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MODULES DEFINED BY GENERIC SYMMETRIC AND ALTERNATING MAPS

WINFRIED BRUNS and UDO VETTER

Universität Osnabrück, Abt. Vechta
Fachbereich Naturwissenschaften, Mathematik, Vechta, F.R.G.

The object of this note is to prove a conjecture of the first author ([Br], p. 192) concerning the perfection of modules defined by generic symmetric or alternating maps of a given rank.

Let B be a commutative noetherian ring, X_{ij}, $1 \leqslant i \leqslant j \leqslant n$, a family of indeterminates, $X_{ji} = X_{ij}$, and X the symmetric matrix (X_{ij}). For an integer r, $0 \leqslant r \leqslant n$, put $R_r = B[X]/I_{r+1}(X)$, $I_{r+1}(X)$ denoting the ideal generated by the $(r+1)$-minors of X. Then the residue class x of the matrix X defines an R_r-homomorphism $x\colon F \to F^*$, $F = (R_r)^n$, which for obvious reasons may be called the *generic symmetric map of rank* r over B. The modules we are interested in are $M = \operatorname{Coker} x$, $M^* = \operatorname{Ker} x$, and $\operatorname{Im} x$.

Analogously a *generic alternating map* is defined with respect to an alternating matrix X of indeterminates, $X_{ij} = -X_{ji}$, $X_{ii} = 0$. The necessarily even rank r is now fixed by the vanishing of the $(r+2)$-pfaffians of X: one considers $R_r = B[X]/\mathrm{Pf}_{r+2}(X)$ and the map $x\colon F \to F^*$ as above.

The results of [Br] for generic $m \times n$ matrices were proved by the method of principal radical systems and a duality argument. In [BV], Section 13 the former has been replaced by filtrations and depth bounds based on the straightening law of the underlying ring. In the alternating case one again has an algebra with straightening law, cf. [DEP], and, mutatis mutandis, the arguments of [BV] go through. In the symmetric case we will draw upon Kutz's paper [Ku]. In both cases local duality will be needed, too. Its application here is easier than in [Br] and [BV], since $B[X]/\mathrm{Pf}_{r+2}(X)$ is Gorenstein (over a Gorenstein B), cf. [KL], and $B[X]/I_{r+1}(X)$, X symmetric, has a very simple canonical module, if it is not Gorenstein [Go].

This paper is in final form and no version of it will be submitted for publication elsewhere.

For general commutative algebra we refer the reader to [Ma]. Our notations follow [Ma] rather closely.

1. Symmetric generic maps

We use notations as introduced above. The case $r = 0$ being without interest, we will assume $r > 0$ throughout.

THEOREM 1. *Let x: $F \to F^*$, rank $F = n$, be the generic symmetric map of rank r, $0 < r \leqslant n$, over B.*

(a) *If $n = r$ or $n \equiv r+1$ (2), then $M = \operatorname{Coker} x$, its R_r-dual $M^* = \operatorname{Ker} x$, and the self-dual module $\operatorname{Im} x$ are perfect $B[X]$-modules.*

(b) *If $r < n$ and $n \not\equiv r+1$ (2), then M is not perfect, but almost perfect: $\operatorname{pd} M = \operatorname{grade} M + 1$. M^* and $\operatorname{Im} x$ are perfect.*

It is of course by no means accidental that the cases (a) and (b) correspond to the ones in which, over a Gorenstein ring B, R_r is Gorenstein resp. non-Gorenstein.

Note first that one may assume $r < n$. In fact, the statements on M^* and $\operatorname{Im} x$ are trivial for $r = n$, and M is annihilated by $\det X$. Therefore M doesn't change if we replace $R_n = B[X]$ by $R_{n-1} = B[X]/(\det X)$.

Next we want to reduce the theorem to the proposition below. This reduction works in the same way as in [Br] or [BV]. For the convenience of the reader we give a complete, though brief treatment.

