TOPICS IN ALGEBRA
BANACH CENTER PUBLICATIONS, VOLUME 26, PART 2
PWN–POLISH SCIENTIFIC PUBLISHERS
WARSAW 1990

PROJECTIVE SIMPLICITY OF GROUPS OF RATIONAL POINTS OF SIMPLY CONNECTED ALGEBRAIC GROUPS DEFINED OVER NUMBER FIELDS

G. TOMANOV

Institute of Mathematics, Bulgarian Academy of Sciences
Sofia, Bulgaria

§ 0. Introduction

(0.1). Let G be an absolutely simple, simply connected, algebraic group, defined over a field K and $G(K)$ be the group of K-rational points of G. It is known that the structure of $G(K)$ depends in an essential way on the properties of the base field K. As was proved by V. P. Platonov [10], the quotient of $G(K)$ by its center $Z(G(K))$ may not be a simple group even when G is a K-isotropic group. Therefore, it is natural to investigate first the cases when K is a local or a global field. It is known that if K is a local or a global field and G is K-isotropic then $G(K)$ is projectively simple, i.e., the group $G(K)/Z(G(K))$ is simple as an abstract group [12], [13], [23]. When K is a local field and G is a K-anisotropic algebraic group then every infinite normal subgroup of $G(K)$ is a congruence subgroup [15].

From now on we assume that G is anisotropic over K and that K is a global field. In this case there exists the following conjecture due to V. P. Platonov [11]: $G(K)$ is projectively simple if and only if the groups $G(K_v)$ are projectively simple for all nonarchimedean completions K_v of the basefield K. (Recall that the group of K-rational points $G(K)$ is called projectively simple iff the quotient $G(K)/Z(G(K))$ is a simple abstract group.) One of the important aspects of the above conjecture is its close relation with the famous congruence-subgroup problem (see [22]). Note that by a general result of G. A. Margulis [8] every infinite normal subgroup of $G(K)$ has a finite index in $G(K)$.

(0.2). At present the projective simplicity of $G(K)$ for K-anisotropic algebraic groups G is proved for groups of type A_1 [9], [14], for Spin(f), where f is a regular quadratic form over a number field of $n \geqslant 5$ variables [7],

This paper is in final form and no version of it will be submitted for publication elsewhere.

for the special unitary group $SU(F^n, h)$, where F is a quadratic extension of a number field K and h is an hermitian form on F^n, $n \geqslant 3$, [2]. In [2] the simplicity of $G(K)/Z(G(K))$ is also announced for groups of type C_n, F_4, G_2. In [4] it is announced that $G(K)$ is projectively simple, provided K is a number field and the K-algebraic group G splits over a quadratic extension of K. The proof is based on a general idea and it is carried out by successive consideration of the types $A_n - G_2$ which split over a quadratic extension of K. Finally, in the work [3] the proof of the Platonov conjecture for groups of type D_n, $n \geqslant 4$, different from ${}^3D_4$ and ${}^6D_4$, is reduced to the proof of this conjecture for groups of type A_3 (not yet proved)(1). A direct proof of the projective simplicity of $G(K)$ for groups of type D_n is announced in the author's note [22].

(0.3). The aim of this paper is to prove that $G(K)$ is projectively simple if K is a number field and G is of type C_n, D_n, F_4 or G_2. Our proofs are based on different approaches from the above mentioned announcements [2], [4] and the partial result [3]. For groups of type D_n, $n > 4$, we give a second proof of the simplicity of $G(K)/Z(G(K))$ which is also valid when K is any global field of characteristic $\neq 2$. The "exceptional" cases ${}^3D_4$ and ${}^6D_4$ (see [19] for the notation), which are an object of a separate investigation, are not considered in this paper.

(0.4). THEOREM. *Let G be an absolutely simple, simply connected, algebraic group defined over a number field K. Assume that G is of type C_n, D_n, F_4 or G_2. Then $G(K)/Z(G(K))$ is a simple group.*

The main part of the paper is devoted to the consideration of the case D_n. We shall assume that G is anisotropic over K, although a slight modification of our proof is valid for isotropic groups too.

§ 1. Notation and recollections

(1.1). In the sequel F will denote a quaternion division algebra over a number field K. Let $*$: $F \to F$ be the standard involution of F over K and N: $F \to K$, $x \to xx^*$ is the usual reduced norm on F. By Σ, Σ_f, Σ_∞ we denote the set of all places of K, the set of all nonarchimedean places of K, the set of all archimedean places of K. We let (V, h) denote an n-dimensional, regular, ε-hermitian, right vector space over F. Assume that $\varepsilon = \pm 1$. Set $G(K) = SU(V, h)$ for $\varepsilon = 1$ and $G(K) = \mathrm{Spin}(V, h)$ for $\varepsilon = -1$. (For the definition of the group $\mathrm{Spin}(V, h)$ in the case when F is a quaternion skew field we refer the reader to [17] or [20].) According to the classification of the algebraic groups over number fields, $SU(V, h)$ is a group of type C_n and $\mathrm{Spin}(V, h)$ is a group of type D_n. In view of the Kneser result [7] for $\mathrm{Spin}(f)$ and the classification [19], in order to prove the theorem for anisotropic

(1) The case A_3 was recently considered by the author in *Remarques sur la structure des groupes algébriques définis sur des corps de nombres*, C. R. Acad. Sci. Paris 310 (1990), 33 36.

algebraic groups of type C_n and D_n we have to prove that $\mathrm{SU}(V, h)$ and $\mathrm{Spin}(V, h)$ are projectively simple.

