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THE CONNECTEDNESS OF SYMMETRIC DEGENERACY LOCI: ODD RANKS

Appendix to "**The connectedness of degeneracy loci**" by Loring W. Tu

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This appendix is in a sense a continuation of [13], in which we proved a conjecture of Fulton and Lazarsfeld ([11, Remark 2, p. 50]) on the connectedness of symmetric and skew-symmetric degeneracy loci, when the rank is even. We now deal with the remaining case of odd-rank symmetric degeneracy loci. We are able to prove the following.

THEOREM. *Let E be a vector bundle of rank e and L a line bundle over an irreducible variety X. Suppose $u\colon E\otimes E\to L$ is a symmetric bundle map and r a positive odd integer $\leqslant e$. If $(\mathrm{Sym}^2 E^*)\otimes L$ is ample and $\dim_{\mathbf{C}} X-\binom{e-r+1}{2}\geqslant e-r$, then the degeneracy locus $D_r(u)$ is connected.*

This is not quite the result that we would like, for in the conjecture the dimension hypothesis is that $\dim_{\mathbf{C}} X-\binom{e-r+1}{2}\geqslant 1$.

The key observation in our proof is the linear-algebra fact that just as the set of symmetric bilinear maps of rank at most an even integer can be characterized by the existence of an isotropic subspace of a suitable dimension, so the symmetric bilinear maps of rank at most an odd integer can be characterized by the existence of a *pair* of subspaces $V_1\subset V_2$ of suitable dimensions such that the bilinear map vanishes on $V_1\times V_2$. Section 7 is devoted to a proof of this characterization. For the proof of the main theorem, we found it useful to introduce for two subspaces A, B of a vector space E the concept of *symmetric bilinear maps* on $A\times B$ and dually that of the *symmetric product* $\mathrm{Sym}(A, B)$ of A and B. This is done in Section 8. Using the linear algebra developed in Sections 7 and 8, it is then possible to represent an odd-rank symmetric degeneracy locus as the image of a zero locus on a flag bundle.

To prove the connectedness of this zero locus, we proceed more or less as in [13], replacing the Grassmann bundle there by the flag bundle. The cohomology comparison lemma ([13, Lemma 3.6]) again applies, but curiously the numbers yield only the result above, instead of the full conjecture.

§ 7. Symmetric maps of rank at most an odd integer

Let E be a vector space of dimension e and ϕ: $E \times E \to \mathbf{C}$ a symmetric bilinear form. An isotropic subspace of ϕ is a subspace V such that ϕ vanishes on $V \times V$. Proposition 1.4 says that $\operatorname{rk}\phi \leqslant 2p$ if and only if E contains such a subspace of dimension $e-p$.

PROPOSITION 7.1. *The symmetric bilinear form ϕ: $E \times E \to \mathbf{C}$ has rank $\leqslant 2p+1$ if and only if E contains a pair of subspaces $V_1 \subset V_2$ of dimensions $e-p-1$ and $e-p$ respectively such that ϕ vanishes on $V_1 \times V_2$.*

Proof. ($\Rightarrow$) If $\operatorname{rk}\phi \leqslant 2p$, choose V_2 to be an isotropic subspace of dimension $e-p$ and V_1 to be any subspace of V_2 of codimension one. If $\operatorname{rk}\phi = 2p+1$, then relative to some basis $\{v_1, \ldots, v_e\}$, ϕ is represented by the matrix

$$\begin{bmatrix} 1 & & & \\ & 0_p & I_p & \\ & I_p & 0_p & \\ & & & 0_{e-2p-1} \end{bmatrix},$$

where 0_k denotes the $k \times k$ zero matrix, and I_p the $p \times p$ identity matrix. Choose V_1 to be the subspace with basis $\{v_{p+2}, \ldots, v_e\}$ and V_2 the subspace with basis $\{v_1, v_{p+2}, \ldots, v_e\}$. The matrix of $\phi|_{V_2 \times V_2}$ relative to this basis is

$$\begin{bmatrix} 1 & 0 \\ 0 & 0_{e-p-1} \end{bmatrix},$$

which shows that ϕ vanishes on $V_1 \times V_2$.