PROPOSITION. *Let $B = \mathbf{Z}$ and $0 < r < n$.*

(a) *If $n \equiv r+1$ (2), then M is a maximal Cohen–Macaulay module over R_r.*

(b) *If $n \not\equiv r+1$ (2), then $\operatorname{depth} M_p \geqslant \operatorname{depth} (R_r)_p - 1$ for all $p \in \operatorname{Spec} R_r$.*

(c) *If $n \not\equiv r+1$ (2), then $\operatorname{depth} M_m < \operatorname{depth} (R_r)_m$ for $m = I_r(X)$.*

(*Note that R_r is a Cohen–Macaulay ring* [Ku].)

Let $B = \mathbf{Z}$, $R = R_r$. We have an exact sequence

$$\mathscr{C}\colon\ 0 \to M^* \to F \xrightarrow{x} F^* \to M \to 0.$$

For all prime ideals p of R, $p \not\supset I_r(x)$, the R_p-module M_p is free. Since (by [Ku])

$$\operatorname{grade} I_r(x) = \operatorname{grade} I_r(X) - \operatorname{grade} I_{r+1}(X) = n - r + 1 \geqslant 2,$$

it follows easily in both cases (a) and (b) that M is a torsionfree R-module, so torsionsfree over $\mathbf{Z}$. Thus $\mathscr{C} \otimes_{\mathbf{Z}} B$ is exact for all rings B, and we see that the formation of $\operatorname{Im} x$ and $\operatorname{Ker} x$ commutes with the extension from $\mathbf{Z}$ to B. Using standard results about generic perfection (cf. [BV], Section 3 for example) one now concludes the theorem's assertions on perfection of modules from those in the special case $B = \mathbf{Z}$ in which they are covered by the proposition:

By Kutz's result R, hence F and F^*, are perfect $B[X]$-modules. In case (a) the proposition implies that M is a perfect $\mathbf{Z}[X]$-module of the same grade, and this in turn yields the perfection of $\operatorname{Im} x$ and M^*. In case (b) one obtains at least that $\operatorname{Im} x$ and, hence, M^* are perfect. Conversely, the perfection of $\operatorname{Im} x$ for general B yields that $\operatorname{pd} M \leqslant \operatorname{grade} M + 1$.

In order to prove equality in the last inequality under the assumption of part (b) of the theorem, let p be a prime ideal of R, $p \supset I_r(x)$, and q its preimage in $\mathbf{Z}[X]$ (under the natural extension $\mathbf{Z}[X] \to B[X]$). Let $\mathscr{F}$ be a $\mathbf{Z}[X]$-free resolution of M of length $\operatorname{pd} M$. Parts (b) and (c) of the proposition imply that $\operatorname{pd} M_q = \operatorname{grade} M + 1$, and that $\mathscr{F}_q$ is a free $\mathbf{Z}[X]_q$-resolution of length $\operatorname{pd} M_q$. Considering the extension $\mathbf{Z}[X] \to B[X] \to B[X]_p$, one sees that $\mathscr{F} \otimes_{\mathbf{Z}[X]} B[X]_p$ is a free resolution of M_p. Since the extension $\mathbf{Z}[X]_q \to B[X]_p$ is local, the length of $\mathscr{F} \otimes_{\mathbf{Z}[X]} B[X]_p$ must be the projective dimension of M_p.

We now introduce a standard induction argument which will be useful for all three parts of the proposition. B being arbitrary, take any prime ideal $p \not\supset I_1(x)$ in R_r. Then there is (i) an element $x_{ii} \notin p$ or (ii) a 2-minor $x_{ii}x_{jj} - (x_{ij})^2 \notin p$, by symmetry $x_{11} \notin p$ or $x_{11}x_{22} - (x_{12})^2 \notin p$. Over $B[X][X_{11}^{-1}]$ one performs elementary row and column transformations to obtain

$$\begin{bmatrix} X_{11} & 0 & & 0 \\ 0 & Y_{11} & \cdots & Y_{1,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & Y_{1,n-1} & \cdots & Y_{n-1,n-1} \end{bmatrix},$$

$Y_{ij} = Y_{ji} = X_{i+1,j+1}X_{11} - X_{1,i+1}X_{1,j+1}$. It is easy to see that the elements Y_{ij}, $1 \leqslant i \leqslant j \leqslant n$, are algebraically independent over B and that $R_r[x_{11}^{-1}]$ is a Laurent polynomial extension of $B[Y]/I_r(Y)$. A similar argument works in case (ii), now reducing both n and r by 2, cf. [Jo].

