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S-FUNCTIONS AND CHARACTERS OF LIE ALGEBRAS AND LIE GROUPS

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S-function methods are used to develop the notion of universal characters of the classical Lie algebras and groups. The associated modification rules are described and exemplified. They are then used, along with certain infinite series of *S*-functions, to derive the Newell–Littlewood rule for decomposing tensor products of both $O(N)$ and $Sp(N)$. A new approach to the Macdonald identities is proposed which also involves universal characters.

0. Introduction

This paper is concerned with the universal characters [Mu, N, K2, KT] of the unitary, orthogonal and symplectic groups, certain infinite series of *S*-functions [L1, K3] and the Macdonald identities [Ma1]. An attempt is made to exploit the well-known algebra of *S*-functions [L1, Ma2] and various notational devices involving partitions and Young diagrams [K1, K2] to give a unified approach to these topics.

The underlying formula on which all else is based in Weyl's character formula [W1]. In the case of the unitary group this leads directly to the fact that universal characters are *S*-functions, so that the famous Littlewood–Richardson rule [LR] may by used to decompose tensor products and to evaluate branching rules. This is demonstrated in § 1, whilst the extension to mixed tensor representations involving composite Young diagrams [K1] is made in § 2. Infinite *S*-function series are defined in § 3 and used to relate the universal characters of both $O(N)$ and $Sp(N)$ to *S*-functions by exploiting long-established generating functions for these group characters [Mu, W2, L1].

The most efficient procedure for the decomposition of tensor products of irreducible representations of the orthogonal and symplectic groups is almost certainly that due to Newell and Littlewood [N, L2]. Their key theorem is

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This paper is in final form and no version of it will be submitted for publication elsewhere.

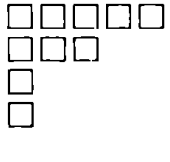
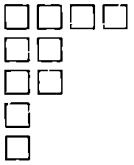
stated and proved [C, CK] in § 4. Finally it is pointed out that the Macdonald identities [Ma1] can be presented in a way which involves just those same infinite S-function series already introduced. The results are presented as a sequence of conjectures but in a footnote to the paper it is pointed out that subsequent to the original presentation of this work it has been possible to prove all but one of the conjectures by making use of an additional type of Lemma on S-function series.

It is first necessary to establish some notation [L1, Ma2, K2]. Let $\lambda = (\lambda_1, \ldots, \lambda_p)$, with $\lambda_i \in \{1, 2, \ldots\}$ and $\lambda_i \geqslant \lambda_{i+1}$ for $i = 1, 2, \ldots, p-1$, denote a partition of weight $|\lambda| = \lambda_1 + \lambda_2 + \ldots + \lambda_p$ and of length $l(\lambda) = p$. It is sometimes convenient to adopt the Frobenius notation [L1, Ma2]

$$\lambda = \begin{pmatrix} a_1 a_2 \ldots a_r \\ b_1 b_2 \ldots b_r \end{pmatrix},$$

with a_i and b_i non-negative integers for $i = 1, 2, \ldots, r$, where $a_i > a_{i+1}$ and $b_i > b_{i+1}$ for $i = 1, 2, \ldots, r-1$ and r is the Frobenius rank of the partition.

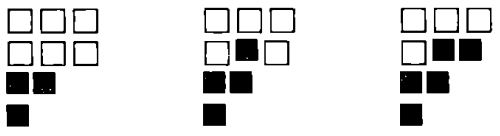
To each partition λ there corresponds a Young diagram or frame F^λ consisting of $l(\lambda)$ left-adjusted rows of boxes of lengths λ_i. The partition λ' conjugate to λ specifies the Young diagram $F^{\lambda'}$, obtained from F^λ by interchanging rows and columns. For example if $\lambda = (531^2)$ then $\lambda' = (42^2 1^2)$ as can be seen from the diagrams

$F^\lambda =$  $\qquad F^{\lambda'} =$ 

The corresponding Frobenius symbols giving the arm and leg lengths relative to the main diagonals of these diagrams are

$$\lambda = \begin{pmatrix} 41 \\ 30 \end{pmatrix} \quad \text{and} \quad \lambda' = \begin{pmatrix} 30 \\ 41 \end{pmatrix}.$$

For what follows later it is convenient to define an operation [K2] on a Young diagram F^λ corresponding to the removal of a continuous boundary strip of boxes of length h starting at the foot of the first column and extending over c columns. The resulting diagram is denoted by $F^{\lambda-h}$. If this diagram is itself a Young diagram specified by some partition μ then the symbol $\lambda-h$ is identified with μ, otherwise $\lambda-h$ is said to be null. For example in the case $\lambda = (3^2 21)$ the following diagrams correspond to $F^{\lambda-h}$ for $h = 3, 4$ and 5.



where □ represents a box and ■ a box that has been removed, so that in these three cases:

$$\begin{array}{lccc} (\lambda-h) = (3^2) & \phi & (31) \\ h \quad = 3 & 4 & 5 \\ c \quad = 2 & 2 & 3\,. \end{array}$$

1. Unitary group characters and S-functions

The connection between Young diagrams and representations of Lie algebras and the corresponding Lie groups comes about through a consideration of their characters. Each finite dimensional irreducible representation (irrep) of a semi-simple or reductive Lie algebra may be specified by means of its highest weight vector Λ. The character of this irrep is given by Weyl's character formula [W1]

$$\mathrm{ch}(\Lambda) = \sum_{w\in W} \varepsilon(w)\, e^{w(\Lambda+\varrho)} / \sum_{w\in W} \varepsilon(w)\, e^{w\varrho} \tag{1.1}$$

where W is the associated Weyl group, $\varepsilon(w)$ is the parity of the element w of W, and ϱ is half the sum of the positive roots of the Lie algebra.

In the case of the Lie algebras A_n, B_n, C_n and D_n the corresponding compact Lie groups are $\mathrm{SU}(n+1)$, $\mathrm{SO}(2n+1)$, $\mathrm{Sp}(2n)$ and $\mathrm{SO}(2n)$, respectively. Since these groups are subgroups of the group $\mathrm{U}(N)$ of unitary transformations in an N-dimensional space for $N = n+1$, $2n+1$, $2n$ and $2n$, respectively, the eigenvalues of their group elements can be written in the form

$$\begin{array}{rll} \mathrm{SU}(n+1) & x_1, x_2, \ldots, x_n, x_{n+1} & \text{with} \quad x_1 x_2 \ldots x_{n+1} = 1, \\ \mathrm{SO}(2n+1) & x_1, x_2, \ldots, x_n, \bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n, 1, & \\ \mathrm{Sp}(2n) & x_1, x_2, \ldots, x_n, \bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n, & \\ \mathrm{SO}(2n) & x_1, x_2, \ldots, x_n, \bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n, & \end{array} \tag{1.2}$$

where $x_j = \exp(i\phi_j)$ and $\bar{x}_j = x_j^{-1}$ for $j = 1, 2, \ldots, n$. The character of the corresponding group element in the conjugacy class labelled by the real parameters ϕ_j for $j = 1, 2, \ldots, n$ is then also given by Weyl's character formula (1.1). All that is required is the interpretation of the formal exponentials in (1.1) as

$$e^{\mu} = x_1^{\mu_1} x_2^{\mu_2} \ldots \quad \text{for any } \mu = (\mu_1, \mu_2, \ldots).$$

