TOPICS IN ALGEBRA
BANACH CENTER PUBLICATIONS, VOLUME 26, PART 2
PWN POLISH SCIENTIFIC PUBLISHERS
WARSAW 1990

REPRESENTATIONS OF S_n AND GL_n AND THE q-SCHUR ALGEBRA

GORDON JAMES*

Imperial College, London, England

1. Introduction

Recent work of Richard Dipper and the author has revealed a close connection between the following topics:

(i) the representation theory of the symmetric groups S_n,

(ii) the representation theory over F of the general linear groups $GL_n(F)$ ("representations in the describing characteristic"), and

(iii) the representation theory of $GL_n(q)$ over fields whose characteristic does not divide q ("representations in the nondescribing characteristic").

This article will survey our results concisely, with the intention of emphasizing the remarkable fact that, in a sense, the first two topics named above are "special cases" of the third. In particular if we could find the p-modular decomposition matrices of $GL_n(q)$ for those primes p which do not divide q, then we would be able to determine the decomposition matrices of S_n, and the p-modular decomposition matrices of the groups $GL_n(q)$ where q is a power of p. The key lies in an algebra which we call the *q-Schur algebra*. The p-modular decomposition matrices of the q-Schur algebra for primes p which do not divide q completely determine all the decomposition matrices which we have mentioned.

It has long been known how the pioneering work of Schur connects the representation theory of S_n to the representation theory of GL_n in the describing characteristic; our methods express both these theories as the case $q = 1$ of the representation theory of the q-Schur algebra.

---

* This article reports on joint work with R. Dipper, supported by NATO Grant Number 0222/87.

This paper is in final form and no version of it will be submitted for publication elsewhere.

2. Partitions

We say that λ is a *partition* of n (and write $\lambda \vdash n$) if $\lambda = (\lambda_1, \lambda_2, \ldots)$ is a non-increasing sequence of nonnegative integers whose sum is n. The conjugacy classes of S_n may be indexed by partitions of n.

EXAMPLE. $n = 4$

Conjugacy class representative	*Partition*
$(1)(2)(3)(4)$,	$(1, 1, 1, 1, 0, \ldots) = (1^4)$,
$(12)(3)(4)$,	$(2, 1, 0, 0, 0, \ldots) = (21^2)$,
$(12)(34)$,	$(2, 2, 0, 0, 0, \ldots) = (2^2)$,
$(123)(4)$,	$(3, 1, 0, 0, 0, \ldots) = (31)$,
(1234),	$(4, 0, 0, 0, 0, \ldots) = (4)$.

A matrix is said to be *unipotent* if and only if all its eigenvalues are 1. The unipotent conjugacy classes of general linear groups may be indexed by partitions of n, using the Jordan canonical form.

EXAMPLE. $n = 4$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \leftrightarrow (1^4), \qquad \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \leftrightarrow (21^2),$$

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \leftrightarrow (2^2), \qquad \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \leftrightarrow (31),$$

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \leftrightarrow (4).$$

3. Where to look for representations

We address the problem of investigating the representation theory of symmetric and general linear groups from scratch. For simplicity, we shall always assume that F is an algebraically closed field. The representation theory of arbitrary groups shows that the number of irreducible representations is related to the number of conjugacy classes, so the remarks in the last section suggest that we should look out for ways of using partitions.

(i) The symmetric group

For the symmetric group (indeed, for an arbitrary group) we can begin with the group algebra. We known that FS_n is a right FS_n-module.

If $\lambda \vdash n$, let $x_\lambda \in FS_n$ be defined by $x_\lambda = \sum \{\pi\colon \pi \in S_{\lambda_1} \times S_{\lambda_2} \times \ldots\}$.

We can improve upon our initial idea of considering just the group algebra FS_n by looking instead at the right ideals $x_\lambda FS_n$.

(ii) The general linear group

The group algebra of $\mathrm{GL}_n(q)$ is complicated, so we start by considering the permutation representation of $F\mathrm{GL}_n(q)$ on the subgroup B of upper triangular matrices

$$\begin{bmatrix} * & & * \\ & * & \\ & & \ddots \\ & & & * \end{bmatrix}.$$

The composition factors of this permutation representation are known as *unipotent representations*.

Once more, we can improve upon our first idea if we take the permutation representation on the subgroup of matrices of the form

$$\begin{array}{cc} & \begin{array}{ccc} \lambda_1 & \lambda_2 & \ldots \end{array} \\ \begin{array}{c} \lambda_1 \\ \lambda_2 \\ \\ \end{array} & \begin{bmatrix} \boxed{*} & & * \\ & \boxed{*} & \\ 0 & & \ddots \end{bmatrix} \end{array}.$$

It turns out that there are several difficulties with this approach in the case where the characteristic of F divides q. For example, consider what happens when we rearrange the order of the blocks on the diagonal. This does not affect the dimension of our permutation representation. However, the new permutation representation is isomorphic to the first one if and only if the characteristic of F does not divide q. In view of the problems, we reserve this approach for the situation where the characteristic of F does not divide q.

