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THE DIMENSION OF A QUASI-HEREDITARY ALGEBRA

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Quasi-hereditary algebras have been introduced by L. Scott [S] in order to study highest weight categories as they arise in the representation theory of complex Lie algebras and algebraic groups. They have been studied by Cline, Parshall and Scott [CPS], [PS], and in [DR1], [DR2]. Here, we are going to give lower and upper bounds for the dimension of a quasi-hereditary algebra in terms of its species, and we characterize those algebras where one of these bounds is attained: we call them the shallow and the deep quasi-hereditary algebras, respectively.

1. Definitions and results

Let A be a basic semiprimary ring with radical N, let $e_1, \ldots, e_n$ be a complete set of orthogonal primitive idempotents. The simple right A-module which is not annihilated by e_i will be denoted by $E(i)$, its projective cover by $P(i) = P_A(i)$. The simple left A-module not annihilated by e_i is denoted by $E^*(i)$. The *species* of A is, by definition, $\mathcal{S} = \mathcal{S}(A) = (F_i, {}_iM_j)_{1 \leqslant i,j \leqslant n}$, where $F_i = e_iAe_i/e_iNe_i$, and ${}_iM_j = e_iNe_j/e_iN^2e_j$. In our considerations, the total ordering of the index set $\{1, \ldots, n\}$ of the species will usually be of importance, and in order to stress this, we will speak of a *labelled* species.

We recall that an ideal J of A is called a *heredity ideal* provided $J^2 = J$, $JNJ = 0$, and the right module J_A (or, equivalently, the left module ${}_AJ$) is

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projective. And A is said to be *quasi-hereditary* provided there exists a chain $\mathscr{J} = (J_i)_i$ of ideals

$$0 = J_0 \subset J_1 \subset \ldots \subset J_m = A$$

such that J_i/J_{i-1} is a heredity ideal of A/J_{i-1}; such a chain will be called a *heredity chain* of A. Observe that any heredity ideal J is generated (as an ideal) by an idempotent, and if e is any idempotent in J, then the ideal $\langle e \rangle$ generated by e is a heredity ideal of A, and $J/\langle e \rangle$ is a heredity ideal of $A/\langle e \rangle$. It follows that we can refine any heredity chain of A to a heredity chain $\mathscr{J}$ such that, in addition, J_i/J_{i-1} is generated by a primitive idempotent, and we call such a heredity chain a *saturated* one. So, let $\mathscr{J}$ be a saturated heredity chain of A, and we always asume that the idempotents e_i are chosen in such a way that $J_i = \langle e_{n-i+1} + \ldots + e_n \rangle$, for $0 \leqslant i \leqslant n$. In this way, the quasi-hereditary algebra A together with the fixed saturated heredity chain determines uniquely $\mathscr{S}(A)$ as a labelled species. Note that $\mathscr{S}(A)$ is a species without loops.

Assume that A is quasi-hereditary, with heredity chain $\mathscr{J} = (J_i)_i$, where $J_i = \langle e_{n-i+1} + \ldots + e_n \rangle$. Let $A_i = A/J_{n-i}$. Note that $E(i)$ and $E^*(i)$ are A_i-modules, and we denote their A_i-projective covers by $\Delta(i) = \Delta_A(i)$ and $\Delta^*(i) = \Delta_A^*(i)$, respectively. Since we deal with a quasi-hereditary algebra, it follows that J_i/J_{i-1}, as a right A-module, is the direct sum of copies of $\Delta(n-i+1)$ (so the modules $\Delta(i)$ are just those modules which occur as building blocks in the standard filtrations of the projective right A-modules: the "Verma modules", or "induced modules"). Similarly, J_i/J_{i-1} is, as left A-module, the direct sum of copies $\Delta^*(n-i+1)$.