($\Leftarrow$) Suppose E contains such a pair of subspaces $V_1 \subset V_2$. Then V_1 is an isotropic subspace of dimension $e-p-1$, and $\operatorname{rk}(\phi|_{V_2 \times V_2}) \leqslant 1$. By [12, Prop. 4], whenever the dimension of a subspace drops by 1, the rank of the restriction of a quadratic form to the subspace drops by at most 2, so that $\operatorname{rk}(\phi|_{V_2 \times V_2}) \geqslant \operatorname{rk}\phi - 2p$. Hence, $\operatorname{rk}\phi \leqslant 2p+1$. ∎

§ 8. The symmetric product of two subspaces

Given a vector space E and two subspaces A and B, we say that a linear map ϕ: $A \otimes B \to \mathbf{C}$ is *symmetric* if it is the restriction of a symmetric linear map: $E \otimes E \to \mathbf{C}$. The space of all symmetric linear maps on $A \otimes B$ is denoted

$\mathrm{Hom}^s(A \otimes B, \mathbf{C})$. In other words,

$$\mathrm{Hom}^s(A \otimes B, \mathbf{C}) := \mathrm{image}(\mathrm{Sym}^2 E^* \hookrightarrow (E \otimes E)^* \to (A \otimes B)^*).$$

We also define the *symmetric product* $\mathrm{Sym}(A, B)$, a subspace of $\mathrm{Sym}^2 E$, to be

$$\mathrm{Sym}(A, B) := \mathrm{image}(A \otimes B \hookrightarrow E \otimes E \xrightarrow{j} \mathrm{Sym}^2 E),$$

where j: $E \otimes E \to \mathrm{Sym}^2 E$ is the natural projection.

PROPOSITION 8.1. *The dual of* $\mathrm{Sym}(A, B)$ *is canonically isomorphic to the space of all symmetric linear maps on* $A \otimes B$:

$$\mathrm{Sym}(A, B)^* \simeq \mathrm{Hom}^s(A \otimes B, \mathbf{C}).$$

Proof. The definition of $\mathrm{Sym}(A, B)$ may be rephrased in terms of the following exact commutative diagram:

$$\begin{array}{ccccc} 0 \to & A \otimes B & \to & E \otimes E \\ & \downarrow & & \downarrow j \\ 0 \to & \mathrm{Sym}(A, B) & \to & \mathrm{Sym}^2 E \\ & \downarrow & & \downarrow \\ & 0 & & 0. \end{array}$$

Dualizing gives

$$\begin{array}{ccccc} 0 \leftarrow & (A \otimes B)^* & \leftarrow & (E \otimes E)^* \\ & \uparrow & & \uparrow j^* \\ 0 \leftarrow & \mathrm{Sym}(A, B)^* & \leftarrow & \mathrm{Sym}^2 E^* \\ & \uparrow & & \uparrow \\ & 0 & & 0, \end{array}$$

which shows that

$$\begin{aligned} \mathrm{Sym}(A, B)^* &= \mathrm{image}(\mathrm{Sym}^2 E^* \hookrightarrow (E \otimes E)^* \to (A \otimes B)^*) \\ &= \mathrm{Hom}^s(A \otimes B, \mathbf{C}). \ \blacksquare \end{aligned}$$

In general the dimension of $\mathrm{Sym}(A, B)$ depends on the dimension of $A \cap B$; in the special case when $A \subset B$, we have the following formula.

LEMMA. *If* $A \subset B$, *then*

$$\dim \mathrm{Sym}(A, B) = (a+1)a/2 + a(b-a),$$

where $a = \dim A$ *and* $b = \dim B$.