What, then, can be done about representations of $\mathrm{GL}_n(F)$ over F? Here we have a module already to hand, namely the n dimensional vector space V over F on which $\mathrm{GL}_n(F)$ acts. Consider

$$V^{\otimes n} := V \otimes V \otimes \ldots \otimes V \quad (n \text{ times}).$$

Then $\mathrm{GL}_n(q)$ acts on this space by

$$(v_1 \otimes v_2 \otimes \ldots \otimes v_n)g = v_1 g \otimes v_2 g \otimes \ldots \otimes v_n g.$$

Note that S_n acts on $V^{\otimes n}$ by place permutations. For example,

$$(\sum_{\pi \in S_n} \pi) V^{\otimes n}$$

could be called the "symmetric part" of $V^{\otimes n}$. If $\lambda = (\lambda_1, \lambda_2, \ldots) \vdash n$, then we can consider

$$x_\lambda V^{\otimes n} = (\text{the symmetric part of } V^{\otimes \lambda_1}) \otimes (\text{the symmetric part of } V^{\otimes \lambda_2}) \otimes \ldots$$

Since the action on $V^{\otimes n}$ of S_n commutes with the action of $\mathrm{GL}_n(F)$, $x_\lambda V^{\otimes n}$ gives us an $F\mathrm{GL}_n(F)$-module.

(iii) Summary

The following table records the tools we have introduced so far for investigating representations.

Table 1

	Symmetric group	GL_n in describing characteristic	$F\mathrm{GL}_n(q)$ when char $F \nmid q$
Group algebra	$A = FS_n$	$A = F\mathrm{GL}_n(F)$	$A = F\mathrm{GL}_n(q)$
Main module	$M = FS_n$	$M = V^{\otimes n}$	$M =$ perm. rep. on B
Refinement	$x_\lambda M$	$x_\lambda M$	Perm. rep. on $\begin{pmatrix} * & & * \\ & \boxed{*} & \\ 0 & & \ddots \end{pmatrix}$

In this table, we have used the same letters A and M three times to denote different objects. The plan is to pursue this threefold notation, until the theories coalesce and we have three cases of the *same* object.

The following remarks should reassure the reader that we are going in a sensible direction.

(i) The problem of finding the composition factors of the modules in the last row of Table 1 is "equivalent" to determining their indecomposable direct summands, and

(ii) if this problem can be solved then we can find the decomposition matrices of the symmetric and finite general linear groups.

We shall expand upon these comments later, but the first remark motivates the discussion in the next section.

4. Endomorphism algebras

Remember that an algebra is a vector space which is also a ring. If A is an algebra and M is a right A-module, then we may consider

$$\mathscr{H} := \mathrm{End}_A(M).$$

Thus, $\mathscr{H}$ consists of those endomorphisms h of M which satisfy

$$(hm)a = h(ma) \quad \text{for all } m \in M, a \in A.$$

We see that $\mathscr{H}$ is an algebra. It is often called an *endomorphism algebra*, or a *Hecke algebra.*

The structure of M is related to the structure of $\mathscr{H}$. One of the most important results in this vein is:

FITTING'S THEOREM (See, for example, [15, 1.4]). *For each way of writing M as a direct sum of A-modules,*

$$M = M_1 \oplus M_2 \oplus \dots \oplus M_m$$

we get a way of writing $\mathscr{H}$ as a direct sum of right ideals,

$$\mathscr{H} = \mathscr{H}_1 \oplus \mathscr{H}_2 \oplus \dots \oplus \mathscr{H}_m$$

and vice versa. Moreover, M_i is indecomposable if and only if $\mathscr{H}_i$ is indecomposable. Also, M_i is isomorphic to M_j if and only if $\mathscr{H}_i$ is isomorphic to $\mathscr{H}_j$.

We indicated at the end of the last section that indecomposable direct summands of certain modules will be of interest, so Fitting's Theorem motivates us to look at Hecke algebras.

EXAMPLE 1. Let $A = FS_n$ and $M = FS_n$. Then $\mathscr{H} := \mathrm{End}_A(M)$ is isomorphic to FS_n. We have achieved little by this manoeuvre, so we again seek a refinement which involves partitions. Therefore, let

$$\mathscr{S} := \mathrm{End}_A(\bigoplus_{\lambda \vdash n} x_\lambda M).$$

Notice that $\mathscr{S}$ contains the endomorphisms of each $x_\lambda M$ and the A-homomorphisms from $x_\lambda M$ to $x_\mu M$ (where $\lambda, \mu \vdash n$). The algebra $\mathscr{S}$ is a version of the usual *Schur algebra* [8].