By definition, both $\Delta(i)$ and $\Delta^*(i)$ are local A-modules. In case all the modules $\Delta(i)$ and $\Delta^*(i)$, with $1 \leqslant i \leqslant n$, have Loewy length at most 2, we call A *shallow*. Thus, A is shallow if and only if all the modules $\operatorname{rad}\Delta(i)$ and $\operatorname{rad}\Delta^*(i)$ are semisimple. Observe that these modules are actually A_{i-1}-modules, and we call A *deep* provided $\operatorname{rad}\Delta(i)$ is a projective right A_{i-1}-module and $\operatorname{rad}\Delta^*(i)$ is a projective left A_{i-1}-module, for all $1 \leqslant i \leqslant n$.

Now, conversely, let $\mathscr{S}$ be a labelled species without loops, say $\mathscr{S} = (F_i, {}_iM_j)_{1 \leqslant i,j \leqslant n}$, with ${}_iM_i = 0$ for all i. The tensor algebra $\mathscr{T}(\mathscr{S})$ can be decomposed as follows. Let $T = T(n)$ be the set of all sequences $(t_0, t_1, \ldots, t_m)$ where the t_i are integers with $1 \leqslant t_i \leqslant n$, and $m \geqslant 1$, such that, moreover, $t_{i-1} \neq t_i$ for $1 \leqslant i \leqslant m$. For $t = (t_0, t_1, \ldots, t_m) \in T$, let

$$M(t) = {}_{t_0}M_{t_1} \underset{F_{t_1}}{\otimes} {}_{t_1}M_{t_2} \underset{F_{t_2}}{\otimes} \cdots \underset{F_{t_{m-1}}}{\otimes} {}_{t_{m-1}}M_{t_m},$$

and for $T' \subseteq T$, let

$$M(T') = \bigoplus_{t \in T'} M(t).$$

Let $\mathscr{T}_0(\mathscr{S}) = \prod_{i=1}^n F_i$ and $\mathscr{T}_+(\mathscr{S}) = M(T)$, thus $\mathscr{T}(\mathscr{S}) = \mathscr{T}_0(\mathscr{S}) \oplus \mathscr{T}_+(\mathscr{S})$.

We are going to define two factor algebras of $\mathscr{T}(\mathscr{S})$ which will turn out to be quasi-hereditary. Both algebras will be of the form $\mathscr{T}(\mathscr{S})/M(T')$ for suitable choices of T'. In order to define the first one, we define complementary subsets U, U^0 of T as follows: Let

$$U = U(n) = \{(t_0, t_1) \in T\} \cup \{(t_0, t_1, t_2) \in T \mid t_0 < t_1 > t_2\},$$

thus

$$U^0 = \mathscr{T} \setminus U = \{(t_0, t_1, \ldots, t_m) \in T \mid \text{there is } 0 < i < m \text{ with } t_i < \max(t_{i-1}, t_{i+1})\}.$$

Obviously, $M(U^0)$ is an ideal of $\mathscr{T}(\mathscr{S})$, and

$$(\mathscr{T}_+(\mathscr{S}))^3 \subseteq M(U^0) \subseteq (\mathscr{T}_+(\mathscr{S}))^2,$$

thus $M(U^0)$ is an admissible ideal. We define $S(\mathscr{S}) = T(\mathscr{S})/M(U^0)$. Note that as abelian groups, we can identify $S(\mathscr{S})$ and $\mathscr{T}_0(\mathscr{S}) \oplus M(U)$.

For the second algebra, we define complementary subsets V, V^0 of T as follows: Let

$$V = V(n) = \{(t_0, \ldots, t_m) \in T \mid \text{given } i < j \text{ with } t_i = t_j, \text{ there exists } l \text{ with } i < l < j \text{ and } t_i < t_l\},$$

$$V^0 = T \setminus V = \{(t_0, \ldots, t_m) \in T \mid \text{there are } i < j \text{ with } t_i = t_j \text{ and } t_l < t_i \text{ for all } i < l < j\}.$$