Recall the standard realizations of the (K-anisotropic) groups of type F_4 and G_2. Let $C = F \oplus Fl$ be a vector space direct sum of F and an isomorphic image Fl of F. Let $\mu \in K^\times$ ($\equiv$ the multiplicative group of K) and define multiplication in C by

$$(a+bl)(c+dl) = (ac+\mu d^* b)+(da+bc^*)l,$$

where a, b, c, d are in F. The alternative algebra we get in this way is called *Cayley algebra*. For $x = a+bl$ we define $x^* = a^*-bl$. Clearly, $x \to x^*$ is an involution on C. Also, $xx^* = Na-\mu Nb$, where N is the reduced norm on F. The map $x \to xx^*$ is called a *norm* on C and it is also denoted by N. The Cayley algebra C is a division algebra if and only if the quadratic form N (of eight variables) is anisotropic. The group $G(K)$ of K-automorphisms of the algebra C is the group of K-rational points of a K-algebraic group G of type G_2 and, conversely, every K-algebraic group of type G_2 can be realized in this way [19]. G is K-anisotropic iff the quadratic form N is anisotropic.

Recall the definition of exceptional central simple reduced Jordan algebra (see [1] for details). Let $\gamma = \mathrm{diag}\{\gamma_1, \gamma_2, \gamma_3\}$, $\gamma_i \neq 0$ in K, be a diagonal matrix and let C_3 be the set of 3×3 matrices with coefficients from C. Any exceptional central simple reduced Jordan algebra is isomorphic to the algebra $J = J(C_3, \gamma)$, the set of γ-hermitian 3×3 Cayley matrices. Thus, $A \in J(C_3, \gamma)$ if and only if $A = \gamma^{-1}(\bar{A})'\gamma$ and the multiplication in $F(C_3, \gamma)$ is $A.B = \frac{1}{2}(AB+BA)$. (Here $\bar{A} = (a_{ij}^*)$, where $A = (a_{ij})$, and A' is the matrix transposed to A.) Now, the group $G(K)$ of K-automorphisms of $J(C_3, \gamma)$ is a group of K-rational points of a K-algebraic group of type F_4. Conversely, the group of K-rational points of every K-algebraic group of type F_4 can be obtained in such a way [19].

§ 2. Proof of the theorem for groups of type D_n

(2.1). Let V be a vector space over F of dimension n and let h be a regular skew hermitian form in V with respect to the standard involution $*$ of F. During the course of the proof we shall often extend the basefield K. Let L/K be a field extension and let $F_L = F\otimes_K L$ splits, i.e., $F_L \cong M_2(L)$. Denote by (V_L, h_L) the hermitian module obtained by the extension of the basefield L/K. (Here and later on we use the terminology from [16] which is generally accepted.) The study of the hermitian form h_L: $V_L \times V_L \to F_L$ is entirely reduced to the study of some quadratic form f_L. We briefly explain this phenomenon (for more detailed discussion, see [16: 10.3.4]). Recall that the involution $*$ acts on F_L as follows:

$$\begin{bmatrix} x & y \\ z & t \end{bmatrix}^* = \begin{bmatrix} t & -y \\ -z & x \end{bmatrix}.$$

Put

$$e_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad e = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad W_L = V_L e_1.$$

We define a quadratic form f_L on W_L by the equality $h_L(xe_1, ye_1) = f_L(xe_1, ye_1)ee_1$, where $x, y \in V_L$. (When $L = K_v$ we abbreviate the notation and write f_v, h_v, W_v etc.) With $x = xe_1 + ye_2 = xe_1 + xee_1 e$ and $y = ye_1 + yee_1 e$ we have

$$h_L(x, y) = \begin{bmatrix} -f_L(xee_1, ye_1) & -f_L(xee_1, yee_1) \\ f_L(xe_1, ye_1) & f_L(xe_1, yee_1) \end{bmatrix}. \tag{1}$$

We identify the elements from V_L with the pairs from $W_L \times W_L$ by means of the bijective map $x \to (xe_1, xee_1)$. Let $\mathscr{P}_L(x)$ (resp. $\mathscr{P}_v(x)$ when $L = K_v$) be the plane spanned by xe_1 and xee_1. It follows from (1) that the determinant of the quadratic form $f_L|_{\mathscr{P}_L(x)}$ is equal to $\det(h_L(x, x))$. In particular, if K_v is the field of reals $\mathbf{R}$ then $\det(h_v(x, x)) < 0$ iff $\mathscr{P}_v(x)$ is a hyperbolic plane. Recall that the hermitian module (V_L, h_L) is called isotropic if there exists a basis $e_1, e_2, \ldots, e_n$ of the free F_L-module V_L such that $h_L(e_1, e_1) = 0$. (V_L, h_L) is isotropic iff f_L is of Witt index $\geqslant 2$ [16: 10.3.5]. Denote by $\mathrm{SU}(V_L, h_L)$ the group of isometries of (V_L, h_L) with determinant 1. It can be easily deduced from (1) that the map $\sigma \to \sigma|_{W_L}$, $\sigma \in \mathrm{SU}(V_L, h_L)$, is an isomorphism between $\mathrm{SU}(V_L, h_L)$ and $\mathrm{SO}(W_L, f_L)$..

(2.2). As usual, by Cl we denote the Clifford algebra of (V, h). Let $\mathrm{SU}(V, h)'$ be the subgroup in $\mathrm{SU}(V, h)$ of all elements with spinornorm 1. (For the definitions of Clifford algebra and spinornorm we refer the reader to [17] or [20].)