Proof. Choose a basis $v_1, \ldots, v_a$ for A, and extend it to a basis $v_1, \ldots, v_a$, $v_{a+1}, \ldots, v_b$ for B. Then a basis for $A \otimes B$ is

$$v_i \otimes v_j, \quad 1 \leqslant i \leqslant a, 1 \leqslant j \leqslant b.$$

The images of these vectors in $\mathrm{Sym}^2 B$ are

$$(*) \qquad v_i v_j := v_i \otimes v_j + v_j \otimes v_i, \quad 1 \leqslant i \leqslant a, 1 \leqslant j \leqslant b.$$

Because of redundancies, for example $v_1 v_2 = v_2 v_1$, the vectors (*) are obviously not linearly independent in $\mathrm{Sym}^2 B$, but at least they span $\mathrm{Sym}(A, B)$. Deleting redundant vectors from (*), we are left with

$$v_i v_j, \quad 1 \leqslant i \leqslant j \leqslant a,$$

and

$$v_i v_j, \quad 1 \leqslant i \leqslant a, \ a+1 \leqslant j \leqslant b,$$

which are linearly independent in $\mathrm{Sym}^2 B$. So they form a basis of $\mathrm{Sym}(A, B)$. Consequently,

$$\dim \mathrm{Sym}(A, B) = (a+1)a/2 + a(b-a). \ \blacksquare$$

PROPOSITION 8.2. *Suppose $A \subset B$. Then there is an exact sequence*

$$0 \to \mathrm{Sym}(A, B) \to \mathrm{Sym}^2 B \to \mathrm{Sym}^2(B/A) \to 0.$$

Proof. Tensoring the exact sequence

$$0 \to A \to B \to B/A \to 0$$

by B yields the exact sequence

$$0 \to A \otimes B \to B \otimes B \to (B/A) \otimes B \to 0,$$

which fits into the commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & A \otimes B & \to & B \otimes B & \to & (B/A) \otimes B & \to & 0, \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \mathrm{Sym}(A, B) & \to & \mathrm{Sym}^2 B & \to & \mathrm{Sym}^2(B/A) & \to & 0. \end{array}$$

A little diagram-chasing shows that $\mathrm{Sym}(A, B)$ is contained in the kernel of the natural surjection α: $\mathrm{Sym}^2 B \to \mathrm{Sym}^2(B/A)$. Since

$$\begin{aligned} \dim \ker \alpha &= \tfrac{1}{2}(b+1)b - \tfrac{1}{2}(b-a+1)(b-a) \\ &= \tfrac{1}{2}(a+1)a + a(b-a) = \dim \mathrm{Sym}(A, B), \end{aligned}$$

the two spaces $\mathrm{Sym}(A, B)$ and $\ker \alpha$ are actually equal. This proves the exactness of the sequence in the proposition. ■

PROPOSITION 8.3. *Suppose $A \subset B$ and $\phi \in \mathrm{Sym}^2 B$. Let $\tilde{\phi}$: $B^* \otimes B^* \to \mathbf{C}$ be the symmetric linear map associated to ϕ. Then ϕ lies in* $\mathrm{Sym}(A, B)$ *if and only if $(B/A)^*$ is an isotropic subspace of $\tilde{\phi}$.*

Proof. By the exact sequence of Proposition 8.2, an element ϕ of $\mathrm{Sym}^2 B$ lies in $\mathrm{Sym}(A, B)$ iff its image in $\mathrm{Sym}^2(B/A)$ is zero iff it is zero as a symmetric map: $(B/A)^* \times (B/A)^* \to \mathbf{C}$ iff $(B/A)^*$ is an isotropic subspace of ϕ. ■

If E is a vector space of dimension e and ψ: $E \times E \to \mathbf{C}$ is a symmetric bilinear map, we define the *isotropic Grassmannian* $G_\psi(k, E)$, sometimes written $G_\psi(k, e)$, to be

$$G_\psi(k, E) := \{V \subset G(k, E) \mid V \text{ is an isotropic subspace of } \psi\}.$$

Note that for $\dim_{\mathbf{C}} E \geqslant 2$, $G_{\psi}(1, E)$ is precisely the quadric defined by ψ in the projective space $\mathbf{P}(E)$.