EXAMPLE 2. Let $A = F\mathrm{GL}_n(F)$ and $M = V^{\otimes n}$. Then $\mathscr{H} := \mathrm{End}_A(M)$ consists of those endomorphisms of $V^{\otimes n}$ which commute with the action of $\mathrm{GL}_n(F)$. We have already seen that the symmetric group, acting by place permutations on $V^{\otimes n}$ commutes with the action of the general linear group. And, in fact, every element of $\mathscr{H}$ comes from the symmetric group:

$$\mathscr{H} \cong FS_n.$$

This time, we can make progress by considering $\mathscr{H}$; indeed, Schur was able to tie together the representation theory of symmetric groups and general linear groups (represented over the natural field) using $\mathscr{H}$.

In this situation, too, we may consider an endomorphism algebra which involves partitions. Let

$$\mathscr{S} := \mathrm{End}_A(\bigoplus_{\lambda \vdash n} x_\lambda M).$$

EXAMPLE 3. Let $A = F\,\mathrm{GL}_n(q)$ where char $F \nmid q$ and let M be the permutation representation on the subgroup of upper triangular matrices. It is well-known that $\mathscr{H} := \mathrm{End}_A(M)$ has a basis $\{T_\pi : \pi \in S_n\}$ over F where the multiplication is determined as follows. If $\tau = (i, i+1) \in S_n$ and $\pi \in S_n$ then

$$(*) \qquad T_\pi T_\tau = \begin{cases} T_{\pi\tau} & \text{if } l(\pi\tau) > l(\pi), \\ (q-1)\,T_\pi + qT_{\pi\tau} & \text{if } l(\pi\tau) < l(\pi). \end{cases}$$

Here l denotes the usual length function on S_n.

Note that $\mathscr{H}$ is a "q-analogue of FS_n" in the following sense. We may take a vector space $\mathscr{H}$ over F with basis $\{T_\pi : \pi \in S_n\}$, then choose $q \in F$ and define a multiplication on $\mathscr{H}$ by $(*)$, to obtain an algebra. If $q = 1$, then the algebra is FS_n and if q is the prime power which appears in $\mathrm{GL}_n(q)$ then the algebra is $\mathrm{End}_A(M)$.

To achieve our refinement, involving partitions, in this situation, we first define $x_\lambda \in \mathscr{H}$ by

$$x_\lambda = \sum \{T_\pi : \pi \in S_{\lambda_1} \times S_{\lambda_2} \times \ldots\}.$$

Recall that M is here the permutation representation of $A = F\,\mathrm{GL}_n(q)$ on the subgroup B of upper triangular matrices. It turns out that we have already encountered $x_\lambda M$, because $x_\lambda M$ is the permutation representation on the subgroup of matrices of the form:

$$\begin{array}{c|c} & \lambda_1 \quad \lambda_2 \quad \cdots \\ \hline \begin{array}{c}\lambda_1 \\ \lambda_2 \\ \vdots\end{array} & \left[\begin{array}{c|c|c} * & & * \\ \hline & * & \\ \hline 0 & & \ddots \end{array}\right]. \end{array}$$

Again, we let

$$\mathscr{S} := \mathrm{End}_A\Big(\bigoplus_{\lambda \vdash n} x_\lambda M\Big).$$

Hereafter, $\mathscr{H}$ will denote the algebra over F whose basis is $\{T_\pi : \pi \in S_n\}$ and whose multiplication is defined by $(*)$. We may therefore describe FS_n as "$\mathscr{H}$ with $q = 1$". To keep track of what we have done, we update Table 1 in Table 2.

Table 2

	Symmetric group	GL_n in describing characteristic	$F\,\mathrm{GL}_n(q)$ when char $F \nmid q$
Group algebra	$A = FS_n$	$A = F\,\mathrm{GL}_n(F)$	$A = F\,\mathrm{GL}_n(q)$
Main module	$M = FS_n$	$M = V^{\otimes n}$	$M =$ perm. rep. on B
Refined module	$x_\lambda M$	$x_\lambda M$	$x_\lambda M$
$\mathrm{End}_A(M)$	$\mathscr{H}$ with $q = 1$	$\mathscr{H}$ with $q = 1$	$\mathscr{H}$
Refined endomorphism algebra	$\mathrm{End}_A(\bigoplus_{\lambda \vdash n} x_\lambda M)$	$\mathrm{End}_A(\bigoplus_{\lambda \vdash n} x_\lambda M)$	$\mathrm{End}_A(\bigoplus_{\lambda \vdash n} x_\lambda M)$

It seems that the three algebras which appear in the last row of Table 2 are different, since they depend on M, which is either FS_n or $V^{\otimes n}$ or a permutation representation. But the next theorem ties everything together. (Notice that in the case where $A = M = FS_n$, the two algebras in the conclusion of the Theorem are equal. The hypothesis of the Theorem also happens to be valid in this case!)