LEMMA. *Let* $\dim_F V = 2$, $\det(h) = 1$, *and let* (V_v, h_v) *be isotropic for each* $v \in \Sigma_f$. *Then* $\mathrm{SU}'(V, h) = [\mathrm{SU}(V, h), \mathrm{SU}(V, h)]$.

Proof. Since $\det(h) = 1$, we have an isomorphism of K-algebras $Cl \cong A_1 \oplus A_2$, where A_1 and A_2 are quaternion K-algebras [17: 6.1 and 9.1]. The standard involution * on Cl induces the standard involutions on A_1 and A_2. Denote $C = \{x \in Cl \mid xx^* \in K^\times\}$. Then $\mathrm{Spin}(V, h) = \{x \in C \mid xx^* = 1\}$ [17: 9.2]. The natural homomorphisms $C \to \mathrm{SU}(V, h)$ and $\mathrm{Spin}(V, h) \to \mathrm{SU}'(V, h)$ are surjective [17] p. 348. Therefore, in order to prove the lemma it is enough to show that $[C, C] = \mathrm{Spin}(V, h)$. Let $S_i = \{v \in \Sigma_f \mid A_{iv} = A_i \otimes_K K_v$ is a division algebra$\}$. It follows from our hypotheses that $S_1 \cap S_2 = \emptyset$. Let Nrd_i be the reduced norm of A_i. Set $R_i = \{x \in A_i \mid \mathrm{Nrd}_i(x) > 0$ at every real place of $K\}$, and $A_i^1 = \{x \in A_i \mid \mathrm{Nrd}_i(x) = 1\}$. By the description of the normal subgroups in A_i^1 [9], we get $[R_i, R_i] = A_i^1$. Let $R = \{(x, y) \in R_1 \times R_2 \mid \mathrm{Nrd}_1(x) = \mathrm{Nrd}_2(x)\}$. Obviously, $R \subset C$ and, therefore, it is enough to prove that $[R, R] = \mathrm{Spin}(V, h)$. It follows from the Hasse–Minkowski theorem that for every $x \in R_1$ (resp. $y \in R_2$) there exists an $y \in R_2$ (resp. $x \in R_1$) with $(x, y) \in R$. Therefore,

$[R, R]$ projects onto A_1^1 and A_2^1, respectively. Since $[R, R]$ is a normal subgroup of $A_1^1 \times A_2^1$, we get $[R, R] \supset [A_1^1, A_2^1] \times [A_2^1, A_2^1]$. On the other hand,

$$[A_1^1, A_1^1] \times [A_2^1, A_2^1] = A_1^1 \times A_2^1 \cap \prod_{v \in S_1 \cup S_2} [A_{1v}^1, A_{2v}^1] \times [A_{2v}^1, A_{2v}^1].$$

Since $[R, R]$ is dense in $\prod_{v \in S_1 \cup S_2} A_{1v}^1 \times A_{2v}^1$ and

$$\prod_{v \in S_1 \cup S_2} [A_{1v}^1, A_{1v}^1] \times [A_{2v}^1, A_{2v}^1] \quad \text{is open in} \quad \prod_{v \in S_1 \cup S_2} A_{1v}^1 \times A_{2v}^1,$$

it follows that $[R, R] = \mathrm{Spin}(V, h)$. This completes our proof.

(2.3). We denote by T the set of all real places v such that $F_v \cong M_2(\mathbf{R})$ and f_v is of Witt index 1, and we denote by T' the set of all real places v such that $F_v \cong M_2(\mathbf{R})$ and f_v is anisotropic. (As fixed in Section 2.1, f_v is the quadratic form corresponding to h_v.) A vector $x \in V$ will be called *definite* iff for all $v \in T$ the plane $\mathscr{P}_v(x)$ is definite, i.e., the corresponding quadratic space of two variables is definite. Let $a \in V$, $\alpha \in F$ and $\tau_{a,\alpha}$ be an unitary reflection, i.e., $\tau_{a,\alpha}(a) = a\alpha$ and $\tau_{a,\alpha}(x) = x$ for $x \in \langle a \rangle^{\perp}$. The unitary reflection $\tau_{a,\alpha}$ will be called *definite* iff a is a definite vector. Denote by $\mathrm{SU}(V, h)^{\sim}$ the subgroup of $\mathrm{SU}(V, h)$ generated by the definite reflections.

(2.4). PROPOSITION. *Let $n \geqslant 3$. Then*

$$\mathrm{SU}(V, h)^{\sim} = \mathrm{SU}(V, h) \cap \textstyle\prod_{v \in T} \mathrm{SU}'(V_v, h_v).$$

The proof of this proposition is based on the following two lemmas, which can be proved by direct computations.