PROPOSITION 8.4. *Let B be a vector space of dimension b, $\phi \in \mathrm{Sym}^2 B$, and $\tilde{\phi}$: $B^* \otimes B^* \to \mathbf{C}$ the symmetric linear map associated to ϕ. Then the variety W of all a-dimensional subspaces A of B such that $\phi \in \mathrm{Sym}(A, B)$ is isomorphic to the isotropic Grassmannian $G_{\tilde{\phi}}(b-a, B^*)$.*

Proof. First observe that every subspace of B^* is of the form $(B/A)^*$ for some subspace A of B. By Proposition 8.3, the map: $W \to G_{\tilde{\phi}}(b-a, B^*)$ defined by $A \mapsto (B/A)^*$ is an isomorphism. ∎

§ 9. A flag bundle construction

We now begin the proof of the main theorem, assuming r to be an odd integer, say $2p+1$. By the argument of [13, Section 5], we may take X to be a smooth irreducible projective variety and L to be the trivial line bundle over X.

Using the characterization in Proposition 7.1 of symmetric maps of rank at most an odd integer, one can represent an odd-rank symmetric degeneracy locus as the image of a zero locus on a flag bundle, as follows. If V is a vector space of dimension e, let $F(a_1, a_2, V)$ be the flag manifold

$$\{V_1 \subset V_2 \subset V \mid \dim_{\mathbf{C}} V_i = a_i\}.$$

The dimension of this flag manifold is easily shown to be

$$a_1(a_2-a_1)+a_2(e-a_2). \tag{9.1}$$

Now let $E \to X$ be a vector bundle of rank e, and let π: $F(e-p-1, e-p, E) \to X$ be its associated flag bundle. Over $F := F(e-p-1, e-p, E)$ there are two universal subbundles S_1 and S_2 of ranks $e-p-1$ and $e-p$ respectively. By the construction of Section 8, $\mathrm{Sym}(S_1, S_2)$ is a subbundle of $\pi^* \mathrm{Sym}^2 E$ and therefore, $\mathrm{Sym}(S_1, S_2)^*$ is a quotient bundle of $\pi^* \mathrm{Sym}^2 E^*$. The section u of $\mathrm{Sym}^2 E^*$ pulls back under π to a section $\pi^* u$ of $\pi^* \mathrm{Sym}^2 E^*$ over f, which in turn projects to a section t of $\mathrm{Sym}(S_1, S_2)^*$:

$$t(x, V_1 \subset V_2 \subset E_x) = u(x)|_{V_1 \times V_2}.$$

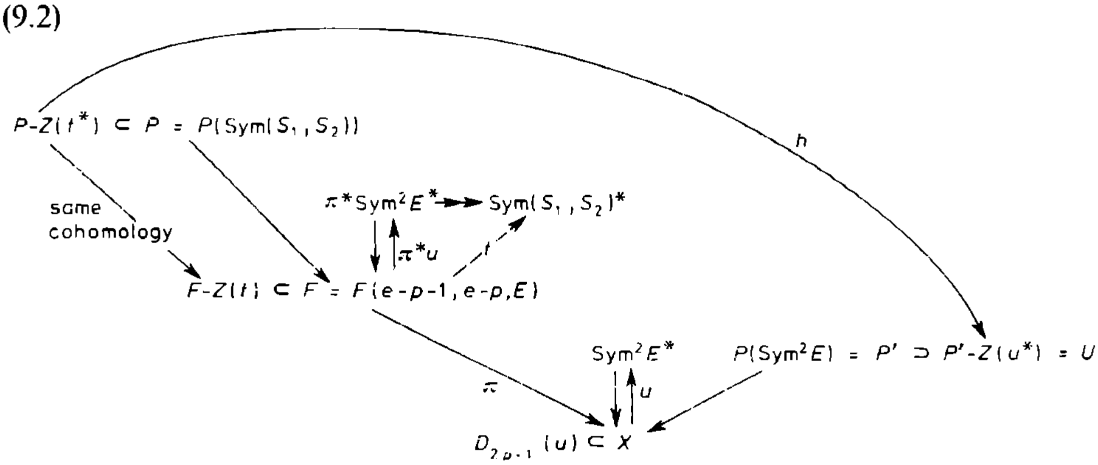
By Proposition 7.1, π maps the zero locus $Z(t)$ in F surjectively onto the degeneracy locus $D_{2p+1}(u)$ in X. Hence, it suffices to prove the connectedness of $Z(t)$. This we do by following the same strategy as in Section 5.