As usual, we may consider a product on T by using the juxtaposition, thus $(t_0, \ldots, t_m) \cdot (t'_0, \ldots, t'_{m'}) = (t_0, \ldots, t_m, t'_0, \ldots, t'_{m'})$. Of course, for subsets T', T'' of T, we define $T' \cdot T'' = \{t' \cdot t'' \mid t' \in T', t'' \in T'' \text{ and } t' \cdot t'' \in T\}$ and so on. Then, obviously, for $n \geqslant 2$

$$V(n) = V(n-1) \cup V(n-1) \cdot n \cup n \cdot V(n-1) \cup V(n-1) \cdot n \cdot V(n-1).$$

By induction on n, we see that $V(n)$ is finite. In particular, the sequences $(t_0, \ldots, t_m) \in V(n)$ are of bounded length, say $m \leqslant v(n)$ for some $v(n)$. Thus

$$(\mathscr{T}_+(\mathscr{S}))^{v(n)+1} \subseteq M(V^0) \subseteq (\mathscr{T}_+(\mathscr{S}))^2,$$

so that $M(V^0)$ is an admissible ideal. We define $D(\mathscr{S}) = \mathscr{T}(\mathscr{S})/M(V^0)$, and note that $D(\mathscr{S})$ can be identified, as an abelian group, with $\mathscr{T}_0(\mathscr{S}) \oplus M(V)$.

THEOREM 1. *Let $\mathscr{S}$ be a labelled species without loops. The rings $S(\mathscr{S})$ and $D(\mathscr{S})$ are quasi-hereditary, with labelled species $\mathscr{S}$. The ring $S(\mathscr{S})$ is shallow, the ring $D(\mathscr{S})$ is deep.*

In particular, we see that the nonexistence of loops is the only condition on a species for being realizable as the species of a quasi-hereditary ring.

Let k be a (commutative) field. In case $\mathscr{S}$ is a finite-dimensional k-species, labelled and without loops, we denote by $s_k(\mathscr{S})$ and $d_k(\mathscr{S})$ the k-dimension of $S(\mathscr{S})$ and $D(\mathscr{S})$, respectively. We are going to formulate an estimate for the Cartan invariants of a quasi-hereditary algebra A in terms of the Cartan invariants of the corresponding algebras $S(\mathscr{S})$ and $D(\mathscr{S})$. In this way, we deduce that the dimension of A is bounded from below by $s_k(\mathscr{S})$ and from above by $d_k(\mathscr{S})$.

THEOREM 2. *Let A be a basic, finite-dimensional k-algebra which is quasi-hereditary with labelled species $\mathscr{S}$. Then, for any i, j*

$$\dim_k(e_i S(\mathscr{S}) e_j) \leqslant \dim_k(e_i A e_j) \leqslant \dim_k(e_i D(\mathscr{S}) e_j).$$

In particular,

$$s_k(\mathscr{S}) \leqslant \dim_k A \leqslant d_k(\mathscr{S}).$$

We have $s_k(\mathscr{S}) = \dim_k A$ if and only if A is shallow, and $d_k(\mathscr{S}) = \dim_k A$ if and only if A is deep.

The proof of Theorem 1 is given in Section 2, the proof of Theorem 2 in Section 3. We add examples showing that besides the algebras $S(\mathscr{S})$ and $D(\mathscr{S})$, there are other shallow or deep algebras. A detailed study of the ring-theoretical and homological properties of quasi-hereditary rings which are shallow or deep will be given in a subsequent publication.

2. The rings $S(\mathscr{S})$ and $D(\mathscr{S})$

The aim of this section is a proof of Theorem 1. Thus, let $\mathscr{S}$ be a labelled species without loops, with index set $\{1, \ldots, n\}$. The proof is by induction on n. If $n = 1$, then $S(\mathscr{S}) = D(\mathscr{S}) = F_1$, thus quasi-hereditary (and trivially both shallow and deep). Thus, let $n \geqslant 2$, and let $\mathscr{S}'$ be the restriction of $\mathscr{S}$ to $\{1, \ldots, n-1\}$.

