T with $\gamma(a) = \operatorname{diag}(a^r, a^s, a^{-r-s})$. Then (by Lemmas 1 and 2 in § 6) we have $\gamma \in \Gamma(G/U;\, T)$ if and only if $r \geqslant 0$ and $r+s \geqslant 0$. We partition these one-parameter subgroups into six classes and, in each case, give a geometric interpretation for those f in V such that $\operatorname{limit}_{a \to 0} \gamma(a) f$ exists.

(1) $r > 0$, $s > 0$, $r = s$: $\operatorname{limit}_{a \to 0} \gamma(a) f$ exists if and only if X_2 divides f.

(2) $r > 0$, $s > 0$, $r \neq s$: $\operatorname{limit}_{a \to 0} \gamma(a) f$ exists, then X_2 divides f.

(3) $r > 0$, $s = 0$: $\operatorname{limit}_{a \to 0} \gamma(a) f$ exists if and only if $(1, 0, 0)$ is a singular point of f and X_3 is a tangent at $(1, 0, 0)$.

(4) $r = 0$, $s > 0$: $\operatorname{limit}_{a \to 0} \gamma(a) f$ exists if and only if $(0, 1, 0)$ is a singular point of f and X_2 is a tangent at $(0, 1, 0)$.

(5) $r > 0$, $s < 0$, $r+2s = 0$: $\operatorname{limit}_{a \to 0} \gamma(a) f$ exists if and only if $(1, 0, 0)$ is a singular point of f.

(6) $r > 0$, $s < 0$, $r+s \geqslant 0$, $r+2s \neq 0$: $\operatorname{limit}_{a \to 0} \gamma(a) f$ exists, then $(1, 0, 0)$ is a singular point of f.

THEOREM 1. *Let $f \in V$. Then* (C1) $\sim$ (C2) *at f.*

Proof. Suppose there is an $f \in V$ such that (C1) holds at f but not (C2). According to Theorem 5, § 6, there is a one-parameter subgroup γ in one of the six classes above and a $u \in U$ such that (i) $\gamma(a) uf = uf$ for all $a \in k^*$ and (ii) $U(uf) \supset uf + V(\gamma)$. If γ is of types (2)–(6), then (a direct computation shows that) $\dim V(\gamma) \geqslant 4 > 3 = \dim U$ so (ii) cannot hold.

Let us assume γ is of type (1). Then uf must be a linear combination of the monomials $X_1^2 X_2$, $X_0 X_1 X_2$, and $X_0^2 X_2$. Furthermore, $V(\gamma)$ has basis consisting of $X_1 X_2^2$, X_2^3, and $X_0 X_2^2$. A direct computation (which we omit) shows that (ii) cannot hold.

THEOREM 2. *Let f be a plane cubic. The orbit Uf is not U-separated if and only if one of the following conditions holds*:

(a) *X_2 divides f*;

(b) *$(1, 0, 0)$ is a singular point of f*;

(c) *there is an $a \in k$ such that $(a, 1, 0)$ is a singular point of f and X_2 is tangent at $(a, 1, 0)$.*

Proof. According to Theorem 1, conditions (C1) and (C2) are equivalent at each $f \in V$. Hence, we may apply Theorem 4, § 6. The proof then follows from the geometric descriptions given above.

§ 9. Plane cubics: H = unipotent radical

We follow the notation introduced in § 8. Let $H = \{(a_{ij}) \in U\colon\ a_{12} = 0\}$. Then H is the unipotent radical of a parabolic subgroup of G and has roots

α_2, $\alpha_1+\alpha_2$ (in the usual notation). In light of Lemma 1, § 6, we wish to describe all the characters χ in $X(T)$ such that $(\chi, \alpha_2) \geqslant 0$ and $(\chi, \alpha_1+\alpha_2) \geqslant 0$. It is easy to check that this set is precisely the set of all linear combinations $a\omega_1+b\omega_2+c(\omega_2-\omega_1)$ where a, b, c are nonnegative integers and $\omega_1(\operatorname{diag}(a_{11}, a_{22}, a_{33})) = a_{11}$, $\omega_2(\operatorname{diag}(a_{11}, a_{22}, a_{33})) = a_{11}a_{22}$. If γ is a one-parameter subgroup in T such that $\langle\gamma, \omega_1\rangle \geqslant 0$, $\langle\gamma, \omega_2\rangle \geqslant 0$, and $\langle\gamma, \omega_2-\omega_1\rangle \geqslant 0$, then $\gamma(a) = \operatorname{diag}(a^r, a^s, a^{-r-s})$ where $r, s \geqslant 0$.