THEOREM [6, 2.24]. LONG HYPOTHESIS. *Let A be an algebra and let M be a cyclic right A-module, say $M = mA$. Let $\mathscr{H} = \mathrm{End}_A(M)$, acting on the left.*

Assume that there exists an idempotent e in A such that

(i) *$me = m$,*

(ii) *$Me = \mathscr{H}m$, and*

(iii) *$Ue \neq 0$ for every nonzero A-submodule U of M.*

Let $I_1, I_2, \ldots, I_m$ be right ideals of $\mathscr{H}$, and assume that for every I_i, I_j and $\phi \in \mathrm{Hom}_{\mathscr{H}}(I_i, I_j)$ there exists $h_\phi \in \mathscr{H}$ such that $\phi(y) = h_\phi y$ for all $y \in I_i$.

CONCLUSION.

$$\mathrm{End}_A\Big(\bigoplus_{i=1}^{m} I_i M\Big) \cong \mathrm{End}_{\mathscr{H}}\Big(\bigoplus_{i=1}^{m} I_i\Big).$$

To apply the Theorem, we take as our right ideals $I_1, I_2, \ldots, I_m$ of $\mathscr{H}$ the right ideals $x_\lambda \mathscr{H}$ $(\lambda \vdash n)$, and note that $(x_\lambda \mathscr{H})M = x_\lambda M$.

In the case of $F\,\mathrm{GL}_n(q)$ where $\mathrm{char}\, F \nmid q$ (column 3 of Table 2), the Long Hypothesis is valid (see [12] or [6, 2.19]) and we obtain

$$\mathrm{End}_A\Big(\bigoplus_{\lambda \vdash n} x_\lambda M\Big) \cong \mathrm{End}_{\mathscr{H}}\Big(\bigoplus_{\lambda \vdash n} x_\lambda \mathscr{H}\Big).$$

DEFINITION. $\mathrm{End}_{\mathscr{H}}(\bigoplus_{\lambda \vdash n} x_\lambda \mathscr{H})$ is called the *q-Schur algebra.*

Note that when $q = 1$, the q-Schur algebra becomes the usual Schur algebra. We have now achieved our main objective of exhibiting an algebra, depending on q, which relates to the representation theory of the symmetric groups and the general linear groups in *all* characteristics.

Remark. For $\lambda \vdash n$, define $y_\lambda \in \mathscr{H}$ by

$$y_\lambda = \sum \{(-q)^{-l(\pi)} T_\pi \colon \pi \in S_{\lambda_1} \times S_{\lambda_2} \times \ldots\}.$$

It is easy to prove [6, 2.9] that the q-Schur algebra is isomorphic to

$$\mathrm{End}_{\mathscr{H}}\Big(\bigoplus_{\lambda \vdash n} y_\lambda \mathscr{H}\Big).$$

In order to apply the Theorem with the Long Hypothesis to the case of $F\,\mathrm{GL}_n(F)$, it seems to be necessary to work with the right ideals $y_\lambda \mathscr{H}$ in place of $x_\lambda \mathscr{H}$. (The Long Hypothesis in this situation is proved in [9]; see also [12, § 5].) In this article, we are assuming that F is infinite, but there is a well-known procedure for obtaining the irreducible representations of $\mathbf{F}_q \mathrm{GL}_n(q)$ from those of $F\,\mathrm{GL}_n(F)$.

5. Indecomposable direct summands

We indicated earlier that the indecomposable direct summands of the various modules $x_\lambda M$ were important, and we shall now amplify this remark.

First, we observe that knowledge of the indecomposable direct summands of the $x_\lambda M$, as λ varies over the partitions of n, is equivalent to knowing the indecomposable direct summands of $\bigoplus_{\lambda \vdash n} x_\lambda M$.

But we have explained that there are bijections (by way of Fitting's Theorem and the Theorem with the Long Hypothesis) between the following sets.

(i) Indecomposable direct summands of $\bigoplus_{\lambda \vdash n} x_\lambda M$.
(ii) Indecomposable direct summands of $\mathrm{End}_{\mathscr{A}}(\bigoplus_{\lambda \vdash n} x_\lambda M)$.
(iii) Indecomposable direct summands of $\mathrm{End}_{\mathscr{H}}(\bigoplus_{\lambda \vdash n} x_\lambda \mathscr{H})$.
(iv) Indecomposable direct summands of $\bigoplus_{\lambda \vdash n} x_\lambda \mathscr{H}$.

The main point

The modules $x_\lambda M$ depend upon the case we are considering; the symmetric group, or the general linear group in the describing characteristic, or the general linear group in the non-describing characteristic. But the right ideals $x_\lambda \mathscr{H}$ of $\mathscr{H}$ are independent of the case we are considering.

We are able to deal with all three theories at once, because we need study only the indecomposable direct summands of $\bigoplus_{\lambda \vdash n} x_\lambda \mathscr{H}$.

The following results are proved in [6].