Let $P = \mathbf{P}(\mathrm{Sym}(S_1, S_2))$ and $P' = \mathbf{P}(\mathrm{Sym}^2 E)$. There is a natural map h: $P \to P'$ defined by

$$h(x, V_1 \subset V_2 \subset E_x, \phi \in \mathrm{Sym}(V_1, V_2)) = (x, \phi \in \mathrm{Sym}^2 E_x).$$

We then have the diagram

(9.2)



PROPOSITION 9.3. *The natural map h: $P \to P'$ sends $P-Z(t^*)$ to $P'-Z(u^*)$.*

Proof. Since

$$\begin{aligned} t^*(x, V_1 \subset V_2 \subset E_x, \phi \in \mathrm{Sym}(V_1, V_2)) &= t(x, V_1 \subset V_2 \subset E_x)^*(\phi) \\ &= \left(u(x)|_{V_1 \times V_2}\right)^*(\phi) \\ &= u(x)^*(\phi) \\ &= u^*(x, \phi) \\ &= u^*\left(h(x, V_1 \subset V_2, \phi)\right), \end{aligned}$$

$t^*(\ ) \neq 0$ iff $u^*(h(\ )) \neq 0$. Hence h sends $P-Z(t^*)$ to $P'-Z(u^*)$. ■

To apply the cohomology lemma (Lemma 5.4) it is now necessary to compute the fiber dimension of h.

§ 10. The fibers of h

The map h: $\mathbf{P}(\mathrm{Sym}(S_1, S_2)) \to \mathbf{P}(\mathrm{Sym}^2 E)$ can be factored into a composition of two natural maps h_1 and h_2:

$$\begin{array}{ccccc} \mathbf{P}(\mathrm{Sym}(S_1, S_2)) & \xrightarrow{h_1} & \mathbf{P}(\mathrm{Sym}^2 S) & \xrightarrow{h_2} & \mathbf{P}(\mathrm{Sym}^2 E) \\ \downarrow & & \downarrow & & \downarrow \\ F(e-p-1, e-p, E) & \to & G(e-p, E) & \to & X, \end{array}$$

where S is the universal subbundle over the Grassmann bundle $G(e-p, E)$,

$$h_1(x, V_1 \subset V_2 \subset E_x, \phi \in \mathrm{Sym}(V_1, V_2)) = (x, V_2 \subset E_x, \phi \in \mathrm{Sym}^2 V_2),$$

and

$$h_2(x, V_2 \subset E_x, \phi \in \mathrm{Sym}^2 V_2) = (x, \phi \in \mathrm{Sym}^2 E).$$

In [13, Section 3] we analyzed the fibers of h_2, and found that if $(x, \phi \in \mathrm{Sym}^2 E_x) \in \mathbf{P}(\mathrm{Sym}^2 E)$, then

$$\begin{aligned} h_2^{-1}(x,\phi) &\simeq \{V_2 \in G(e-p, E_x) \mid \operatorname{im}\phi \subset V_2 \subset E_x\} \\ &\simeq G(e-p-\operatorname{rk}\phi, e-\operatorname{rk}\phi). \end{aligned}$$

We now analyze the fibers of h_1. Let $(x, V_2 \subset E_x, \phi \in \operatorname{Sym}^2 V_2)$ be an element of $\mathbf{P}(\operatorname{Sym}^2 S)$. Denote by $\tilde{\phi}$: $V_2^* \otimes V_2^* \to \mathbf{C}$ the symmetric linear map associated to ϕ. Then

$$\begin{aligned} h_1^{-1}(x, V_2, \phi) &= \{V_1 \in G(e-p-1, V_2) \mid \phi \in \operatorname{Sym}(V_1, V_2)\} \\ &\simeq G_{\tilde{\phi}}(1, V_2^*) \quad \text{(by Prop. 8.4)} \\ &\simeq \text{a quadric in } \mathbf{P}^{e-p-1}. \end{aligned}$$