(2.5). LEMMA. *Let W be a regular quadratic $2n$-dimensional ($n \geqslant 2$) space nover $\mathbf{R}$ of Witt index 1. Let $x, y \in W$ and suppose that x and y span a hyperbolic plane. Then there exists a $g \in \mathrm{SO}(W)$ such that the vectors $x - gx$ and $y - gy$ span a definite plane.*

(2.6). LEMMA. *Let $x, y \in V$, $x \neq y$, and $h(x, x) = h(y, y)$. Set $\delta = h(x-y, x)^{-1} h(x-y, y)$. Then $\tau_{x-y,\delta}(x) = y$.*

Proof of Proposition 2.4. The inclusion $\mathrm{SU}(V, h)^{\sim} \subset \mathrm{SU}(V, h) \cap \prod_{v \in T} \mathrm{SU}'(V_v, h_v)$ is obvious (see 2.1). Let $\sigma \in \mathrm{SU}(V, h) \cap \prod_{v \in T} \mathrm{SU}'(V_v, h_v)$. Fix a vector $x \in V$ with $\det(h_v(x, x)) < 0$ at every $v \in T$. It follows from Lemma 2.5 and the weak approximatuon theorem that there exists a $g \in \mathrm{SU}(V, h)$ with $\det(h_v(x - g\sigma x, x - g\sigma x)) > 0$ at every $v \in T$. Let $\tau = \tau_{x - g\sigma x, \delta}$ be a unitary reflection given by Lemma 2.6. Since $\tau \in \mathrm{SU}'(V_v, h)^{\sim}$ for $v \in T$ we obtain $g \in \mathrm{SU}'(V_v, h_v)$ when $v \in T$. The density of $\mathrm{SU}(V, h)^{\sim}$ in $\prod_{v \in T} \mathrm{SU}'(V_v, h_v)$ implies the existence of a $g' \in \mathrm{SU}(V, h)^{\sim}$ such that $\det(h_v(x - g'\sigma x, x - g'\sigma x)) > 0$ for any $v \in T$. By Lemma 2.6 there exists a definite reflection τ' such that $\tau'(x) = g'\sigma x$. Denote $V_1 = x^{\perp}$ and $h_1 = h|_{V_1}$. Since $\mathrm{SU}(V_1, h_1)$ is generated by definite reflections, we get $\tau'^{-1} g' \sigma \in \mathrm{SU}(V, h)^{\sim}$, i.e., $\sigma \in \mathrm{SU}(V, h)^{\sim}$. The lemma is proved.

(2.7). PROPOSITION. *Let* $\dim_F V \geqslant 4$ *and let* $x \in V$ *be a definite vector. Then there is a 2-dimensional plane* $V_0 \subset V$ *with the following properties*: (a) $x \in V_0$; (b) $\det(V_0) = 1$; (c) V_0 *is isotropic over* K_v *for all* $v \notin T \cup T'$; (d) *for every definite vector* $y \in V$ *there is a* $z \in V_0$ *such that* $h(y, y) = h(z, z)$.

We precede the proof of Proposition 2.7 by the following lemma.

(2.8). LEMMA. *Let* $F_v \cong M_2(K_v)$, $v \in \Sigma$. *Let* λ *be a skew symmetric element (i.e.,* $\lambda^* = -\lambda$) *with* $\lambda^2 = a \neq 0$ *and let* $c \in K_v^x$. *Then the* 1*-dimensional skew hermitian forms* $\langle\lambda\rangle$ *and* $\langle c\lambda\rangle$ *are isometric if and only if* $(c, a)_v = 1$, *where* $(\ ,\ )_v$ *is the Hilbert symbol.*

Proof. Let $\mu \in F_v$, $\mu^* = -\mu$, $\mu\lambda = -\lambda\mu$ and $\mu^2 = b \neq 0$. By the Scolem–Noether theorem such a μ exists. Now $\langle\lambda\rangle \cong \langle c\lambda\rangle$ iff c is represented by one of the quadratic forms $\langle 1, -a\rangle$ or $\langle b, -ab\rangle$ [16: 10.3.4]. The following equivalences hold: $\langle\lambda\rangle \cong \langle c\lambda\rangle$ as hermitian forms $\Leftrightarrow$ $\langle c, -ac\rangle \cong \langle 1, -c\rangle$ or $\langle c, -ac\rangle \cong \langle b, -ab\rangle$ as quadratic forms $\Leftrightarrow$ $(c, a)_v = (1, -c)_v = 1$ or $(c, a)_v = (a, b)_v$. On the other hand, $F_v \cong M_2(K_v)$ implies that $(a, b)_v = 1$. The lemma is proved.

Proof of Proposition (2.7). Let $R = \{v \in \Sigma \mid F_v \text{ is a skew field}\}$. Set $\lambda = h(x, x)$ and $a = \lambda^2$. Let $\varepsilon_v = -1$ for $v \in T \cup T'$ and $\varepsilon_v = 1$ for $v \notin R \cup T \cup T'$. Since $R = \emptyset$ and $a < 0$ for every $v \in T \cup T'$, there exists a $c \in K^x$ such that $(c, a)_v = \varepsilon_v$ for every $v \in R$. Let $V_1 = x^\perp$ and $h_1 = h|_{V_1}$. Then $-c\lambda$ is represented locally by h_1 at all places of K. Indeed, for $v \in R$ the assertion follows from the basic properties of skew hermitian forms over local skew fields [16: 10.3.6 and 10.3.7]. Since $c < 0$ at every place $v \in T \cup T'$, we get that h_1 represents $-c\lambda$ over K_v whenever $v \in T \cup T'$. For any $v \notin R \cup T \cup T'$ he quadratic form g_v corresponding to h_{1_v} (see (2.1)) represents over K_v every regular 2-dimensional form, that is h_1 represents $-c\lambda$ for $v \notin R \cup T \cup T'$. By the local-global principle for skew hermitian forms [16: Theorem 10.4.1], h_1 represents $-c\lambda$ globally, i.e., $-c\lambda = h_1(y, y)$ for some $y \in V_1$.