In the case of the group $\mathrm{U}(N)$ for which the associated Weyl group is the symmetric group S_N, the character of the irrep labelled by the partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_N)$ in the conjugacy class specified by $\phi = (\phi_1, \phi_2, \ldots, \phi_N)$ is then given by

$$\mathrm{ch}_{\phi}(\lambda) = \sum_{\pi\in S_N} \varepsilon(\pi)\, e^{\pi(\lambda+\varrho)\cdot\phi} / \sum_{\pi\in S_N} \varepsilon(\pi)\, e^{\pi\varrho\cdot\phi} \tag{1.3}$$

with $\varrho = (N-1, N-2, \ldots, 1, 0)$. Setting $x_j = \exp(i\phi_j)$ for $j = 1, 2, \ldots, N$ then yields

$$(1.4)\quad \mathrm{ch}(\lambda) = \sum_{\pi \in S_N} \varepsilon(\pi)\, x_{\pi_1}^{\lambda_1+N-1}\, x_{\pi_2}^{\lambda_2+N-1} \ldots x_{\pi_N}^{\lambda_N} / \sum_{\pi \in S_N} \varepsilon(\pi)\, x_{\pi_1}^{N-1}\, x_{\pi_2}^{N-2} \ldots x_{\pi_N}^{0}$$

so that

$$(1.5)\qquad \mathrm{ch}_\phi(\lambda) = |x_i^{\lambda_j+N-j}| / |x_i^{N-j}| = s_\lambda(x_1, x_2, \ldots, x_N)$$

where the resulting ratio of bialternants is nothing other than the well known Schur function [L1, Ma2].

This Schur function can be given a combinatorial definition [Sta] by invoking the notion of a standard Young tableau, T^λ, which is a numbering of the boxes of the Young diagram F^λ in which (i) the entries are taken from the set $\{1, 2, \ldots, N\}$, (ii) the entries are nondecreasing from left to right across each row, and (iii) the entries are strictly increasing from top to bottom down each column. For example, the Young diagram F^λ and a standard Young tableau T^λ are illustrated in the case $\lambda = (531^2)$ by

$$F^\lambda = \begin{array}{|c|c|c|c|c|}\hline & & & & \\ \hline & & \\ \cline{1-3} \\ \cline{1-1} \\ \cline{1-1} \end{array} \qquad T^\lambda = \begin{array}{ccccc} 1 & 1 & 2 & 4 & 4 \\ 2 & 3 & 3 \\ 4 \\ 5 \end{array}$$

To each such standard Young tableau, T^λ, there corresponds a weight vector $\boldsymbol{m} = (m_1, m_2, \ldots, n_N)$, where m_j is the number of entries j in the tableau. With this terminology the Schur function (5) can be expressed in the form

$$(1.6)\qquad s_\lambda(x_1, x_2, \ldots, x_N) = \sum_{T^\lambda} x_1^{m_1}\, x_2^{m_2} \ldots x_N^{m_N} = \sum_{T^\lambda} \boldsymbol{x}^{\boldsymbol{m}}.$$

It follows that the irrep of $U(N)$ in question has character

$$(1.7)\qquad \mathrm{ch}_\phi(\lambda) = \{\lambda\}(\boldsymbol{x})_N = s_\lambda(\boldsymbol{x})_N = s_\lambda(x_1, x_2, \ldots, x_N),$$

where a variety of notations has been used to specify the character. It follows from (1.6) that

$$(1.8)\qquad \{\lambda\}(\boldsymbol{x})_N = s_\lambda(\boldsymbol{x})_N = 0 \quad \text{if } \lambda'_1 > N$$

as can be seen from condition (iii) for the standardness of the Young tableaux. This is our first example [L1], albeit a trivial one, of a modification rule. Furthermore on restriction from $U(N)$ to $SU(N)$ each irrep remains irreducible and its character is unchanged save for the fact that the condition $x_1 x_2 \cdot \ldots \cdot x_N = 1$ now applies. Then having applied (1.8) an additional modification rule applies:

$$(1.9)\qquad \{\lambda\}(\boldsymbol{x})_N = \{\mu\}(\boldsymbol{x})_N \quad \text{where } \mu_i = \lambda_i - \lambda_N \text{ for } i = 1, 2, \ldots, N$$

corresponding to the removal of all columns of length N from the Young diagram F^λ.

The characters (1.7) may be generalised to give universal characters defined by taking the inverse limit [Ma2]

$$\{\lambda\} = \{\lambda\}(\boldsymbol{x}) = s_\lambda(\boldsymbol{x}) = \lim_{\leftarrow N} \{\lambda\}(\boldsymbol{x})_N. \tag{1.10}$$

The characters of $U(N)$ for any finite N are then recovered from the universal characters by means of the specialisation:

$$\{\lambda\}(\boldsymbol{x})_N = \{\lambda\}(x_1, x_2, \ldots, x_N, 0, 0, \ldots). \tag{1.11}$$

The algebra of universal characters then coincides with the algebra of Schur functions $s_\lambda(\boldsymbol{x})$ of arbitrarily many indeterminates $x_1, x_2, \ldots$. This algebra is such that products are given by

$$\{\mu.\nu\} = \{\mu\}.\{\nu\} = \sum_\lambda c^\lambda_{\mu\nu} \{\lambda\} \tag{1.12}$$

and quotients by

$$\{\lambda/\mu\} = \{\lambda\}/\{\mu\} = \sum_\nu c^\lambda_{\mu\nu} \{\nu\}, \tag{1.13}$$

where the coefficients $c^\lambda_{\mu\nu}$ are the famous Littlewood–Richardson coefficients [LR, L1, Ma2]. The dependence on $(\boldsymbol{x})$ has, for typographical convenience, been suppressed in (1.12) and (1.13). Thus the tensor product formula appropriate to $U(N)$ takes the form

$$\mathrm{ch}_\phi(\mu)\,\mathrm{ch}_\phi(\nu) = \{\mu\}(\boldsymbol{x})_N.\{\nu\}(\boldsymbol{x})_N = \sum_\lambda c^\lambda_{\mu\nu} \{\lambda\}(\boldsymbol{x})_N = \sum_{\lambda,\lambda_1 \leq N} c^\lambda_{\mu\nu}\, \mathrm{ch}_\phi(\lambda) \tag{1.14}$$

It is almost as straightforward to deal with the branching rule appropriate to the restriction from $U(M+N)$ to $U(M)\times U(N)$. This time one obtains

$$\begin{aligned} \mathrm{ch}_{\phi,\psi}(\lambda) &= s_\lambda(\boldsymbol{x}, \boldsymbol{y})_{M+N} = \sum_\mu s_\mu(\boldsymbol{x})_M\, s_{\lambda/\mu}(\boldsymbol{y})_M \\ &= \sum_{\mu,\nu} c^\lambda_{\mu\nu}\, s_\mu(\boldsymbol{x})_M\, s_\nu(\boldsymbol{y})_N = \sum_{\mu,\mu_1 \leq M} \sum_{\nu,\nu_1 \leq N} c^\lambda_{\mu\nu}\, \mathrm{ch}_\phi(\mu)\,\mathrm{ch}_\psi(\nu) \end{aligned} \tag{1.5}$$