LEMMA 1. *There are families Y_{ij}, $1 \leqslant i \leqslant j \leqslant n-1$, and Z_{ij}, $1 \leqslant i \leqslant j \leqslant n-2$, of algebraically independent elements over B such that*

(a) *$R_r[x_{11}^{-1}]$ is a Laurent polynomial extension of $S_{r-1} = B[Y]/I_r(Y)$, and*

(b) *$R_r[(x_{11}x_{22} - x_{12}^2)^{-1}]$ is a Laurent polynomial extension of $T_{r-2} = B[Z]/I_{r-1}(Z)$.*

In both cases M is the extension of the modules defined by Y and Z resp.

The easiest part of the proposition is (c). Lemma 1 reduces its proof to the case $r = 1$. Since $(\mathbf{Z} \setminus \{0\}) \cap m = \emptyset$, we may then work over $\mathbf{Q}$. Now m is the irrelevant maximal ideal of the graded $\mathbf{Q}$-algebra $R = R_1$. By the local duality theorem it is enough to prove that $\operatorname{Ext}_R^1(M, \omega_R) \neq 0$. This is obvious since ω_R is generated by the entries of the first row (or column) of x, cf. [Go].

A crucial step in the proof of both (a) and (b) is a weaker depth bound:

LEMMA 2. *Let $B = \mathbf{Z}$ (or any Cohen–Macaulay ring), and $\mathfrak{p} \subset R = R_r$ a prime ideal, $\mathfrak{p} \supset I_1(x)$. Then*

$$\operatorname{depth} M_{\mathfrak{p}} \geqslant \operatorname{depth} R_{\mathfrak{p}} - r \geqslant \tfrac{1}{2} \operatorname{depth} R_{\mathfrak{p}}.$$

Proof. By [Ku]

$$\operatorname{depth} R_{\mathfrak{p}} \geqslant nr - r(r-1)/2,$$

implying the second inequality.

In order to prove the first inequality we put $M_i = \sum_{j=i+1}^{n} R\bar{e}_j$, $\bar{e}_j$ denoting the residue class in M of the jth canonical basis element of F^*. One has a filtration

$$M = M_0 \supset M_1 \supset \ldots \supset M_r.$$

We claim: (i) M_r is a free R-module.

(ii) The annihilator J_i of M/M_i is the ideal generated by the i-minors of the first i columns of x.

(iii) The generator $\bar{e}_i$ of M_{i-1}/M_i is linearly independent over R/J_i.

It follows from these claims that M has a filtration with quotients R and R/J_i, $i = 1, \ldots, r$. By [Ku] all these rings are Cohen–Macaulay, and $\dim R/J_i = \dim R + i - r - 1$.

Claim (i) is clear: $\operatorname{rank} x = r$, and the first r columns are linearly independent, hence $\operatorname{rank} M/M_r = 0 = \operatorname{rank} M - (n-r)$. Since M/M_i is represented by the matrix $(x|i)$ consisting of the first i columns of x, $\operatorname{Ann} M/M_i \supset J_i$. On the other hand the first $i-1$ columns of $(x|i)$ are linearly independent over R/J_i (again by [Ku]), and by the same argument as used for (i) one concludes (iii) and (ii). ∎