Put $V_0 = \langle x, y\rangle$. It follows from the discussions in (2.1) that V_0 is isotropic for each $v \notin R \cup T \cup T'$. Since $\det(V_0) = 1$, V_0 is also isotropic for $v \in R$ [16: 10.3.6 and 10.3.7]. Therefore V_0 is the plane we need and the proof is completed.

(2.9). PROPOSITION. *Let* $n \geqslant 4$. *Then* $\mathrm{SU}'(V, h)$ *is generated by the commutators* $[\tau_1, \tau_2]$, *where* τ_1 *and* τ_2 *are definite unitary reflections.*

Proof. Let $g \in \mathrm{SU}'(V, h)$. By Proposition 2.4 there exist definite unitary reflections $\tau_i = \tau_{a_i, \alpha_i}$ such that $g = \prod_{i=1}^r \tau_i$. By Proposition (2.7) there is a 2-dimensional plane V_0 with the following properties: (a) $\dim(V_0) = 1$; (b) V_0 is isotropic over K_v for each $v \in \Sigma_f$; (c) if $y \in V$ is a definite vector then there exists a $z \in V_0$ with $h(y, y) = h(z, z)$. For each $i = 1, 2, \ldots, r$ fix $a_i, b_i \in V_0$ such that $h(a_i, a_i) = h(b_i, b_i)$. According to the Witt theorem there exist

$n_i \in \mathrm{SU}(V, h)$ such that $n_i a_i = b_i$ for every i. It is easy to see that n_i can be chosen in such a way that $n_i \in \mathrm{SU}'(V_v, h_v)$ for each $v \in T$. By Proposition 2.4 $n_i \in \mathrm{SU}(V, h)^{\sim}$. We have

$$\tau_{a_i,\alpha_i} = \tau_{b_i,\alpha_i} \tau_{b_i,\alpha_i}^{-1} \tau_{a_i,\alpha_i} = \tau_{b_i,\alpha_i} (\tau_{b_i,\alpha_i}^{-1} n_i^{-1} \tau_{b_i,\alpha_i} n_i).$$

Let N be the subgroup of $\mathrm{SU}'(V, h)$ generated by the commutators of definite reflections. Using the identity $[x, y_1 y_2] = [x, y_2]\ [x, y_1]\ [[x, y_1], y_2]$ one easily sees that $[\tau_{b_i,\alpha_i}, n_i] \in N$. Therefore $g \equiv \prod_{i=1}^{r} \tau_{b_i,\alpha_i} \in [\mathrm{SU}(V_0), \mathrm{SU}(V_0)]$. But $[\mathrm{SU}(V_0), \mathrm{SU}(V_0)] \subset N$. Therefore $g \in N$, which completes our proof.

(2.10). *First proof of the simplicity for groups of type D_n.* Let N be a normal subgroup in $\mathrm{SU}'(V, h)$ such that $N \not\subset Z(\mathrm{SU}'(V, h))$. Due to Proposition (2.9), it is enough to prove that N contains all commutators $[\tau_1, \tau_2]$, where τ_1 and τ_2 are definite reflections. Let $\tau_i = \tau_{a_i,\alpha_i}$, $a_i \in V$ and $\alpha_i \in F^{\times}$. Let $a_1 \in V_0$, where V_0 is given by Proposition (2.7). Note that N is dense subgroup of $\prod_{v\in T} \mathrm{SU}'(V_v, h_v)$. Therefore there is a $g \in N$ such that V_0 and ga_2 span a 3-dimensional subspace V_1 with $\det(V_1)_v > 0$ for all $v \in T$. Since $g\tau_2 g^{-1} = \tau_{ga_2g^{-1},\alpha_i}$, without loss of generality, we can replace τ_2 by $g\tau_2 g^{-1}$ and assume that V_1 is spanned by V_0 and a_2.

We contend that $\mathrm{SU}(V_1)$ is a group of K-rational points of a K-algebraic group, which splits over a quadratic extension of K. Indeed, it is known that $Cl(V_0) \cong A_1 \oplus A_2$, where A_1 and A_2 are quaternion skew fields over K [17: 9.1]. Denote $R = \{v \in \Sigma \mid F_v$ is a skew field$\}$ and $R' = \{v \in \Sigma \mid A_1 \otimes_K K_v$ or $A_2 \otimes_K K_v$ is a skew field$\}$. It follows readily from the isotropy of V_0 at the places $v \in T \cup T_1$ that $R' = R \cup T \cup T'$. Let x be a nonzero vector from V_1 such that $x \in V_0^{\perp}$. Set $\lambda = h(x, x)$ and $L = K(\lambda)$. Since $\det(V_1)_v > 0$ for every $v \in T \cup T'$, we get $-\lambda^2 = \det(h(x, x)) > 0$ for every $v \in T \cup T'$. On the other hand, $F \otimes_K L \cong M_2(L)$. Hence $A_1 \otimes_K L \cong M_2(L)$ and $A_2 \otimes_K L \cong M_2(L)$. Since the plane $\mathscr{P}_L(x)$ (see 2.1) is isotropic, our contention is proved.

The group $\mathrm{SU}(V_1)$ is a group of K-rational points of a simple algebraic group of type A_3. By [2] $\mathrm{SU}'(V_1)$ does not contain infinite normal subgroups. Since N has a finite index in $\mathrm{SU}'(V, h)$ [8], we get that $\mathrm{SU}'(V_1) \subset N$. In particular, $[\tau_1, \tau_2] \in N$. Our proof is finished.