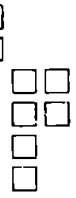
where use has been made of the notation and combinatorics of skew Young diagrams $F^{\lambda/\mu}$ and the corresponding skew standard Young tableaux $T^{\lambda/\mu}$ [Sta, Ma2]. Remarkably everything is governed yet again by the Littlewood–Richardson coefficients. In terms of universal characters one can then write the branching rule in the form [K2]

$$U(M+N) \to U(M)\times U(N) \qquad \{\lambda\} = \sum_\mu \{\mu\} \times \{\lambda/\mu\}. \tag{1.16}$$

2. Composite Young diagrams and unitary group representations

Quite apart from the irreps of $U(N)$ whose highest weight vectors are partitions as in (1.4) there exist other irreps with highest weight vectors Λ whose characters are again given by (1.4) with λ replaced by Λ. Amongst

these other irreps are the so-called mixed tensor irreps for which $\Lambda = (\mu_1, \mu_2, \ldots, \mu_p, 0, 0, \ldots, 0, -\nu_q, \ldots, -\nu_2, -\nu_1)$ where $\mu = (\mu_1, \mu_2, \ldots, \mu_p)$ and $\nu = (\nu_1, \nu_2, \ldots, \nu_q)$ are partitions. and $p+q \leqslant N$. One can associate with such irreps various generalisations of the original Young diagrams. Of these perhaps the most useful is the composite Young diagram $F^{\bar{\nu};\mu}$ [K1]. Varying the notation slightly, this is exemplified by the diagram

$F^{\bar{\nu};\mu} =$  in the case $\Lambda = (2^2 1^2 0^{N-6} - 2 - 3)$.

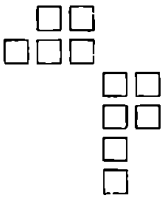
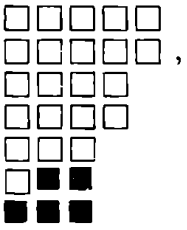
Correspondingly the character of the irrep of $U(N)$ with highest weight $\Lambda = (\mu_1, \mu_2, \ldots, \mu_p, 0, 0, \ldots, 0, -\nu_q, \ldots, -\nu_2, -\nu_1)$ is given, thanks to (1.4), by

$$\mathrm{ch}_\phi(\Lambda) = \{\bar{\nu};\mu\}(\boldsymbol{x})_N$$

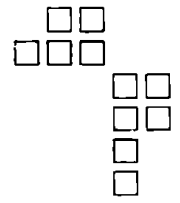
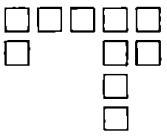
$$= \sum_{\pi \in S_N} \varepsilon(\pi)\, x_{\pi_1}^{\mu_1+N-1} x_{\pi_2}^{\mu_2+N-2} \ldots x_{\pi_N}^{-\nu_1} \Big/ \sum_{\pi \in S_N} \varepsilon(\pi)\, x_{\pi_1}^{N-1} x_{\pi_2}^{N-2} \ldots x_{\pi_N}^{0} \tag{2.1}$$

$$= (x_1, x_2, \ldots, x_N)^{-\nu_1} \{\varrho\}(\boldsymbol{x})_N = \{\bar{\varepsilon}\}(\boldsymbol{x})_N^{\nu_1} . \{\varrho\}(\boldsymbol{x}) \tag{2.2}$$

where $\varrho = (\mu_1 + \nu_1, \mu_2 + \nu_1, \ldots, -\nu_2 + \nu_1, 0)$ and $\varepsilon = (1^N)$. The correspondence between Λ and ϱ is such that F^ϱ is obtained from $F^{\bar{\nu};\mu}$ by adjoining F^μ to the complement in $F^{\nu_1^N}$ of F^ν as illustrated in the case $N = 7$ and $\Lambda = (2^2 1^2 0 - 2 - 3)$ by

$F^{\bar{\nu};\mu} =$  and $F^\varrho =$  ,

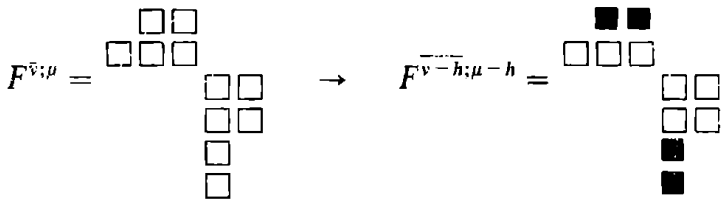
where $\mu = (2^2 1^2)$ and $\nu = (32)$, leading to $\varrho = (5^2 4^2 31)$. It is clear that if the sum of the number of parts of μ and of ν is greater than N then the above correspondence leads to non-standard diagrams. For example, taking $N = 3$ rather than $N = 7$ in the previous example gives

$F^{\bar{\nu};\mu} =$  and $F^\varrho =$ 

with F^{ϱ} non-standard. In fact, just as $\{\lambda\}(x)_N$ is standard in $U(N)$ if and only if $\lambda'_1 \leqslant N$ but requires modification in accordance with the rule (1.8) if $\lambda'_1 > N$, so $\{\bar{\nu};\mu\}(x)_N$ is standard if and only if $\mu'_1 + \nu'_1 \leqslant N$ but requires modification if $\mu'_1 + \nu'_1 > N$ in accordance with the rule [K2]

$$\{\bar{\nu};\mu\}(x)_N = (-1)^{c+\bar{c}+1}\{\overline{\nu-h};\mu-h\}(x)_N \quad \text{where } h = \mu'_1 + \nu'_1 - N - 1. \tag{2.3}$$

This modification rule is considerably more complicated than (1.8), involving as it does two applications of the boundary strip removal procedure covering c and $\bar{c}$ columns. However in practice this is easy to apply. For example, in the case $\mu = (2^2\,1^2)$, $\nu = (32)$ and $N = 3$ referred to above, one has $\mu'_1 = 4$, and $\nu'_1 = 2$ so that $h = 2$ leading to the diagram modification



and hence to the result

$$\{\overline{32};\,2^2\,1^2\}(x)_3 = +\{\bar{3};\,2^2\}(x)_3.$$

In general this modification procedure may have to be repeated more than once in order to reduce the original composite diagram either to one which occupies a total of not more than N rows and is thus standard, or to a null shape.

Universal characters corresponding to mixed tensor representations may be defined once again by taking an inverse limit:

$$\{\bar{\nu};\mu\} = \{\bar{\nu};\mu\}(x) = \lim_{\leftarrow N}\{\bar{\nu};\mu\}(x)_N. \tag{2.4}$$

The character of $U(N)$ for any finite N may be recovered from the universal character by setting $x_j = 0$ for $j > N$. This restriction may be shown to imply the validity of the modification rule (2.3) by, for example, making use of certain determinantal expansions of the relevant characters [K2].