(i) *The number of pairwise non-isomorphic indecomposable direct summands of $\bigoplus_{\lambda \vdash n} x_\lambda \mathscr{H}$ is equal to the number of partitions of n.*

(ii) *For each $\lambda \vdash n$ there exists an indecomposable right ideal Y^λ of $\mathscr{H}$ with the properties that Y^λ is a summand of $x_\lambda \mathscr{H}$ with multiplicity* 1 *and Y^λ is isomorphic to a summand of $x_\mu \mathscr{H}$ only if $\lambda \trianglerighteq \mu$.*

We call the indecomposable right ideals, Y^λ, *Young modules.* In the case where $q = 1$ (so $\mathscr{H} \cong FS_n$), these modules were discovered independently by several people, of whom Klyachko [14] was probably the first.

The main result in [6] connects the decomposition matrix of the q-Schur algebra with the decomposition matrices of S_n and $\mathrm{GL}_n(q)$. This result is best explained by means of an example.

EXAMPLE. Assume that $n = 5$ and $\mathrm{char}\, F$ divides $1 + q$. The right ideals $x_\lambda \mathscr{H}$ of $\mathscr{H}$ can be written as direct sums of Young modules in the following way:

$$\begin{aligned}
x_{(5)}\mathscr{H} &= Y^{(5)},\\
x_{(41)}\mathscr{H} &\cong Y^{(5)} \oplus Y^{(41)},\\
x_{(32)}\mathscr{H} &\cong (1-\alpha)\, Y^{(5)} \oplus Y^{(41)} \oplus Y^{(32)},\\
x_{(31^2)}\mathscr{H} &\cong 2Y^{(41)} \oplus Y^{(31^2)},\\
x_{(2^21)}\mathscr{H} &\cong (1-\alpha)\, Y^{(5)} \oplus 2Y^{(41)} \oplus Y^{(32)} \oplus Y^{(2^21)},\\
x_{(21^3)}\mathscr{H} &\cong 2Y^{(41)} \oplus Y^{(31^2)} \oplus 2Y^{(2^21)} \oplus Y^{(21^3)},\\
x_{(1^5)}\mathscr{H} &\cong (5-\alpha)\, Y^{(2^21)} \oplus 4Y^{(21^3)} \oplus Y^{(1^5)}.
\end{aligned}$$

Here,

$$\alpha = \begin{cases} 1 & \text{if } \operatorname{char} F \mid (q-1);\ \text{that is, } \operatorname{char} F = 2,\\ 0 & \text{otherwise.} \end{cases}$$

We record the coefficients which appear here in a matrix B (the last row of B is determined by the multiplicities of $Y^{(5)}$, etc.).

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0\\ 4 & 1 & 0 & 0 & 0 & 0 & 0\\ 5-\alpha & 2 & 1 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 1 & 0 & 0\\ 0 & 2 & 2 & 2 & 1 & 1 & 0\\ 0 & 0 & 1-\alpha & 0 & 1-\alpha & 1 & 1 \end{bmatrix}.$$

Let A denote the matrix of Kostka number for $n = 5$. (Kostka numbers [16] are defined in a purely combinatorial way.)

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0\\ 4 & 1 & 0 & 0 & 0 & 0 & 0\\ 5 & 2 & 1 & 0 & 0 & 0 & 0\\ 6 & 3 & 1 & 1 & 0 & 0 & 0\\ 5 & 3 & 2 & 1 & 1 & 0 & 0\\ 4 & 3 & 2 & 2 & 1 & 1 & 0\\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

Let $\Delta = AB^{-1}$. Thus,

$$\Delta = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0 & 0\\ \alpha & 0 & 1 & 0 & 0 & 0 & 0\\ \alpha+1 & 0 & 1 & 1 & 0 & 0 & 0\\ \alpha & 0 & 1 & 1 & 1 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 1 & 0\\ 1 & 0 & 0 & 1 & \alpha & 0 & 1 \end{bmatrix}.$$

We have (for $n = 5$ and $\operatorname{char} F \mid (1+q)$):

(i) Δ is the decomposition matrix of the q-Schur algebra.

(ii) Δ is the part of the decomposition matrix of $F\,\mathrm{GL}_n(q)$ which corresponds to unipotent representations.

(iii) The first three columns of Δ give the decomposition matrix of $\mathscr{H}$.

If $\alpha = 1$ (so $\operatorname{char} F = 2$), then, further,

(iv) Δ is the decomposition matrix for Weyl modules corresponding to partitions of n. (Weyl modules relate to the representation theory of $F\,\mathrm{GL}_n(F)$.)

(v) The first three columns of Δ give the 2-modular decomposition matrix of S_n.