(2.11). One checks easily that the assertions proved in (2.1)–(2.9) remain true if the field K is replaced by any global field of characteristic $\neq 2$. The result [2], which we have used in the final step (2.10) of the above proof, is proved only for number fields. Now, we are going to prove the simplicity for D_n, $n > 4$, in another way. The new proof is valid for any global field of characteristic $\neq 2$ and it is based on an idea of M. Kneser applied in [7] for the orthogonal groups. (It is likely that the same idea works also for $n = 4$.)

(2.12). *Second proof of the simplicity for groups of type D_n, $n > 4$.*

PROPOSITION. *Let $n > 4$, let $\tau = \tau_{c,\alpha}$ be a definite reflection, let $a \in V$ be a definite vector, and let N be an infinite normal subgroup of* $\mathrm{SU}'(V, h)$. *Assume that a and τa are linearly independent. There exists then an $n \in N$ such that $na = \tau a$.*

Proof. By a result of Margulis [8: 2.4.9] we may assume that N is a normal subgroup of $\mathrm{SU}(V, h)$. Set $b = \tau a$, $V_1 = \{a, b\}^{\perp}$, and $h_1 = h|_{V_1}$. For $n \in N$ we set

$$A = \begin{bmatrix} h(a, a) & h(a, b) & h(a, na) \\ h(b, a) & h(a, a) & h(a, na) \\ h(na, a) & h(na, a) & h(a, a) \end{bmatrix}$$

Assume that $\mathrm{Nrd}(A) \neq 0$. Let $(\bar{V}, g)$ be an abstractly defined 3-dimensional hermitian F-space whose scalar product is given by A with respect to a fixed basis a_1, b_1, z_1. Choose $t_1 \in \bar{V}$ such that a_1, b_1, t_1 generate V and t_1 is orthogonal both to a_1 and b_1. Set $\delta = g(t_1, t_1)$. Assume that h_1 represents δ. Then there exists $az \in V$ such that $\{a, b, z\}$ have A as their matrix of scalar products. By Witt's theorem there exist s and $t \in \mathrm{SU}(V, h)$ with $s(a) = a$, $s(na) = z$, $t(a) = b$, and $t(na) = z$. An easy computation shows that $(tn^{-1}t^{-1}sns^{-1})a = b$. Therefore in order to prove the lemma it is enough to find an $n \in N$ with $\mathrm{Nrd}\, A \neq 0$ and such that δ from the construction above is representable by h_1.

Since c is a definite vector one can easily derive from the local-global principle that there exists $z' \in c^{\perp}$ with $h(z', z') = h(a, a)$. Let $\tilde{S}$ be the set of all places v of K which are real (when K is a number field) or for which F_v is a skew field. Since a is a definite vector, for every $v \in S$ there exists an $n_v \in \mathrm{SU}'(V_v, h_v)$ such that $n_n a = z'$. Take $n \in N$ close to n_n for all $v \in \tilde{S}$. Now h_1 represents δ for every $v \in \tilde{S}$. In fact h_1 will also represent δ for any $v \notin \tilde{S}$. This follows from our discussions in (2.1) and the following assertion [16: 6.4.7]: Let φ and ψ be quadratic forms over K_v with $\dim \varphi \geqslant 4$, $\dim \psi = 2$, and $\det \varphi \neq -\det \psi$ if $\dim \varphi = 4$. Then φ represents ψ.

Thus by the local global principle h_1 represents δ globally. This proves the proposition.

Now we are able to complete the second proof for D_n, $n > 4$. By Proposition (2.9) $\mathrm{SU}'(V, h)$ is generated by commutators $[\tau_{a,\alpha}, \tau_{b,\beta}]$, where a and b are definite vectors. Without loss of generality we can assume that a and b are linearly independent, since otherwise, $[\tau_{a,\alpha}, \tau_{b,\beta}] = 1$. According to the above proposition there exists an $n \in N$ such that $\tau_{a,\alpha} b = nb = c$. We have $[\tau_{a,\alpha}, \tau_{b,\beta}] = \tau_{c,\beta} \tau^{-1 b,\beta} = h \tau_{b,\beta} h^{-1} \tau_{b,\beta}^{-1}$. Since N can be assumed to be normal in $\mathrm{SU}(V, h)$ we obtain that $N = \mathrm{SU}'(V, h)$. The proof is completed.

§ 3. Proof of the theorem for groups of type C_n, F_n, G_2

(3.1). First we shall consider the case when G is of type C_n ($n \geqslant 2$), i.e., $G(K) = \mathrm{SU}(V, h)$. It is well known that $G(K)$ is generated by unitary reflections and that every reflection is contained in a subgroup $H(K) \subset G(K)$, where H