The importance of universal characters and the associated modification rule is illustrated by the fact that tensor products in the unitary groups are governed by the universal rule

$$\{\bar{\nu};\mu\} \times \{\bar{\lambda};\varkappa\} = \sum_{\sigma,\tau}\{\overline{(\nu/\sigma).(\lambda/\tau)};\,(\mu/\tau).(\varkappa/\sigma)\}. \tag{2.5}$$

For example, (2.5) gives the universal tensor product formula

$$\{\bar{1};\,1\} \times \{\bar{1};\,1\} = \{\bar{2};\,2\} + \{\bar{2};\,1^2\} + \{\bar{1}^2;\,2\} + \{\bar{1}^2;\,1^2\} + 2\{\bar{1};\,1\} + \{0\}.$$

In the case of U(3) the modification rule (2.3) implies that $\{\bar{1}; 1^2\}_3 = 0$ so that

$$\{\bar{1}; 1\}_3 \times \{\bar{1}; 1\}_3 = \{\bar{2}; 2\}_3 + \{\bar{2}; 1^2\}_3 + \{\bar{1}^2; 2\}_3 + 2\{\bar{1}; 1\}_3 + \{0\}_3.$$

Similarly for U(2) we have $\{\bar{2}; 1^2\}_2 = \{\bar{1}^2; 2\}_2 = 0$ and $\{\bar{1}^2; 1^2\}_2 = -\{\bar{1}; 1\}_2$ so that

$$\{\bar{1}; 1\}_2 \times \{\bar{1}; 1\}_2 = \{\bar{2}; 2\}_2 + \{\bar{1}; 1\}_2 + \{0\}_2.$$

3. Infinite *S*-function series and characters of O(N) and Sp(N)

Littlewood [L1] gave a number of generating functions for infinite series of *S*-functions, and the list can be extended [K3] to give

$$A_q(\boldsymbol{x}) = \prod_{i<j}(1-qx_ix_j) = \sum_{\alpha\in A}(-1)^{|\alpha|/2} q^{|\alpha|/2} s_\alpha(\boldsymbol{x}), \tag{3.1a}$$

$$B_q(\boldsymbol{x}) = \prod_{i<j}(1-qx_ix_j)^{-1} = \sum_{\beta\in B} q^{|\beta|/2} s_\beta(\boldsymbol{x}), \tag{3.1b}$$

$$C_q(\boldsymbol{x}) = \prod_{i\leqslant j}(1-qx_ix_j) = \sum_{\gamma\in C}(-1)^{|\gamma|/2} q^{|\gamma|/2} s_\gamma(\boldsymbol{x}), \tag{3.1c}$$

$$D_q(\boldsymbol{x}) = \prod_{i\leqslant j}(1-qx_ix_j)^{-1} = \sum_{\delta\in D} q^{|\delta|/2} s_\delta(\boldsymbol{x}), \tag{3.1d}$$

$$E_q(\boldsymbol{x}) = \prod_i(1-qx_i)\prod_{i<j}(1-q^2x_ix_j) = \sum_{\varepsilon\in E}(-1)^{(|\varepsilon|+r)/2} q^{|\varepsilon|} s_\varepsilon(\boldsymbol{x}), \tag{3.1e}$$

$$F_q(\boldsymbol{x}) = \prod_i(1-qx_i)^{-1}\prod_{i<j}(1-q^2x_ix_j)^{-1} = \sum_{\zeta\in F} q^{|\zeta|} s_\zeta(\boldsymbol{x}), \tag{3.1f}$$

$$G_q(\boldsymbol{x}) = \prod_i(1+qx_i)\prod_{i<j}(1-q^2x_ix_j) = \sum_{\varepsilon\in E}(-1)^{(|\varepsilon|-r)/2} q^{|\varepsilon|} s_\varepsilon(\boldsymbol{x}), \tag{3.1g}$$

$$H_q(\boldsymbol{x}) = \prod_i(1+qx_i)^{-1}\prod_{i<j}(1-q^2x_ix_j)^{-1} = \sum_{\zeta\in F}(-1)^{|\zeta|} q^{|\zeta|} s_\zeta(\boldsymbol{x}), \tag{3.1h}$$

$$L_q(\boldsymbol{x}) = \prod_i(1-qx_i) = \sum_m(-1)^m q^m s_{1^m}(\boldsymbol{x}), \tag{3.1l}$$

$$M_q(\boldsymbol{x}) = \prod_i(1-qx_i)^{-1} = \sum_m q^m s_m(\boldsymbol{x}), \tag{3.1m}$$

$$P_q(\boldsymbol{x}) = \prod_i(1+qx_i)^{-1} = \sum_m(-1)^m q^m s_m(\boldsymbol{x}), \tag{3.1p}$$

$$Q_q(\boldsymbol{x}) = \prod_i(1+qx_i) = \sum_m q^m s_{1^m}(\boldsymbol{x}), \tag{3.1q}$$

where, in Frobenius notation, A, C and E are the sets of partitions of the form

$$\begin{pmatrix} a_1 & a_2 & \dots \\ a_1+1 & a_2+1 & \dots \end{pmatrix}, \quad \begin{pmatrix} a_1+1 & a_2+1 & \dots \\ a_1 & a_2 & \dots \end{pmatrix}, \quad \begin{pmatrix} a_1 & a_2 & \dots \\ a_1 & a_2 & \dots \end{pmatrix},$$

D is the set of partitions all of whose parts are even, B is the set of partitions all of whose distinct parts are repeated an even number of times and F is the set

of all partitions. The Frobenius rank of the self-conjugate partition ε has been denoted by r. It is convenient in what follows to denote the $q = 1$ series $A_1(\boldsymbol{x})$ by $A(\boldsymbol{x})$, $B_1(\boldsymbol{x})$ by $B(\boldsymbol{x})$ and so on, and from time to time suppress the explicit dependence upon $(\boldsymbol{x})$.

These expansions are universally valid in the sense that if the products on the left are taken over all positive integer values of i and j, then on the right the S-functions $s_\lambda(\boldsymbol{x})$ are the universal S-functions, (1.10), involving an infinite number of variables $(x_1, x_2, \ldots)$. Restricting the domain of the indices i and j on the left to $(1, 2, \ldots, N)$ leads on the right to S-functions $s_\lambda(x_1, x_2, \ldots, x_N)$ which are of course subject to the modification rule (1.8).

The irreducible tensor representations of the orthogonal and symplectic groups, $O(N)$ and $Sp(N)$, like the covariant irreducible representations of the unitary group, $U(N)$, are specified by highest weight vectors which are partitions. The notation used for the corresponding group characters [Mu, L1, K3] is:

(3.2a) $\qquad U(N) \qquad \mathrm{ch}_\phi(\lambda) = \{\lambda\}(\boldsymbol{x})_N,$

(3.2b) $\qquad O(N) \qquad \mathrm{ch}_\phi(\lambda) = [\lambda](\boldsymbol{x})_N,$

(3.2c) $\qquad Sp(N) \qquad \mathrm{ch}_\phi(\lambda) = \langle\lambda\rangle(\boldsymbol{x})_N.$

Making use of Weyl's character formula (1.1) and the link between formal exponentials and eigenvalues of group elements described by (1.2) it is possible to derive the following generating formulae [Mu, W2, L1]:

$$\prod_{i,a}(1-x_i y_a)^{-1} = \sum_\lambda \{\lambda\}(\boldsymbol{x})\{\lambda\}(\boldsymbol{y}), \tag{3.3}$$

$$\prod_{i,a}(1-x_i y_a)^{-1} \prod_{a\leqslant b}(1-y_a y_b) = \prod_\lambda [\lambda](\boldsymbol{x})\{\lambda\}(\boldsymbol{y}), \tag{3.4}$$

$$\prod_{i,a}(1-x_i y_a)^{-1} \prod_{a<b}(1-y_a y_b) = \sum_\lambda \langle\lambda\rangle(\boldsymbol{x})\{\lambda\}(\boldsymbol{y}), \tag{3.5}$$

where the summations are over all partitions λ.