Therefore, for $(x,\phi) \in \mathbf{P}(\operatorname{Sym}^2 E)$,

$$\begin{aligned} \text{(10.1)} \qquad \dim_{\mathbf{C}} h^{-1}(x,\phi) &= \dim_{\mathbf{C}} G_{\tilde{\phi}}(1, e-p) + \dim_{\mathbf{C}} G(e-p-\operatorname{rk}\phi, e-\operatorname{rk}\phi) \\ &= e-p-2+(e-p-\operatorname{rk}\phi)p \\ &= (e-p-\operatorname{rk}\phi)(p+1)+\operatorname{rk}\phi-2. \end{aligned}$$

§ 11. Completing the proof

Returning to Diagram 9.2, our goal now is to compute the cohomology of $P-Z(t^*)$ by applying Lemma 5.4. Stratifying $U = P' - Z(u^*)$ by rank, we let $Y_k = U_{e-p-k} = \mathbf{P}(D_{e-p-k}(\operatorname{Sym}^2 E)) - Z(u^*)$ be the locus of rank $\leqslant e-p-k$ in U as in Section 3. Since $\operatorname{Sym}^2 E^*$ is ample, U is affine, and each Y_k, being a closed subvariety of U, is also affine. Then

$$\ldots \subset Y_{k+1} \subset Y_k \subset \ldots \subset Y_0$$

and if $(x,\phi) \in Y_k - Y_{k+1}$,

$$\dim_{\mathbf{C}} h^{-1}(x,\phi) = k(p+1)+e-p-k-2 = (k-1)p+e-2$$

by (10.1). In the cohomology comparison lemma (5.4) set $d(k) = (k-1)p+e-2$. Then

$$\begin{aligned} R &= \max\nolimits_{k\geqslant 0} \{\dim_{\mathbf{C}} Y_k + 2(k-1)p+2e-4\} \\ &= \max\nolimits_{k\geqslant 0} \left\{\dim_{\mathbf{C}} P' - \binom{p+k+1}{2} + 2(k-1)p+2e-4\right\} \\ &= \max\nolimits_{k\geqslant 0} \left\{\dim_{\mathbf{C}} P' - \binom{p-k}{2} + 2e-3p-4\right\} \\ &= \dim_{\mathbf{C}} P' + 2e-3p-4 \\ &= \dim_{\mathbf{C}} X + \binom{e+1}{2} + 2e-3p-5 \\ &= \dim_{\mathbf{C}} X + \frac{e^2+5e}{2} - 3p-5. \end{aligned}$$

By Lemma 5.4,

$$H^q(P-Z(t^*);\, Z) = 0 \qquad \text{for } q \geqslant \dim_{\mathbf{C}} X + \frac{e^2+5e}{2} - 3p - 4.$$

By (9.1) the dimension of the flag bundle $F = F(e-p-1, e-p, E)$ is

$$\begin{aligned} \dim_{\mathbf{C}} F &= \dim_{\mathbf{C}} X + (e-p-1)\cdot 1 + (e-p)p \\ &= \dim_{\mathbf{C}} X + (e-p)(p+1) - 1. \end{aligned}$$

A straightforward computation shows that

$$\dim_{\mathbf{C}} X \geqslant \binom{e-2p}{2} + e - 2p - 1 \Leftrightarrow 2\dim_{\mathbf{C}} F - 1 \geqslant \dim_{\mathbf{C}} X + \frac{e^2+5e}{2} - 3p - 4.$$

By hypothesis, $\dim_{\mathbf{C}} X \geqslant \binom{e-(2p+1)+1}{2} + e - (2p+1)$. Hence,

$$H^q(F-Z(t);\, \mathbf{Z}) = H^q(P-Z(t^*);\, \mathbf{Z}) = 0$$

for $q = 2\dim_{\mathbf{C}} F$, $2\dim_{\mathbf{C}} F - 1$. As in Section 5 this implies that $Z(t)$ and hence $D_{2p+1}(u)$ is connected.

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