Consider first $S(\mathscr{S})$. Given $m \in \mathbf{N}$, let $[1, m] = \{i \in \mathbf{N} \mid 1 \leqslant i \leqslant m\}$. Then

$$\begin{aligned}
S(\mathscr{S})e_n &= F_n \oplus M([1, n-1]\cdot n),\\
e_n S(\mathscr{S}) &= F_n \oplus M(n\cdot[1, n-1]),\\
\langle e_n \rangle &= F_n \oplus M(\{t \in U \mid t_i = n \text{ for some } i\})\\
&= F_n \oplus M([1, n-1]\cdot n \cup n\cdot[1, n-1] \cup [1, n-1]\cdot n\cdot[1, n-1])\\
&= (F_n \oplus M([1, n-1]\cdot n)) \otimes_{F_n} (F_n \oplus M(n\cdot[1, n-1]))\\
&= S(\mathscr{S})e_n \otimes_{F_n} e_n S(\mathscr{S}).
\end{aligned}$$

In particular, $e_n S(\mathscr{S}) e_n = F_n$, and the equalities above show that $\langle e_n \rangle$ is a heredity ideal. Of course, $\operatorname{rad} \Delta(n) = M(n\cdot[1, n-1])$ is a semisimple right module, $\operatorname{rad}\Delta^*(n) = M([1, n-1]\cdot n)$ is a semisimple left module. Since $S(\mathscr{S})/\langle e_n\rangle = S(\mathscr{S}')$, we use induction and conclude that $S(\mathscr{S})$ is a shallow quasi-hereditary ring.

Next, we consider $D(\mathscr{S})$. We have

$$\begin{aligned} D(\mathscr{S})e_n &= F_n \oplus M(V(n-1)\cdot n), \\ e_n D(\mathscr{S}) &= F_n \oplus M(n\cdot V(n-1)), \\ \langle e_n\rangle &= F_n \oplus M(V(n-1)\cdot n \cup n\cdot V(n-1) \cup V(n-1)\cdot n\cdot V(n-1)) \\ &= (F_n \oplus M(V(n-1)\cdot n)) \otimes_{F_n} (F_n \oplus M(n\cdot V(n-1))) \\ &= D(\mathscr{S})e_n \otimes_{F_n} e_n D(\mathscr{S}), \end{aligned}$$

so that $e_n D(\mathscr{S}) e_n = F_n$, and $\langle e_n\rangle$ is a heredity ideal. Since $D(\mathscr{S})/\langle e_n\rangle = D(\mathscr{S}')$, it follows by induction that $D(\mathscr{S})$ is quasi-hereditary. Now

$$\operatorname{rad}\Delta(n) = M(n\cdot V(n-1)) = \bigoplus_{i=1}^{n-1} {}_nM_i \otimes_{F_i} P_{D(\mathscr{S}')}(i),$$

thus $\Delta(n)$ is a projective right $D(\mathscr{S}')$-module. Similarly, $\operatorname{rad}\Delta^*(n)$ is a projective left $D(\mathscr{S}')$-module. By induction, it follows that $D(\mathscr{S})$ is deep.

3. Quasi-hereditary k-algebras

Let k be a field, and A a basic finite-dimensional quasi-hereditary k-algebra with labelled species $\mathscr{S}$. Let $\{1, \ldots, n\}$ be the index set of $\mathscr{S}$. Note that $e_n A e_n = F_n$, and, in the same way, $e_n S(\mathscr{S}) e_n = e_n D(\mathscr{S}) e_n = F_n$. In particular, for the proof of the dimension inequalities, we may assume $n \geqslant 2$. Let $\mathscr{S}'$ be the restriction of $\mathscr{S}$ to $\{1, \ldots, n-1\}$; clearly, this is the labelled species for $B = A/\langle e_n\rangle$. By induction, we know that

$$\dim_k(e_i S(\mathscr{S}') e_j) \leqslant \dim_k(e_i B e_j) \leqslant \dim_k(e_i D(\mathscr{S}') e_j),$$

for all $i, j \leqslant n-1$.