The pattern of results for this case is completely general; the decomposition matrix Δ of the q-Schur algebra is square and lower unitriangular. Knowing this matrix is equivalent (via $\Delta = AB^{-1}$) to knowing multiplicities of Young modules in right ideals $x_\lambda \mathscr{H}$. The matrix Δ gives the decomposition matrix of $\mathrm{GL}_n(q)$ for unipotent representations, and includes the decomposition matrix of $\mathscr{H}$. When $q = 1$, Δ is the decomposition matrix for Weyl modules (and thus gives representations of $F\,\mathrm{GL}_n(F)$) and Δ includes the decomposition matrix of FS_n.

Note the neat way that "putting $q = 1$" works. We always consider the p-modular decomposition matrix of the q-Schur algebra for primes p which do not divide q. The case where p divides $q-1$ gives us the decomposition matrix for Weyl modules for $F\,\mathrm{GL}_n(F)$ where the characteristic of F equals p. In particular, when p divides $q-1$, the part of the p-modular decomposition matrix of $\mathrm{GL}_n(q)$ which corresponds to unipotent representations coincides with the decomposition matrix for Weyl modules over fields of characteristic p.

There is, as yet, no method known for finding decomposition matrices of symmetric groups, so the decomposition matrices Δ of q-Schur algebras are still a mystery. However, a great deal is known about Δ. For example, the block structure is known [6, 6.7], and the p-modular decomposition matrices of q-Schur algebras have been determined for all q and p (coprime to q) for all n up to and including $n = 10$ [13].

6. The decomposition matrix of $\mathrm{GL}_n(q)$

We have seen that, for p coprime to q, the part of the p-modular decomposition matrix of $\mathrm{GL}_n(q)$ which corresponds to unipotent representations coincides with the p-modular decomposition matrix of the q-Schur algebra. In this section, we outline how the full p-modular decomposition matrix of $\mathrm{GL}_n(q)$ can be found from decomposition matrices of q-Schur algebras.

First we say more about the conjugacy classes of $\mathrm{GL}_n(q)$.

(i) From matrices in $\mathrm{GL}_n(q)$ whose characteristic polynomial is $(f(X))^k$, where $f(X)$ is an irreducible polynomial over $\mathbf{F}_q$ of degree d (and $dk = n$), we get conjugacy classes which are indexed by partitions of k.

EXAMPLE. Suppose that $f(X)$ is an irreducible polynomial of degree d, and let S denote the $d \times d$ companion matrix of $f(X)$. Assume that $4d = n$. We get conjugacy classes of $\mathrm{GL}_n(q)$, which depend upon $f(X)$ and upon partitions of 4:

$$\begin{bmatrix} S & 0 & 0 & 0 \\ 0 & S & 0 & 0 \\ 0 & 0 & S & 0 \\ 0 & 0 & 0 & S \end{bmatrix} \leftrightarrow (1^4), \qquad \begin{bmatrix} S & I & 0 & 0 \\ 0 & S & 0 & 0 \\ 0 & 0 & S & 0 \\ 0 & 0 & 0 & S \end{bmatrix} \leftrightarrow (21^2),$$

$$\begin{bmatrix} S & I & 0 & 0 \\ 0 & S & 0 & 0 \\ 0 & 0 & S & I \\ 0 & 0 & 0 & S \end{bmatrix} \leftrightarrow (2^2), \qquad \begin{bmatrix} S & I & 0 & 0 \\ 0 & S & I & 0 \\ 0 & 0 & S & 0 \\ 0 & 0 & 0 & S \end{bmatrix} \leftrightarrow (31),$$

$$\begin{bmatrix} S & I & 0 & 0 \\ 0 & S & I & 0 \\ 0 & 0 & S & I \\ 0 & 0 & 0 & S \end{bmatrix} \leftrightarrow (4).$$

Here 0 and I denote the $d \times d$ zero and identity matrices, respectively.

(ii) In general, from matrices whose characteristic polynomial is

$$(f_1(X))^{k_1}(f_2(X))^{k_2} \ldots (f_N(X))^{k_N}$$

(with $f_1(X), f_2(X), \ldots, f_N(X)$ distinct irreducible polynomials over $\mathbf{F}_q$ of degrees $d_1, d_2, \ldots, d_N$ and $d_1 k_1 + d_2 k_2 + \ldots + d_N k_N = n$), we get conjugacy classes of $\mathrm{GL}_n(q)$ which are indexed by

$$\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(N)} \quad \text{where } \lambda^{(i)} \vdash k_i.$$

EXAMPLE. Consider $(f_1(X))^3 (f_2(X))^2$, where $3 \deg f_1 + 2 \deg f_2 = n$. For the partitions $(21) \vdash 3$, $(2) \vdash 2$, we get a conjugacy class of $\mathrm{GL}_n(q)$ which is represented by

$$\begin{bmatrix} S_1 & I & 0 & 0 & 0 \\ 0 & S_1 & 0 & 0 & 0 \\ 0 & 0 & S_1 & 0 & 0 \\ 0 & 0 & 0 & S_2 & I \\ 0 & 0 & 0 & 0 & S_2 \end{bmatrix}.$$

Assume that p does not divide q. A matrix whose characteristic polynomial is $(f_1(X))^{k_1}(f_2(X))^{k_2} \ldots (f_N(X))^{k_N}$ is p-regular if and only if the roots of each of $f_1(X), f_2(X), \ldots, f_N(X)$ have multiplicative order coprime to p. Let F be an algebraically closed field of characteristic p coprime to q. By general theory, the number of irreducible $F\mathrm{GL}_n(q)$-modules equals the number of p-regular conjugacy classes of $\mathrm{GL}_n(q)$. Our discussion of the p-regular classes therefore gives the number of irreducible $F\mathrm{GL}_n(q)$-modules.