is a simple algebraic K-group of type $C_2 = B_2$. By [7] $H(K)/Z(H(K))$ is a simple group and by [8] every infinite normal subgroup N of $G(K)$ has a finite index in $G(K)$. Therefore in order to prove the simplicity for $G(K)$, it is enough to show that $Z(H(K)) \subset [H(K), H(K)]$. The group $H(K)$ is isomorphic to $\mathrm{Spin}(W, f)$, where (W, f) is a 5-dimensional quadratic anisotropic space over K. If $W_0 \subset W$ is a 3-dimensional subspace of W then, naturally, $\mathrm{Spin}(W_0) \subset \mathrm{Spin}(W)$ and $-1 \in \mathrm{Spin}(W_0)$. (Here 1 denotes the unit element of the Clifford algebra of W.) Therefore it is enough to find a 3-dimensional subspace W_0 such that $-1 \in [\mathrm{Spin}(W_0), \mathrm{Spin}(W_0)]$. Recall that $\mathrm{Spin}(W_0)$ is isomorphic to the group D^1 of the elements with reduced norm 1 of a certain skew quaternion field D over K. It is proved in [14] that $[D^1, D^1] = D^1 \cap \prod_{v \in R} [D_v^1, D_v^1]$ where $R = \{v \in \Sigma_f \mid D_v = D \otimes_K K_v$ is a division algebra$\}$. Let $v \in R$ and $P_v = \{x \in D_v \mid v(x) > 0\}$. (Here, we identify the place $v \in \Sigma_f$ with its order.) It is known [15] that $[D_v^1, D_v^1] = (1 + P_v) \cap D_v^1$, $v \in R$. Therefore, $-1 \in [\mathrm{Spin}(W_0), \mathrm{Spin}(W_0)]$ if and only if W is K_v-isotropic for every $v \in \Sigma_f$ with $v(2) = 0$. So, the following lemma implies that the group $G(K)/Z(G(K))$ is simple.

(3.2). LEMMA. *Let (W, f) be a 5-dimensional quadratic anisotropic space over K. Then W contains a 3-dimensional subspace W_0 which is K_v-isotropic for every $v \in \Sigma_f$ with $v(2) = 0$.*

Proof. Let $T_0 = \{v \in \Sigma_\infty \mid f$ is anisotropic over $k_v\}$ and let v_0 be a nonarchimedean place over 2, i.e., $v_0(2) > 0$. We set $T = T_0$ if T_0 contains even number of places and we set $T = T_0 \cup \{v_0\}$, otherwise. Multiplying f by an element from $K^\times$, if necessary, we can assume that (a) $f = g \perp g'$, where $g = x^2 - ax^2$, (b) g is anisotropic over K_{v_0} and (c) if v is real and f is K_v-isotropic then g' is also K_v-isotropic. Set $T_1 = T \cup \{v \in \Sigma_f \mid g'$ is anisotropic over $K_v\}$. We fix a $d \in K^\times$ such that if $x \not\equiv d \pmod{K_v^{\times 2}}$ for every $v \in T_1 \cap \Sigma_f$ then g' represents x over K_v for every $v \in T_1 \cap \Sigma_f$. Let $(\ ,\ )_v$, where $v \in \Sigma$, be the usual Hilbert symbol [18]. Since the quotient $K_v^\times / K_v^{\times 2}$ has at least four elements, it follows that there are a, $b_0 \in K$ with $(a, b_0)_v = -1$ if $v \in T_1 - T$, and $-b \not\equiv d \pmod{K_v^{\times 2}}$ if $v \in T_1 \cap \Sigma_f$. Fix a $c \in K^\times$ such that $\prod_{v \in T_1} (b_0, c)_v = 1$ and $(b_0, c)_v = (-d, c)_v$ for $v \in T_1 \cap \Sigma_f$. We set $\varepsilon_v = (a, b_0)_v$ if $v \in T_1$ and $\varepsilon_v = 1$ if $v \notin T_1$. We also set $\eta_v = (b_0, c)_v$ if $v \in T_1$ and $\eta_v = 1$ if $v \notin T_1$. Consider the system of equations $(a, x)_v = \varepsilon_v$ and $(x, c)_v = \eta_v$ to hold simultaneously for all $v \in \Sigma$. Since the system has a local solution (in $K_v^\times$) for every $v \in \Sigma$, it has also a global solution $b \in K$ [18]. It is easy to see that $(b, c)_v = (b_0, c)_v \neq (-d, c)_v$ when $v \in T_1 \cap \Sigma_f$. This yields readily that g' represents $-b$ over K_v for every $v \in \Sigma$. By the Hasse–Minkowski theorem $-b$ is represented by g' over K, i.e., f contains a quadratic form $x^2 - ay^2 - bz^2$. Since $(a, b)_v = 1$ for $v \notin T$, we get that $x^2 - ay^2 - bz^2$ is isotropic over K_v for each $v \in \Sigma_f$ with $v(2) = 0$. The lemma is proved.

Remark. The idea of using the result [14] in proving the simplicity for the groups of type C_n was pointed out by A. S. Rapinchuk.

(3.3) Groups of type F_4. Let $J = J(C_3, \Gamma)$ be a reduced exceptional central simple algebra, where C is a Cayley algebra over K and $\Gamma = \operatorname{diag}(\gamma_1, \gamma_2, \gamma_3)$, $\gamma_i \in K^\times$ [1]. Denote by $G(K)$ the group of automorphisms of $J(C_3, \Gamma)$. We have to prove that $G(K)$ is a simple group. Our main sources of references will be the works [1] and [6].