First, consider $e_n A e_j$, with $1 \leqslant j \leqslant n-1$. Let $X = \bigoplus_{j=1}^{n-1} e_n A e_j$, thus X is the radical of the right A-module $e_n A$; this is a B-module with top $\bar{X} = \bigoplus_{i=1}^{n-1} {}_nM_i$. Let $d_i = \dim({}_nM_i)_{F_i}$. We denote by P the B-projective cover of X, thus P is the direct sum of d_i copies of $e_i B$, for $1 \leqslant i \leqslant n-1$. The epimorphisms $P \to X \to \bar{X}$ yield epimorphisms $Pe_j \to Xe_j \to \bar{X}e_j$. Now, $\bar{X}e_j = {}_nM_j$, $Xe_j = e_n A e_j$, and $Pe_j = \bigoplus_{i=1}^{n-1} (e_i B e_j)^{d_i}$, thus

$$\dim_k({}_nM_j) \leqslant \dim_k(e_n A e_j) \leqslant \sum_{i=1}^{n-1} d_i \cdot \dim_k(e_i B e_j).$$

However, $e_n S(\mathscr{S}) e_j = {}_nM_j$, so the left-hand term is the desired one. Now,

$\operatorname{rad}(e_n D(\mathscr{S})_{D(\mathscr{S})})$ is the $D(\mathscr{S}')$-projective module with top $\bigoplus_{i=1}^{n-1} {}_nM_i$, thus

$$\operatorname{rad}(e_n D(\mathscr{S})_{D(\mathscr{S})}) = \bigoplus_{i=1}^{n-1} (e_i D(\mathscr{S}'))^{d_i}.$$

It follows that $e_n D(\mathscr{S}) e_j = \bigoplus_{i=1}^{n-1} (e_i D(\mathscr{S}') e_j)^{d_i}$, and therefore

$$\sum_{i=1}^{n-1} d_i \cdot \dim_k(e_i B e_j) \leqslant \sum_{i=1}^{n-1} d_i \cdot \dim_k(e_i D(\mathscr{S}') e_j) = \dim_k(e_n D(\mathscr{S}) e_j).$$

This finishes the proof for $e_n A e_j$. The dual proof yields the similar inequality for $e_j A e_n$, where $1 \leqslant j \leqslant n-1$.

It remains to consider $e_i A e_j$, where $1 \leqslant i,j \leqslant n-1$. Since $\langle e_n \rangle = A e_n \otimes_{F_n} e_n A$, there is the exact sequence

$$0 \to e_i A e_n \otimes_{F_n} e_n A e_j \to e_i A e_j \to e_i B e_j \to 0,$$

and similar ones for $S(\mathscr{S})$ and $D(\mathscr{S})$, namely

$$0 \to e_i S(\mathscr{S}) e_n \otimes_{F_n} e_n S(\mathscr{S}) e_j \to e_i S(\mathscr{S}) e_j \to e_i S(\mathscr{S}') e_j \to 0,$$

$$0 \to e_i D(\mathscr{S}) e_n \otimes_{F_n} e_n D(\mathscr{S}) e_j \to e_i D(\mathscr{S}) e_j \to e_i D(\mathscr{S}') e_j \to 0.$$

The desired inequalities follow from the inequalities for $e_i A e_n$, $e_n A e_j$, and $e_i B e_j$, by taking into account that for a right F_n-space X and a left F_n-space Y, we have

$$\dim_k X \otimes_{F_n} Y = \frac{1}{\dim_k F_n} \dim_k X \cdot \dim_k Y.$$

This finishes the proof of the first part of Theorem 2.

Now assume that A is shallow. By induction, we know that $\dim_k(e_i S(\mathscr{S}') e_j) = \dim_k(e_i B e_j)$, for $i,j \leqslant n-1$. Since $X = \bar{X}$, we have $e_n S(\mathscr{S}) e_j = {}_nM_j = e_n A e_j$, for $j \leqslant n-1$, and similarly $e_j S(\mathscr{S}) e_n = e_j A e_n$ for $j \leqslant n-1$. It follows that $\dim_k(e_i S(\mathscr{S}) e_j) = \dim_k(e_i A e_j)$, for all i, j.