Substituting (3.3) into (3.4) and using the series expansion for $C(\boldsymbol{y})$ gives

$$\begin{aligned}\prod_{i,a}(1-x_i y_a)^{-1} \prod_{a\leqslant b}(1-y_a y_b) &= \sum_\mu s_\mu(\boldsymbol{x}) s_\mu(\boldsymbol{y}) \sum_\gamma (-1)^{|\gamma|/2} s_\gamma(\boldsymbol{y}) \\ &= \sum_{\mu,\gamma,\lambda} (-1)^{|\gamma|/2} c^\lambda_{\mu\gamma} s_\mu(\boldsymbol{x}) s_\lambda(\boldsymbol{y}).\end{aligned}$$

Comparing coefficients of $\{\lambda\}(\boldsymbol{y}) = s_\lambda(\boldsymbol{y})$ in this expression and (3.4) then gives

$$[\lambda](\boldsymbol{x}) = \sum_{\mu,\gamma}(-1)^{|\gamma|/2} c^\lambda_{\mu\gamma} s_\gamma(\boldsymbol{x}) = \sum_{\gamma,\lambda}(-1)^{|\gamma|/2} s_{\lambda/\gamma}(\boldsymbol{x}). \tag{3.6}$$

This can conveniently be written as

$$O(N) \to U(N) \qquad [\lambda](\boldsymbol{x}) = \{\lambda/C\}(\boldsymbol{x}). \tag{3.7}$$

The fact that $D(x)C(x) = 1$ then immediately gives

$$\text{(3.8)} \qquad \mathrm{U}(N) \to \mathrm{O}(N) \qquad \{\lambda\}(x) = \{\lambda/DC\}(x) = [\lambda/D](x).$$

In exactly the same way it may be shown that

$$\text{(3.9)} \qquad \mathrm{Sp}(N) \to \mathrm{U}(N) \qquad \langle\lambda\rangle(x) = \{\lambda/A\}(x)$$

and, using the fact that $B(x)A(x) = 1$,

$$\text{(3.10)} \qquad \mathrm{U}(N) \to \mathrm{Sp}(N) \qquad \{\lambda\}(x) = \langle\lambda/B\rangle(x).$$

It must be stressed that these are universal identities [K3] in the sense that they are valid for all N, but when interpreting them it is important to note that the indeterminates x_i in the orthogonal and symplectic group characters are constrained as in (1.2). These constraints for finite N may be shown to lead to the modification rules [K2]:

$$\text{(3.11)} \quad \mathrm{O}(N) \qquad [\lambda](x)_N = (-1)^{\epsilon-1}[\lambda - h](x)_N \qquad \text{with } h = 2\lambda'_1 - N \geqslant 1,$$

$$\text{(3.12)} \quad \mathrm{Sp}(N) \qquad \langle\lambda\rangle(x)_N = (-1)^{\epsilon}\langle\lambda - h\rangle(x)_N \qquad \text{with } h = 2\lambda'_1 - N - 2 \geqslant 0.$$

The role of modification rule (3.12) may be illustrated through a consideration of the following example arising from the use of the branching rule (3.10)

$$\mathrm{U}(N) \to \mathrm{Sp}(N) \qquad \{2^2 1^2\} \to \langle 2^2 1^2/B\rangle = \langle 2^2 1^2/(0 + 1^2 + 2^2 + 1^4 \ldots)\rangle$$
$$= \langle 2^2 1^2\rangle + \langle 2^2\rangle + \langle 21^2\rangle + \langle 1^4\rangle + 2\langle 1^2\rangle + \langle 1\rangle,$$

where the dependence on (x) has been suppressed. This result is universal but, for example, in the case of Sp(4) the modification rule (3.12) gives $\langle 2^2 1^2\rangle_4 = -\langle 2^2\rangle_4$, $\langle 21^2\rangle_4 = -\langle 21^2\rangle_4 = 0$ and $\langle 1^4\rangle_4 = -\langle 1^2\rangle_4$, so that we have

$$\mathrm{U}(4) \to \mathrm{Sp}(4) \qquad \{2^2 1^2\}_4 \to \langle 1^2\rangle_4 + \langle 1\rangle_4.$$

Such modification rules may be avoided if the original branching rule (3.10) is altered to take into account each specific value of N. Two ways have recently been proposed for doing this by Sundaram [Su] and Tokuyama [T]. Both methods involve the introduction of new tableaux whose enumeration amounts to a refinement of the Littlewood–Richardson rule for evaluating the quotients of (3.10). Sundaram's method involves the smallest change in the original rule.

4. The Newell–Littlewood theorem

As indicated earlier, tensor products of unitary group irreps may be decomposed by making use of universal characters. The relevant universal product rule, (1.12), involves just the Littlewood–Richardson coefficients. In the case of any particular unitary group $\mathrm{U}(N)$ it is then only necessary to apply to the output of the universal product rule the modification rule (1.8).

An exactly analagous procedure can be used to deal with tensor products of irreps of both the orthogonal and the symplectic groups. The universal product rules are given by the following

THEOREM (Newell–Littlewood) [N, L2].

$$\text{(4.1)} \qquad O(N) \qquad [\lambda] \times [\mu] = \sum_{\zeta} [(\lambda/\zeta).(\mu/\zeta)]$$

$$\text{(4.2)} \qquad Sp(N) \qquad \langle \lambda \rangle \times \langle \mu \rangle = \sum_{\zeta} \langle (\lambda/\zeta).(\mu/\zeta) \rangle$$

where $\times$ *indicates a tensor product, or equivalently a product of characters,* . *indicates an S-function product* (1.12) *and* / *indicates an S-function quotient* (1.13). *The brackets* [ ] *and* $\langle\ \rangle$ *signify characters of* $O(N)$ *and* $Sp(N)$, *respectively, as in* (3.2b) *and* (3.2c). *The dependence on* (x) *has again been suppressed.*

In order to prove these results it is necessary to prove a succession of small Lemmas.