Suppose that $f(X)$ is an irreducible polynomial of degree d over $\mathbf{F}_q$ and $dk = n$ and $\lambda \vdash k$. We shall explain, in a moment, how to obtain an irreducible $F\,\mathrm{GL}_n(q)$-module $D_F(f, \lambda)$. First, we show how to use these modules to get all the irreducible $F\,\mathrm{GL}_n(q)$-modules.

Let $f_1(X), f_2(X), \ldots, f_N(X)$ be distinct irreducible polynomials over $\mathbf{F}_q$ of degrees $d_1, d_2, \ldots, d_N$ and suppose that $d_1 k_1 + d_2 k_2 + \ldots + d_N k_N = n$, and that $\lambda^{(1)} \vdash k_1, \lambda^{(2)} \vdash k_2, \ldots, \lambda^{(N)} \vdash k_N$. Suppose, too, that the roots of $f_1(X), f_2(X), \ldots, f_N(X)$ have orders coprime to p. Using the irreducible $F\,\mathrm{GL}_{d_i k_i}(q)$-modules $D_F(f_i, \lambda^{(i)})$, we get an irreducible module

$$D_F(f_1, \lambda^{(1)}) \otimes D_F(f_2, \lambda^{(2)}) \otimes \ldots \otimes D_F(f_N, \lambda^{(N)})$$

for

$$\mathrm{GL}_{d_1 k_1}(q) \times \mathrm{GL}_{d_2 k_2}(q) \times \ldots \times \mathrm{GL}_{d_N k_N}(q),$$

that is, for the group of matrices in $\mathrm{GL}_n(q)$ of the form

$$\begin{array}{cc} & \begin{array}{ccc} d_1 k_1 & d_2 k_2 & \end{array} \\ \begin{array}{c} d_1 k_1 \\ d_2 k_2 \\ \end{array} & \begin{bmatrix} * & & 0 \\ & * & \\ 0 & & \ddots \end{bmatrix} \end{array} \quad \text{and hence for} \quad \begin{bmatrix} * & & * \\ & * & \\ 0 & & \ddots \end{bmatrix}.$$

Now, inducing this module to $\mathrm{GL}_n(q)$ gives an irreducible $F\,\mathrm{GL}_n(q)$-module. Moreover, this construction gives a complete set of pairwise non-isomorphic irreducible $F\,\mathrm{GL}_n(q)$-modules.

Our task of describing the irreducible $F\,\mathrm{GL}_n(q)$-modules is therefore reduced to the following. Given an irreducible polynomial $f(X)$ over $\mathbf{F}_q$ of degree d, where $dk = n$, and given $\lambda \vdash k$, we want to define a corresponding irreducible $F\,\mathrm{GL}_n(q)$-module $D_F(f, \lambda)$. (We remark that in the construction of $D_F(f, \lambda)$ we need not assume that the roots of $f(X)$ have order coprime to p. Thus, in fact, we construct more irreducible representations than we require. For further information on this point, see [5].)

Corresponding to $f(X)$ there is a *cuspidal module* C_F for $F\,\mathrm{GL}_d(q)$. (S. I. Gelfand [7] constructed this module in the case where $F = \mathbf{C}$; moreover, $\mathrm{C}_\mathbf{C}$ remains irreducible modulo p, if p does not divide q, see [12, § 3].)

Now,

$$\mathrm{C}_F \otimes \mathrm{C}_F \otimes \ldots \otimes \mathrm{C}_F \quad (k \text{ times}) \text{ is a module for}$$

$$\mathrm{GL}_d(q) \times \mathrm{GL}_d(q) \times \ldots \times \mathrm{GL}_d(q) \quad (k \text{ times}).$$

In such a situation, we have seen how to "extend and induce to $\mathrm{GL}_n(q)$" to obtain an $F\,\mathrm{GL}_n(q)$-module, M say. (In the special case where $f(X) = X - 1$, the module M is just the permutation representation on the subgroup B of upper triangular matrices, as in § 3.)