Similarly, if we assume that A is deep, then, by induction, $\dim_k(e_i B e_j) = \dim_k(e_i D(\mathscr{S}') e_j)$, for $i, j \leqslant n-1$. On the other hand, we have in this case $X = P$, thus $e_n A e_j = \bigoplus_{i=1}^{n-1} (e_i B e_j)^{d_i}$, and therefore

$$\dim_k(e_n A e_j) = \sum_{i=1}^{n-1} d_i \cdot \dim_k(e_i B e_j) = \sum_{i=1}^{n-1} d_i \cdot \dim_k(e_i D(\mathscr{S}') e_j) = \dim_k(e_n D(\mathscr{S}) e_j).$$

It follows that $\dim_k(e_i A e_j) = \dim_k(e_i D(\mathscr{S}) e_j)$.

Note that $\dim_k A = \sum_{i,j} \dim_k(e_i A e_j)$, thus always $s_k(\mathscr{S}) \leqslant \dim_k A \leqslant d_k(\mathscr{S})$. Let us first assme $s_k(\mathscr{S}) = \dim_k A$, thus $\dim_k(e_i A e_j) = \dim_k(e_i S(\mathscr{S}) e_j)$, for all i,j. If $i, j \leqslant n-1$, a proper inequality $\dim_k(e_i S(\mathscr{S}') e_j) < \dim_k(e_i B e_j)$ would yield that $\dim_k(e_i S(\mathscr{S}) e_j) < \dim_k(e_i A e_j)$ for the same pair i, j of indices, since

$$\dim_k(e_i A e_j) - \dim_k(e_i S(\mathscr{S}) e_i) = \dim_k(e_i B e_j) - \dim_k(e_i S(\mathscr{S}') e_j) + a,$$

with

$$a = \dim_k(e_i Ae_n \otimes_{F_n} e_n Ae_j) - \dim_k(e_i S(\mathscr{S})e_n \otimes_{F_n} e_n S(\mathscr{S})e_j) \geqslant 0.$$

Thus $s_k(\mathscr{S}') = \dim_k B$, and B is shallow by induction. On the other hand, $\dim_k(e_n S(\mathscr{S})e_j) = \dim_k(e_n Ae_j)$ implies that $Xe_j = \bar{X}e_j$, for all $1 \leqslant j < n$, and therefore $X = \bar{X}$ is semisimple. This shows that the right A-module $e_n A$ has Loewy length at most 2. Similarly, the left A-module Ae_n has Loewy length at most 2. As a consequence, A is shallow.

In the same way, we proceed in case $\dim_k A = d_k(\mathscr{S})$. We see immediately that $\dim_k(e_i Ae_j) = \dim_k(e_i D(\mathscr{S})e_j)$, for all i, j, and conclude that $\dim_k B = d_k(\mathscr{S}')$. Thus B is deep by induction. On the other hand, $\dim_k(e_n Ae_j) = \dim_k(e_n D(\mathscr{S})e_j)$ implies that $Pe_j = Xe_j$, for all $1 \leqslant j \leqslant n-1$, and therefore $X = P$ is a projective right B-module. Similarly, the radical of the left A-module Ae_n is projective as a left B-module. Thus A is deep.

4. Examples

The bounds $s_k(\mathscr{S}) \leqslant \dim_k A \leqslant d_k(\mathscr{S})$ are optimal, but we should remark that usually $d_k(\mathscr{S}) - s_k(\mathscr{S})$ may be rather large. As an example, consider the k-species $\mathscr{S}_n = (F_i, {}_iM_j)_{1 \leqslant i,j \leqslant n}$ with $F_i = k$ and ${}_iM_i = 0$ for all i, whereas ${}_iM_j = k$ for all $i \neq j$; thus $T(\mathscr{S}_n)$ is the path algebra for the quiver with n vertices, a unique arrow $i \to j$ for $i \neq j$, and no loops. We are going to exhibit $s(n) := s_k(\mathscr{S}_n)$ and $d(n) := d_k(\mathscr{S}_n)$. It suffices to calculate the cardinalities of the index sets $U(n)$ and $V(n)$, since