LEMMA 1.

$$\text{(4.3)} \qquad s_{(\mu.\nu)/\varrho}(x) = \sum_{\sigma} s_{(\mu/\sigma).(\nu/(\varrho/\sigma))}(x).$$

Proof. It suffices to expand a product of S-functions of $(z) = (x, y)$ in two different ways. Firstly,

$$\text{(4.4)} \qquad s_\mu(x, y)\, s_\nu(x, y) = \sum_{\lambda} c^{\lambda}_{\mu\nu}\, s_\lambda(x, y) = \sum_{\lambda,\varrho} c^{\lambda}_{\mu\nu}\, s_\varrho(x)\, s_{\lambda/\varrho}(y)$$
$$= \sum_{\varrho} s_\varrho(x)\, s_{(\mu.\nu)/\varrho}(y),$$

where the last expression serves really to define what is meant by the left-hand side of the lemma, and secondly

$$\text{(4.5)} \qquad s_\mu(x, y)\, s_\nu(x, y) = \sum_{\sigma,\tau} s_\sigma(x)\, s_{\mu/\sigma}(y)\, s_\tau(x)\, s_{\nu/\tau}(y)$$
$$= \sum_{\sigma,\tau,\varrho} c^{\varrho}_{\sigma\tau}\, s_\varrho(x)\, s_{\mu/\sigma}(y)\, s_{\nu/\tau}(y)$$
$$= \sum_{\varrho,\sigma} s_\varrho(x)\, s_{\mu/\sigma}(y)\, s_{(\nu/(\varrho/\sigma))}(y).$$

The required result (4.3) then follows by comparing the coefficients of $s_\varrho(x)$ in (4.4) and (4.5). ■

LEMMA 2.

$$\text{(4.6)} \qquad s_{D/\mu}(x) = s_{(\mu/D).D}(x).$$

Proof. This time one expands $D(\boldsymbol{x}, \boldsymbol{y})$ in two ways:

$$D(\boldsymbol{x}, \boldsymbol{y}) = \prod_{i \leqslant j} (1 - x_i x_j)^{-1} \prod_{i,a} (1 - x_i y_a)^{-1} \prod_{a \leqslant b} (1 - y_a y_b)^{-1} \tag{4.7}$$

$$= D(\boldsymbol{x}) D(\boldsymbol{y}) \sum_{\lambda} s_\lambda(\boldsymbol{x}) s_\lambda(\boldsymbol{y}) \qquad \text{from (3.3)}$$

$$= \sum_{\lambda} D(\boldsymbol{x}) s_\lambda(\boldsymbol{x}) \sum_{\delta \in D} s_\delta(\boldsymbol{y}) s_\lambda(\boldsymbol{y})$$

$$= \sum_{\lambda} D(\boldsymbol{x}) s_\lambda(\boldsymbol{x}) \sum_{\delta \in D, \mu} c^{\mu}_{\delta\lambda} s_\mu(\boldsymbol{y}) \qquad \text{from (1.12)}$$

$$= \sum_{\mu} D(\boldsymbol{x}) s_{\mu/D}(\boldsymbol{x}) s_\mu(\boldsymbol{y}) \qquad \text{from (1.13)}$$

and

$$D(\boldsymbol{x}, \boldsymbol{y}) = \sum_{\delta \in D} s_\delta(\boldsymbol{x}, \boldsymbol{y}) = \sum_{\delta \in D, \mu} s_{\delta/\mu}(\boldsymbol{x}) s_\mu(\boldsymbol{y}) = \sum_{\mu} s_{D/\mu}(\boldsymbol{x}) s_\mu(\boldsymbol{y}). \tag{4.8}$$

This time a comparison of the coefficients of $s_\mu(\boldsymbol{y})$ proves the required result (4.6). ■

LEMMA 3.

$$s_{(\mu.\nu)/D}(\boldsymbol{x}) = \sum_{\zeta} s_{(\mu/(\zeta.D))}(\boldsymbol{x}) s_{(\nu/(\zeta.D))}(\boldsymbol{x}). \tag{4.9}$$

Proof. In this case one can work entirely in terms of S-functions of $(\boldsymbol{x})$, which for convenience may be dropped from the notation on the understanding that all subsequent S-functions depend on these variables. Then

$$S_{(\mu.\nu)/D} = \sum_{\delta \in D} S_{(\mu.\nu)/\delta} = \sum_{\delta \in D, \sigma} S_{(\mu/\sigma)} S_{(\nu/(\delta/\sigma))} \qquad \text{from Lemma 1}$$

$$= \sum_{\sigma} S_{(\mu/\sigma)} S_{(\nu/(D/\sigma))} = \sum_{\sigma} S_{(\mu/\sigma)} S_{(\nu/((\sigma/D).D))} \qquad \text{from Lemma 2}$$

$$= \sum_{\delta \in D, \sigma} S_{(\mu/\sigma)} S_{(\nu/((\sigma/\delta).D))} = \sum_{\delta \in D, \sigma, \zeta} c^{\sigma}_{\zeta\delta} S_{(\mu/\sigma)} S_{(\nu/(\zeta.D))} \qquad \text{from (1.13)}$$

$$= \sum_{\delta \in D, \zeta} S_{(\mu/(\zeta.\delta))} S_{(\nu/(\zeta.D))} \qquad \text{from (1.12)}$$

$$= \sum_{\zeta} S_{(\mu/(\zeta.D))} S_{(\nu/(\zeta.D))}.$$

This proves the Lemma. ■

Now we are in a position to prove the Newell–Littlewood Theorem as follows [C, CK]:

$$[\lambda] \times [\mu] = \{\lambda/C\}.\{\mu/C\} = \{(\lambda/C).(\mu/C)\} \qquad \text{from (3.7)}$$

$$= [((\lambda/C).(\mu/C))/D] \qquad \text{from (3.8)}$$

$$= \sum_{\zeta} [(\lambda/(C.\zeta.D)).(\mu/(C.\zeta.D))] \qquad \text{from Lemma 3}$$

$$= \sum_{\zeta} [(\lambda/\zeta).(\mu/\zeta)] \qquad \text{since C.D} = 1$$

which is the required formula (4.1). Similarly, replacing C and D by A and B throughout gives the formula (4.2)

$$\langle\lambda\rangle \times \langle\mu\rangle = \sum_{\zeta} \langle(\lambda/\zeta).(\mu/\zeta)\rangle. \quad \blacksquare$$

In making use of (4.1) and (4.2) for $O(N)$ and $Sp(N)$, respectively, for some fixed N it is course also necessary to use in addition the modification rules (3.11) and (3.12).

5. Macdonald identities

The infinite series of S-functions encountered in the last section have a role to play in studying the Macdonald identities [Ma1]. Indeed we conjecture that a number of these identities can be recast in the form of expansions of certain infinite products of infinite series of S-functions in terms of infinite sums of universal group characters. Modification rules are then required to recover the Macdonald identities appropriate to a finite number of variables. To make the connection we are seeking it is convenient to note that the Macdonald identity associated with a simple Lie algebra, L, can be written in the form [Ma1]

$$\prod_{k=1}^{\infty} \{(1-q^k)^n \prod_{\alpha\in R} (1-q^k e^{\alpha})\} = \sum_{m\in M} q^{c(m)} \mathrm{ch}(m) \tag{5.1}$$

where n is the rank of L, R is the set of roots of L, and M is a lattice generated by the roots of L, suitably scaled. Whilst $c(m) = \{(m+\varrho, m+\varrho)-(\varrho, \varrho)\}/\{(\varphi+\varrho, \varphi+\varrho)-(\varrho, \varrho)\}$ is the eigenvalue of a second order Casimir operator, where φ is the highest root and ϱ is half the sum of the positive roots. Finally $\mathrm{ch}(m)$ is defined by Weyl's character formula (1.1). It should be borne in mind that the vector m in the lattice M will not in general be a highest weight vector. Weyl reflections are required in order to determine the highest weight of the corresponding irrep.