Now we can already prove part (a) of the proposition under whose hypotheses $R = R_r$ is a Gorenstein ring. Let $\mathfrak{p} \subset R$ be a prime ideal. Arguing inductively one may suppose that $M_{\mathfrak{q}}$ is a maximal Cohen–Macaulay module for all primes $\mathfrak{q}$ strictly contained in $\mathfrak{p}$, and, by Lemma 1, that $\mathfrak{p} \supset I_1(x)$. Let $D = \operatorname{Coker} x^*$ be the Auslander–Bridger dual of M. Of course $D \cong M$. Lemma 2 and the assumptions so far imply that $M_{\mathfrak{p}}$ is a dth syzygy module, $d = \operatorname{depth} M_{\mathfrak{p}}$, hence

$$\operatorname{Ext}^i_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}, R_{\mathfrak{p}}) = \operatorname{Ext}^i_{R_{\mathfrak{p}}}(D_{\mathfrak{p}}, R_{\mathfrak{p}}) = 0 \quad \text{for } i = 1, \ldots, d,$$

(cf. [BV], Section 16 for example). On the other hand $\operatorname{depth} M_{\mathfrak{p}} \geqslant d$ is equivalent to

$$\operatorname{Ext}^i_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}, R_{\mathfrak{p}}) = 0 \quad \text{for } i = \operatorname{depth} R_{\mathfrak{p}} - d + 1, \ldots, \operatorname{depth} R_{\mathfrak{p}}$$

by local duality. Hence $\operatorname{Ext}^i_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}, R_{\mathfrak{p}}) = 0$ for all $i > 0$, and $M_{\mathfrak{p}}$ is a maximal Cohen–Macaulay module.

Part (b) finally follows by induction, the numerical information supplied above, and:

LEMMA 3. *Let* $B = \mathbf{Z}$.

(a) *As an* (R/J_r)*-module* M/M_r *is reflexive.*

(b) *Its dual over* R/J_r *is isomorphic to* J_{r-1}/J_r.

(c) M/M_r *is a maximal Cohen–Macaulay module over* R/J_r.

(*In order to include the case* $r = 1$: *A* 0*-minor has the value* 1.)

Proof. To simplify the notation write $\bar{R}$ for R/J_r and $\bar{M}$ for M/M_r. Let us first observe that (c) holds in case $n \equiv r+1$ (2) since, as has just been proved, M is a maximal Cohen–Macaulay module over R.

Next one notices that the case $r = 1$ is indeed trivial, M/M_1 being free of rank 1 over R/J_1. Suppose that $r > 1$ and proceed by induction. Then, via M and Lemma 1, it follows that $\bar{M}_p$ is a maximal Cohen–Macaulay module over $\bar{R}_p$ for all $p \in \operatorname{Spec} \bar{R}$, $p \not\supset I_1(x)/J_r$.

For (a) it is enough to show that (i) $\bar{M}_p$ is free for all primes p such that $\operatorname{depth} R_p \leqslant 1$, and (ii) $\operatorname{depth} \bar{M}_p \geqslant 2$ for the remaining ones. (i) is clear: $\operatorname{grade} I_{r-1}(x|r)/J_r \geqslant 2$, and $\bar{M}_p$ is free if $p \not\supset I_{r-1}(x|r)/J_r$. In order to verify (ii) one may now assume that $n \geqslant r+2$, $r > 1$, and $p \supset I_1(x)/J_r$. Then Lemma 2 implies (ii).

The dual of $\bar{M}$ is isomorphic to the kernel of the map $\bar{R}^r \to \bar{R}^n$ defined by the transpose y of $(x|r)$. Taking the determinantal relations of the rows of y, one sees that J_{r-1}/J_r is embedded in $\operatorname{Ker} y$ such that this embedding splits at all prime ideals not containing $I_{r-1}(x|r)/J_r$, in particular at all primes p such that $\operatorname{depth} R_p \leqslant 1$. Since J_{r-1}/J_r is a maximal Cohen–Macaulay module over $\bar{R}$, (b) follows easily.

It remains to prove (c) for $n \not\equiv r+1$ (2). In this case J_r is the canonical module of R, so $\bar{R} = R/J_r$ is a Gorenstein ring, cf. [HK], 6.13. By (b), the dual of $\bar{M}$ is Cohen–Macaulay, so is $\bar{M}$ by (a). ∎

Once proved over $\mathbf{Z}$, Lemma 3 holds over any Cohen–Macaulay ring B, in fact over arbitrary B if one replaces the Cohen–Macaulay property by perfection.