Let $e \in J(C_3, \Gamma)$ be a primitive idempotent. For a $g \in G(K)$ we set $e' = ge$. According to the result [1: Theorem 1] we may and we will assume that e and e' have coordinates in a quadratic subfield L of C. In other words, we assume that $e, e' \in J(L_3, \Gamma)$, where $J(L_3, \Gamma)$ is a special Jordan algebra consisting of the matrices $A \in M_3(L)$ invariant under the involution $A \to \Gamma^{-1} \bar{A}' \Gamma$ (see (1.1)). Let (L^3, h) be a 3-dimensional hermitian space over L with a matrix of scalar products Γ. Then $J(L_3, \Gamma)$ can be identified with the set of all h-hermitian linear transformations of L^3 (i.e., the set of all $A \in \operatorname{End}_L(L^3)$ with $h(x, Ay) = h(Ax, y)$ for any $x, y \in L^3$). We fix two orthogonal bases of L^3 $\{w_0, w_1, w_2\}$ and $\{w'_0, w'_1, w'_2\}$ such that $w_0 = w'_0$, $w \in \operatorname{Im}(e)$ and $w' \in \operatorname{Im}(e')$. Since the hermitian subspace spanned by $\operatorname{Im}(e)$ and $\operatorname{Im}(e')$ is nondegenerate, the choice of such bases is possible. Set $r_i = h(w_i, w_i)$ and $r'_i = h(w'_i, w'_i)$, $i = 0, 1, 2$. The following isomorphisms hold: $J(C_3, \Gamma) \cong J(C_3, R)$ and $J(C_3, \Gamma) \cong J(C_3, R')$, where $R = \operatorname{diag}(r_0, r_1, r_2)$ and $R' = \operatorname{diag}(r'_0, r'_1, r'_2)$, respectively [1: Theorem 5]. In view of these isomorphisms, we can fix a set of orthogonal idempotents E_0, E_1, E_2 (resp. E'_0, E'_1, E'_2) in $J(C_3, \Gamma)$ such that E_i (resp. E'_i) maps w_i on w_i and w_j, $j \neq i$, on 0 (resp. w'_i on w'_i and w'_j, $j \neq i$, on 0). Now let $N^\times = N(C^\times)$. (Recall that N denotes the norm map on the Cayley algebra C.) Note that $E_1 = e$, $E'_1 = e'$, and $E_0 = E'_0$. Since E_1 and E'_1 are conjugate, we get $r_0^{-1} r_2 \equiv r'^{-1} r'_2 \pmod{N^\times}$ [6: Corollary, p. 381]. On the other hand, the equality $E_0 = E'_0$ implies $r_1^{-1} r_2 \equiv r_1'^{-1} r'_2 \pmod{N^\times}$. Therefore, the ordered sets of idempotents (E_0, E_1, E_2) are conjugate [1: Theorem 9], i.e., there exists a $\sigma \in G(G)$ such that $\sigma E_i = E'_i$ for $i = 0, 1, 2$. Our main conclusion from the above considerations is that $(\sigma^{-1} g)(e) = e$.

It is well known that the subgroup of elements of $G(K)$ fixing a given primitive idempotent is isomorphic to $\operatorname{Spin}_9(f)$, where f is a regular quadratic form over K [6: Theorem 4, p. 376]. In fact, it is proved in (3.1) and (3.2) that if H is a simply connected K-algebraic group of type B_n ($n \geqslant 2$) then $H(K)$ does not contain proper infinite normal subgroups. On the other hand, every infinite normal subgroup P in $G(K)$ has a finite index in $G(K)$ [8]. Since $\operatorname{Spin}_g(f)$ is a group of type B_4, it follows that $\sigma, \sigma^{-1} g \in P$. In particular, $g \in P$, i.e., $G(K) = P$. Therefore $G(K)$ is a simple group.

(3.4) Groups of type G_2. Let C be a division Cayley algebra over K and $G(K)$ be the group of automorphisms of C. Let N be the norm on C. The bilinear form associated to N is given by

$$(x, y) = \tfrac{1}{2}[N(x+y) - N(x) - N(y)], \qquad x, y \in C.$$

Denote by C_0 the orthogonal complement of 1 (= the unity of C) in C. It is easy to see that $C_0 = \{x \in C | x^* = -x\}$ (see [5]). Let us fix an $a \in C_0$, $a \neq 0$, and $g \in G(K)$. It is well known that a and ga are contained in a quaternion division algebra $F \subset C$ [5]. (In fact, if $a \neq ga$ then the subalgebra generated by a and ga coincides with F.) Let $i \in F^{\perp} \cap C_0$, $i \neq 0$. Set $L = K(i)$. Then $L^{\perp}$ is a 3-dimensional vector space over L. For any $x, y \in L^{\perp}$ we set $h(x, y) = (x, y) + \mu^{-1} i(ix, y)$, where $\mu = i^2$. It is proved in [5], p. 70, that h is a nondegenerate hermitian form on L and that the subgroup $H(K) = \{\sigma \in G(K) | \sigma i = i\}$ is canonically isomorphic to $\mathrm{SU}(L^{\perp}, h)$. It is easy to see that $h(a, a) = h(ga, ga)$. By the Witt theorem there exists a $\sigma \in H(K)$ with $\sigma ga = a$. Note that the group $P(K) = \{\tau \in G(K) | \tau a = a\}$ is isomorphic to $\mathrm{SU}(L_1^{\perp}, h_1)$, where $L_1 = K(a)$ and h_1 is an hermitian form on $L_1^{\perp}$ [5]. By virtue of [2] the group $H(K)$ (resp. $P(K)$) does not contain proper infinite normal subgroups. Therefore if N is an infinite normal subgroup in $G(K)$ then N contains the subgroups $H(K)$ and $P(K)$. (Recall that by [8] N has a finite index in $G(K)$.) This implies that σ, $\sigma g \in N$. In particular, $g \in N$, i.e., $N = G(K)$. The theorem is proved.

Remark. We refer to the work [5, Theorem 8] for another proof of the simplicity for groups of type G_2 modulo the result [2] that every unimodular unitary group in a 3-dimensional vector space over a quadratic extension of K is projectively simple.

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