An example of such an identity (5.1) is provided by the case $L = so(3)$ for which $n = 1$, $R = \{\alpha, -\alpha\}$, $\varphi = \alpha$ and $M = \{m\alpha: m\in 2\mathbf{Z}\}$.

One obtains

$$\prod_{k=1}^{\infty} (1-q^k)(1-q^k e^{\alpha})(1-q^k e^{-\alpha}) = \sum_{m\in 2\mathbf{Z}} q^{m(m+1)/2} \mathrm{ch}(m) \tag{5.2}$$

where $\mathrm{ch}(m) = (e^{m\alpha}-e^{-(m+1)\alpha})/(1-e^{-\alpha}) = -\mathrm{ch}(-m-1)$ so that

$$\prod_{k=1}^{\infty} (1-q^k)(1-q^k e^{\alpha})(1-q^k e^{-\alpha}) = \sum_{m=0}^{\infty} (-1)^m q^{m(m+1)/2} \mathrm{ch}(m) \tag{5.3}$$

The specialisation of this for which $e^{\alpha} = e^{-\alpha} = 1$ yields Jacobi's identity

$$\prod_{k=1}^{\infty} (1-q^k)^3 = \sum_{m=0}^{\infty} (-1)^m (2m+1) q^{m(m+1)/2}. \tag{5.4}$$

The form of the $so(3)$ identity which will be generalised here is that given in (5.3). In the case of $so(2n+1)$ the root system is such that

$$(5.5)\quad \prod_{k=1}^{\infty}\{(1-q^k)^n\prod_{\alpha\in R}(1-q^k e^{\alpha})\} = \prod_{k=1}^{\infty}\{(1-q^k)^n \prod_{1\leqslant i\leqslant n}(1-q^k x_i)(1-q^k x_i^{-1}).$$
$$\prod_{1\leqslant i<j\leqslant n}(1-q^k x_i x_j)(1-q^k x_i x_j^{-1})(1-q^k x_i^{-1}x_j)(1-q^k x_i^{-1}x_j^{-1})\}$$
$$= \prod_{k=1}^{\infty}\{\prod_{1\leqslant i<j\leqslant N}(1-q^k x_i x_j)\} = \prod_{k=1}^{\infty} A_{q^k}(\boldsymbol{x})_N,$$

where $N = 2n+1$ and $x_{n+i} = x_i^{-1}$ for $i = 1, 2, \ldots, n$ and $x_{2n+1} = 1$ as in (1.2). The notation of (3.1a) has been used to emphasise that the left-hand side of Macdonald identity has now been written as a product of S-function series in a finite set of indeterminates $(\boldsymbol{x})_N$. These indeterminates are nothing other than eigenvalues of elements of the group $\mathrm{SO}(2n+1)$ and S-functions in these eigenvalues can be written in terms of $\mathrm{SO}(2n+1)$ group characters by making use of (3.8). It follows that an expansion of the form

$$(5.6)\quad \prod_{k=1}^{\infty} A_{q^k}(\boldsymbol{x})_N = \sum_{\lambda} g_{\lambda}(q)\,[\lambda](\boldsymbol{x})_N$$

must be valid. The problem is to determine the coefficients $g_{\lambda}(q)$. This can be done by using (3.1a), the S-function product rule (1.12), (3.8) and the S-function quotient rule (1.13). Calculations along these lines strongly suggest the validity of the very remarkable formula

$$(5.7)\quad \prod_{k=1}^{\infty} A_{q^k}(\boldsymbol{x})_N = \sum_{\alpha\in A}(-1)^{|\alpha|/2}\,q^{|\alpha|/2}\,[\alpha](\boldsymbol{x})_N.$$

Thus one starts from a product over k of a power series in q^k whose coefficients are S-function and obtains the same series but this time in q with coefficients which are $\mathrm{SO}(N)$ characters. On the basis of these and similar calculations one is led to the following conjectures:

$$(5.8a)\quad \prod_{k=1}^{\infty} A_{q^k}(\boldsymbol{x}) = \sum_{\alpha\in A}(-1)^{|\alpha|/2}\,q^{|\alpha|/2}\,[\alpha](\boldsymbol{x})_N,$$

$$(5.8b)\quad \prod_{k=1}^{\infty} C_{q^k}(\boldsymbol{x})_N = \sum_{\gamma\in C}(-1)^{|\gamma|/2}\,q^{|\gamma|/2}\,\langle\gamma\rangle(\boldsymbol{x})_N,$$

$$(5.8c)\quad \prod_{k=1}^{\infty} A_{q^k}(\boldsymbol{x})_N\,L_{q^k}(\boldsymbol{x})_N = \sum_{\alpha\in A}(-1)^{|\alpha|/2}\,q^{|\alpha|/2}\,[\alpha](\boldsymbol{x})_{N+1},$$

$$(5.8d)\quad \prod_{k=1}^{\infty} C_{q^k}(\boldsymbol{x})_N\,P_{q^k}(\boldsymbol{x})_N = \sum_{\gamma\in C}(-1)^{|\gamma|/2}\,q^{|\gamma|/2}\,[\gamma](\boldsymbol{x})_{N+1},$$

$$\text{(5.8e)}\qquad \prod_{k=1}^{\infty} A_{q^k}(\boldsymbol{x})_N\, Q_{q^k}(\boldsymbol{x})_N = \sum_{\alpha\in A} (-1)^{|\alpha|/2}\, q^{|\alpha|/2}\, \langle\alpha\rangle(\boldsymbol{x})_{N-1},$$

$$\text{(5.8f)}\qquad \prod_{k=1}^{\infty} C_{q^k}(\boldsymbol{x})_N\, M_{q^k}(\boldsymbol{x})_N = \sum_{\gamma\in C} (-1)^{|\gamma|/2}\, q^{|\gamma|/2}\, \langle\gamma\rangle(\boldsymbol{x})_{N-1},$$

$$\text{(5.8g)}\qquad \prod_{k=1}^{\infty} E_{q^k}(\boldsymbol{x})_N\, M_{q^{2k}}(\boldsymbol{x})_N = \sum_{\varepsilon\in E} (-1)^{(|\varepsilon|+r)/2}\, q^{|\varepsilon|}\, [\varepsilon](\boldsymbol{x})_N,$$

$$\text{(5.8h)}\qquad \prod_{k=1}^{\infty} E_{q^k}(\boldsymbol{x})_N\, Q_{q^{2k}}(\boldsymbol{x})_N = \sum_{\varepsilon\in E} (-1)^{(|\varepsilon|+r)/2}\, q^{|\varepsilon|}\, \langle\varepsilon\rangle(\boldsymbol{x})_N,$$

$$\text{(5.8i)}\qquad \prod_{k=1}^{\infty} G_{q^k}(\boldsymbol{x})_N\, P_{q^{2k}}(\boldsymbol{x})_N = \sum_{\varepsilon\in G} (-1)^{(|\varepsilon|-r)/2}\, q^{|\varepsilon|}\, [\varepsilon](\boldsymbol{x})_N,$$

$$\text{(5.8j)}\qquad \prod_{k=1}^{\infty} G_{q^k}(\boldsymbol{x})_N\, L_{q^{2k}}(\boldsymbol{x})_N = \sum_{\varepsilon\in E} (-1)^{(|\varepsilon|-r)/2}\, q^{|\varepsilon|}\, \langle\varepsilon\rangle(\boldsymbol{x})_N,$$

and

$$\text{(5.9)}\qquad \prod_{k=1}^{\infty} U_{q^k}(\boldsymbol{x})_N = \prod_{k=1}^{\infty} \Big\{ \prod_{i,j=1}^{N} (1-q^k x_i x_j^{-1})/(1-q^k) \Big\}$$

$$= \sum_{\zeta\in F} (-1)^{|\zeta|}\, q^{|\zeta|}\, \{\overline{\zeta'};\, \zeta\}(\boldsymbol{x})_N.$$