THEOREM 1. *Let $f \in V$. Then* (C1) $\sim$ (C2) *at f.*

Proof. The proof follows the lines of that for Theorem 1, § 8. It is easier here, though, since for every γ we have $\dim V(\gamma) \geqslant 3 > 2 = \dim H$.

THEOREM 2. *Let f be a plane cubic. The orbit Hf is not H-separated on V if and only if one of the following conditions holds*:

(a) *X_2 divides f*;

(b) *f has a nonzero singular point $(x, y, 0)$ at which X_2 is a tangent.*

If H is the unipotent radical with roots α_1, and $\alpha_1+\alpha_2$, then we may prove Theorem 1 just as above. Furthermore, if $\chi \in X(T)$ and $(\chi, \alpha_1) \geqslant 0$, $(\chi, \alpha_1+\alpha_2) \geqslant 0$, then χ may be written as $a\omega_1+b\omega_2+c(\omega_1-\omega_2)$ where a, b, c are nonnegative intergers, Therefore, $\gamma \in \Gamma(G/H; T)$ if and only if $\gamma(a) = (a^r, a^s, a^{-r-s})$ with $r > 0$, $s \leqslant 0$, and $r+s \geqslant 0$. It follows that an orbit Hf is not H-separated on V if and only if $(1, 0, 0)$ is a singular point of f.

References

[1] A. Borel, *Linear Algebraic Groups*, Benjamin, New York 1969.

[2] N. Bourbaki, *Eléments de mathématique*, Vol. XXXIV, *Groupes et algèbres de Lie*, Chap. VI. Hermann, Paris 1968.

[3] M. Brion, *Invariants d'un sous-groupe unipotent maximal d'un groupe semi-simple*, Ann. Inst. Fourier (Grenoble) 33 (1983) 1–27.

[4] A. Cerezo, *Sur les invariants algébriques du groupe engendré par une matrice nilpotente*, preprint.

[5] J. Dixmier and M. Reynaud, *Sur le quotient d'une variété algébrique par un groupe algébrique*, Mathematical analysis and applications, Part A. Advances in Math. Supplementary Studies, Vol. 7A (1981) 327–344.

[6] J. H. Grace and A. Young, *The Algebra of Invariants*, Cambridge University Press 1903, reprinted in Chelsea Publishing Company, Bronx, New York.

[7] F. Grosshans, *Hilbert's fourteenth problem for nonreductive groups*, Math. Z. 193 (1986) 95–103.

[8] —, *The invariants of unipotent radicals of parabolic subgroups*, Invent. Math. 73, (1983) 1–9.

[9] J. E. Humphreys, *Linear Algebraic Groups*, Springer, New York–Heidelberg–Berlin 1975.

[10] G. R. Kempf, *Instability in invariant theory*, Ann. of Math. 108 (1978) 299–316.

[11] D. Mumford, *Geometric Invariant Theory*, Springer, New York–Heidelberg–Berlin 1965, 2nd ed., with J. Fogarty, 1982.

[12] P. E. Newstead, *Introduction to Moduli Problems and Orbit Spaces*, Tata Institute, Bombay 1978.

[13] K. Pommerening, *Ordered sets with the standardizing property and straightening laws for algebras of invariants*, Adv. in Math. 63 (1987) 271–290.

[14] —, *Observable radizelle Untergruppen von halbeinfachen algebraischen Gruppen*, Math. Z. 165 (1979) 243–250.

[15] R. W. Richardson, *Affine coset spaces of reductive algebraic groups*, Bull. London Math. Soc. 9 (1977) 38–41.

[16] T. A. Springer, *Invariant Theory*, Lecture Notes in Math. 585, Springer, New York–Heidelberg–Berlin 1977.

[17] L. Tan, *The invariant theory of unipotent subgroups of reductive algebraic groups*, PhD dissertation, UCLA, Los Angeles 1986.

[18] J. Tits, *Théorème de Bruhat et sous-groupes paraboliques*, C. R. Acad. Sci. Paris 254 (1962) 2910–2912.