Let $A = F\,\mathrm{GL}_n(q)$ and $\mathscr{H} = \mathrm{End}_A(M)$. Then $\mathscr{H}$ has a basis $\{T_\pi \colon \pi \in S_k\}$, and the multiplication in $\mathscr{H}$ is given by equation $(*)$ in § 4, replacing q by q^d. The Long Hypothesis of the Theorem near the end of § 4 is valid in this more general situation, so we have

(i) the indecomposable direct summands of $\bigoplus_{\lambda \vdash k} x_\lambda M$ are in bijective correspondence with the indecomposable direct summands of $\bigoplus_{\lambda \vdash k} x_\lambda \mathscr{H}$.

Therefore,

(ii) the number of pairwise non-isomorphic indecomposable direct summands of $\bigoplus_{\lambda \vdash k} x_\lambda M$ is equal to the number of partitions of k.

Moreover,

(iii) for $\lambda \vdash k$, the indecomposable direct summand of $\bigoplus_{\lambda \vdash k} x_\lambda M$ which corresponds to the partition λ has a unique top composition factor. We call this top composition factor $D_F(f, \lambda)$.

This completes our description of the irreducible $F\mathrm{GL}_n(q)$-modules.

Just as the decomposition matrix of the q-Schur algebra coincides with the part of the decomposition matrix of $F\mathrm{GL}_n(q)$ which corresponds to unipotent representations, we have, more generally,

(iv) the part of the decomposition matrix of $F\mathrm{GL}_n(q)$ which corresponds to the factors of $C_F \otimes C_F \otimes \ldots \otimes C_F$ extended and induced to $\mathrm{GL}_n(q)$ is equal to the p-modular decomposition matrix of the q^d-Schur algebra.

The last remark, combined with the construction of the general irreducible $F\mathrm{GL}_n(q)$-module (which involves tensor products) shows that the decomposition matrix of $F\mathrm{GL}_n(q)$ has the form

$$\left[\begin{array}{cccc}
\boxed{\begin{matrix}1 & & 0\\ & 1 & \\ & & \ddots \\ * & & 1\end{matrix}} & & & 0\\
 & \boxed{\begin{matrix}1 & & 0\\ & 1 & \\ & & \ddots \\ * & & 1\end{matrix}} & & \\
 & & \boxed{\begin{matrix}1 & & 0\\ & 1 & \\ & & \ddots \\ * & & 1\end{matrix}} & \\
0 & & & \ddots \\
\hdashline
 & * & &
\end{array}\right].$$

Each lower unitriangular submatrix above the dotted line is a tensor product of decomposition matrices of q^d-Schur algebras (for various $d \leqslant n$). The entries below the dotted line can be calculated, using the Littlewood–Richardson Rule (see [6, § 7]).

The task of finding the decomposition matrices of S_n and $\mathrm{GL}_n(q)$ is therefore reduced to the problem of determining decomposition matrices of q-Schur algebras.

References

[1] R. Dipper, *On the decomposition numbers of the finite general linear groups*, Trans. Amer. Math. Soc. 290 (1985), 315–344.

[2] --, *On the decomposition numbers of the finite general linear groups II*, Trans. Amer. Math. Soc. 292 (1985), 123–133.

[3] R. Dipper and G. D. James, *Representations of Hecke algebras of general linear groups*, Proc. London Math. Soc. 52 (1986), 20–52.

[4] –, –, *Blocks and idempotents of Hecke algebras of general linear groups*, Proc. London Math. Soc. 54 (1987), 57–82.

[5] –, –, *Identification of the irreducible modular representations of* $\mathrm{GL}_n(q)$, J. Algebra 104 (1986), 266–288.

[6] –, –, *The q-Schur algebra*, Proc. London Math. Soc. 59 (1989), 23–50.

[7] S. I. Gelfand, *Representations of the full linear group over a finite field*, Math. USSR-Sb. 12 (1) (1970), 13–39.

[8] J. A. Green, *Polynomial Represntations of* GL_n, Lecture Notes in Math. 830, Springer, Berlin (1980).

[9] G. D. James, *The decomposition of tensors over fields of prime characteristic*, Math. Z. 172 (1980), 161–178.

[10] --, *Unipotent representations of the finite general linear groups*, J. Algebra 74 (1982), 443–465.

[11] –, *Representations of General Linear Groups*, London Math. Soc. Lecture Note Ser. 94, (1984).

[12] –, *The irreducible representations of the finite general linear groups*, Proc. London Math. Soc. 52 (1986), 236–268.

[13] –, *The decomposition matrices of* $\mathrm{GL}_n(q)$ *for* $n \leqslant 10$, Proc. London Math. Soc. 60 (1990), 225–265.

[14] A. A. Klyachko, *Direct summands of permutation modules*, Selecta Math. Soviet. 3 (1983/84), 45–55.

[15] P. Landrock, *Finite Group Algebras and their Modules*, London Math. Soc. Lecture Note Ser. 84, (1983).

[16] I. G. Macdonald, *Symmetric Functions and Hall Polynomials*, Oxford University Press (1979).