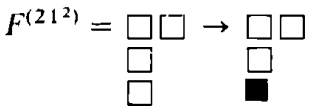
In these identities the indeterminates must be restricted as appropriate for the particular group whose characters appear on the right-hand side. However all the results may be viewed as involving universal characters with the final restrictions being applied through the use of the relevant modification rules (3.11), (3.12) or (2.3). To illustrate this, consider the first identity (5.8a) applied in the case $N = 5$ so that the group characters on the right-hand side are those of SO(5). Thus

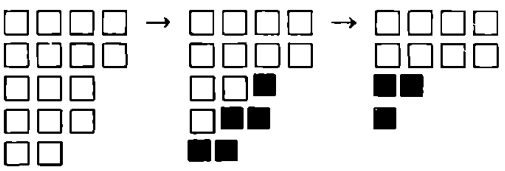
$$\text{(5.10)}\qquad \prod_{k=1}^{\infty} A_{q^k}(\boldsymbol{x})_5 = [0]_5 - q\,[1^2]_5 + q^2\,[21^2]_5 - \ldots$$

$$\ldots + q^8\,[4^2\,3^2\,2]_5 - \ldots$$

All the terms except the first two are non-standard characters of SO(5) and must be modified through the use of (3.11) with $N = 5$. Remembering that all the terms in the resulting series are labelled by partitions $\alpha\in A$, the Frobenius symbol of which has $b_1 = a_1 + 1$, it follows that the continuous boundary strip to be removed has length $h = 2(a_1+2) - N = h_{11} - (N-2)$ where h_{11} is the hook length of the (1, 1)-box of F^α. Removing the strip then leaves along the right-hand boundary a continuous boundary strip of length $(N-2)$ which is 3 in the case of our example. The resulting character may still be non-standard in which case the process must be repeated. For example dealing with the

non-standard terms exhibited in (5.9) gives the diagrams

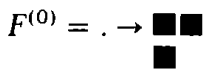
$F^{(21^2)} =$ 

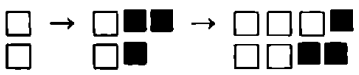
$F^{(4^23^22)} =$ 

so that, taking into account sign factors,

$$\prod_{k=1}^{\infty} A_{q^k}(\boldsymbol{x})_5 = [0]_5 - q[1^2]_5 + q^2[21]_5 - \ldots - q^8[4^2]_5 -$$

It is not difficult to see that the general pattern of terms that survive are those consisting of diagrams with a core specified by either $[0]$ or $[1^2]$ built by adding strips, or slinkies [CGR], of length $(N-2) = 3$ to this core in all possible ways such that each slinky starts in the first row and their successive addition yields a standard diagram at each stage. Thus the two terms obtained above can be viewed as arising in the following way

$F^{(0)} =$ 

$F^{(1^2)} =$  .

This leads to a reformulation of the conjectured Macdonald's identity (5.8a) in terms of standard group characters:

$$\prod_{k=1}^{\infty} A_{q^k}(\boldsymbol{x})_N = \sum_{\substack{\alpha \in A \\ \alpha_1' \leqslant [N/2]}} \sum_{s=0}^{\infty} (-1)^{|\alpha|/2+c} q^{|\alpha|/2+r} [\lambda](\boldsymbol{x})_N \tag{5.11}$$

where $c = c_1 + c_2 + \ldots + c_s$, $r = r_1 + r_2 + \ldots + r_s$ and λ is any partition such that F^{λ} is formed from F^{α} through the addition of s slinkies each of length $(N-2)$ with the ith slinky starting at position $(1, r_i)$ and extending over c_i columns.

As a trivial application of this formulation of the identity it is easy to deal with the case $N = 3$, corresponding to the group SO(3) and the algebra $so(3)$. In this case the only allowed core in the first summation is $\alpha = (0)$. Since $(N-2) = 1$ the added slinkies consist of single boxes added to the first row. It follows that (5.11) gives

$$\prod_{k=1}^{\infty} A_{q^k}(\boldsymbol{x})_3 = \sum_{m=0}^{\infty} (-1)^m q^{m(m+1)/2} [m](\boldsymbol{x})_3. \tag{5.12}$$

Setting $x_1 = e^{\alpha}$, $x_2 = e^{-\alpha}$ and $x_3 = 1$ this gives (5.3), whilst setting $x_1 = x_2 = x_3 = 1$ gives the Jacobi identity (5.4).

The form (5.11) is the optimum form of an expansion of type (5.6) in the sense that all the resulting characters are SO(N) standard. To make contact with the original expansions of type (5.1) it is necessary to examine closely the lattice M and to perform the Weyl reflections bringing each vector m to the dominant chamber. Such an examination lends support to the validity of the conjecture (5.8), and consequently to (5.11).

Note

At the time that this talk was presented at the Stefan Banach International Mathematical Center in Warsaw the above conjectures were only supported by some explicit calculations of the first few terms in the various power series expansions, some checks against the original Macdonald identities and some dimensionality checks involving specialisations of the type $x_i = 1$. It can now be reported that we have devised proofs of all the conjectured results (5.8). These proofs will be published elsewhere. Suffice to say that they involve various generalisations of Lemma 2 in Section 4. One such generalisation takes the form

LEMMA 4.

$$s_{A_p/C_r}(\boldsymbol{x}) = s_{A_p}(\boldsymbol{x})\, s_{A_{rp^2}/C_{p-1}}(\boldsymbol{x}), \tag{5.13}$$

from which it follows that

$$\begin{aligned} s_{A_q/C}(\boldsymbol{x}) &= s_{A_q}(\boldsymbol{x})\, s_{A_{q^2}/C_{q-1}}(\boldsymbol{x}) \\ &= s_{A_q}(\boldsymbol{x})\, s_{A_{q^2}}(\boldsymbol{x})\, s_{A_{q^3}/C_{q-2}}(\boldsymbol{x}) = \ldots = \prod_{k=1}^{\infty} s_{A_{q^k}}(\boldsymbol{x}) \end{aligned} \tag{5.14}$$

where it is necessary to take care that at the nth stage of this iterative process the last factor only contributes $1 + q^{n+1}(\ldots)$.

Thanks to (3.7) this serves to prove (5.8a). All the other conjectures (5.8) can now be proved in a similar way.

The remaining conjecture (5.9) has essentially been proved by Stembridge [Ste] using altogether different methods and without making quite as explicit as we have done the universal nature of the result (5.9) and the related importance of the modification rule (2.3).

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