$$s(n) = n + U(n), \qquad d(n) = n + V(n).$$

Clearly, $|U(1)| = 0 = |V(1)|$. For $n \geqslant 2$, we have

$$U(n) = U(n-1) \cup [1, n-1] \cdot n \cup n \cdot [1, n-1] \cup [1, n-1] \cdot n \cdot [1, n-1],$$

thus

$$|U(n)| = |U(n-1)| + 2(n-1) + (n-1)^2 = |U(n-1)| + n^2 - 1,$$

and consequently,

$$|U(n)| = -n + \sum_{t=1}^{n} t^2 = -n + \tfrac{1}{6}n(n+1)(2n+1).$$

Similarly, from

$$V(n) = V(n-1) \cup V(n-1) \cdot n \cup n \cdot V(n-1) \cup V(n-1) \cdot n \cdot V(n-1)$$

for $n \geqslant 2$, we obtain

$$|V(n)| = 3|V(n-1)| + |V(n-1)|^2.$$

It follows that $s(n) = \frac{1}{6}(n+1)(2n+1)$, and that $d(n)$ is given recursively by

$d(1) = 1$, and $d(n) = d(n-1) + \big(d(n-1)+1\big)^2$ for $n \geqslant 2$. The first values for $s(n)$ and $d(n)$ are the following:

$$\begin{array}{ll} s(1) = 1, & d(1) = 1, \\ s(2) = 5, & d(2) = 5, \\ s(3) = 14, & d(3) = 41, \\ s(4) = 30, & d(4) = 1805, \\ s(5) = 55, & d(5) = 3263441. \end{array}$$

Let $\mathscr{S}$ be a labelled species without loops. Let us assume that there are even no oriented cycles. Then $D(\mathscr{S})$ is the tensor algebra of $\mathscr{S}$. In particular, if $\mathscr{S}$ is, in addition, a finite-dimensional k-algebra where k is a perfect field, then $D(\mathscr{S})$ is the only deep quasi-hereditary algebra with species $\mathscr{S}$. If the labelling is chosen in such a way that ${}_iM_j = 0$ for $i > j$, then $S(\mathscr{S}) = T(\mathscr{S})/T_+(\mathscr{S})^2$, so again $S(\mathscr{S})$ is the only shallow quasi-hereditary algebra with labelled species $\mathscr{S}$. Of course, in general there may be shallow rings which are not of the form $S(\mathscr{S})$, the first example is the path algebra of the quiver of Fig. 1 with the commutativity relation.

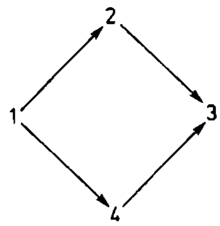

Fig. 1

For a labelled species $\mathscr{S}$ without loops but with oriented cycles there usually also will exist deep rings which are not of the form $D(\mathscr{S})$. For example, consider the algebra A given by the quiver of Fig. 2 with relations $\beta\alpha - \gamma\delta = 0$

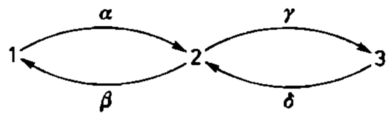

Fig. 2

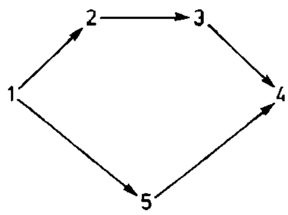

Fig. 3

and $\delta\gamma = 0$. The labelled species corresponding to this quiver will be denoted by $\mathscr{S}$. Then A is deep with labelled species $\mathscr{S}$, but not isomorphic to $D(\mathscr{S})$.

Also, we should remark that there are quasi-hereditary algebras A with radical N such that no ideal $I \subseteq N^2$ yields a shallow algebra A/I. A typical example is the algebra A given by the quiver of Fig. 3 with the commutativity relation. Note that A has a unique minimal nonzero ideal J. An ideal I with A/I shallow must contain J, but there is no ideal I with $J \subseteq I \subseteq N^2$ such that A/I is quasi-hereditary with respect to the given ordering of the vertices.

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