2. Generic alternating maps

Let X be an alternating $n \times n$ matrix of indeterminates over B and $R_r = B[X]/\operatorname{Pf}_{r+2}(X)$, $r \geqslant 0$ an even integer.

THEOREM 2. *Let* x: $F \to F^*$, $\operatorname{rank} F = n$, *be the generic alternating map of rank* r *over* B. *Suppose that* $r < n-1$ *if* n *is odd. Then* $M = \operatorname{Coker} x$, $M^* = \operatorname{Ker} x$, *and the self-dual module* $\operatorname{Im} x$ *are perfect* $B[X]$*-modules.*

The case $r = n-1$, n odd, is a true exception: $\operatorname{Coker} x \cong \operatorname{Pf}_r(X)$ is an ideal of projective dimension 2 [BE].

The proof of Theorem 2 is reduced to the case $B = \mathbf{Z}$ and $r+2 \leqslant n$ by the same arguments as above. It is enough to show that $\operatorname{depth} M_p \geqslant \operatorname{depth} R_p - 1$

for all $\mathfrak{p} \in \operatorname{Spec} R_r$, $r = \operatorname{rank} x$. Together with the fact that $M_\mathfrak{p}$ is free for $\mathfrak{p} \not\supset \operatorname{Pf}_r(x)$, $\operatorname{grade} \operatorname{Pf}_r(x) = 2(n-r)+1 \geqslant 5$, this implies that M is reflexive and M^*, the second syzygy of M, is Cohen–Macaulay, whence $M = M^{**}$ is Cohen–Macaulay, too. After all, R_r is a Gorenstein ring (over any Gorenstein B), cf. [KL].

For indices $i_1, \ldots, i_u$ we denote (the residue class of) the pfaffian of the $u \times u$ matrix $(X_{i_v i_w}\colon 1 \leqslant v, w \leqslant u)$ by $[i_1, \ldots, i_u]$. Note that $[i_1, \ldots, i_u] = 0$ if $i_v = i_w$ for some $v, w, v \neq w$. One has the expansions

$$[i_1, \ldots, i_u] = \sum_{j=1}^{u} (-1)^{v+j} \sigma(j, v) [i_1, \ldots, \hat{i}_j, \ldots, \hat{i}_v, \ldots, i_u] [i_j, i_v]$$
$$= \sum_{j=1}^{u} (-1)^{v+j} \sigma(v, j) [i_1, \ldots, \hat{i}_j, \ldots, \hat{i}_v, \ldots, i_u] [i_v, i_j]$$

along a row or column resp., $\sigma(j, v)$ denoting the sign of $v-j$.

As above we put $M_r = \sum_{j=r+1}^{n} R_r \bar{e}_i$, $\bar{M} = M/M_r$, and, now, $\bar{R} = R_r/R_r[1, \ldots, r]$. By what has just been said, it is enough to show the following lemma:

Lemma 4. *Let $r < n$.*

(a) *M_r is a free R_r-module.*

(b) $[1, \ldots, r]\bar{M} = 0$.

(c) *As an $\bar{R}$-module $\bar{M}$ is isomorphic to the ideal $I \subset \bar{R}$ generated by the pfaffians $[1, \ldots, \hat{i}, \ldots, r+1]$, $1 \leqslant i \leqslant r$.*

(d) *Suppose that $r+2 \leqslant n$. Then I is a maximal Cohen–Macaulay module over $\bar{R}_r$.*

Proof. (a) follows as the corresponding statement in Lemma 1: $\operatorname{rank} M = n-r$ and $[1, \ldots, r | 1, \ldots, r] = [1, \ldots, r]^2$ is not a zero-divisor.

(b) follows from arguments analogous to those detailed below in the proof of (c).

For the proof of (c) and (d) we use the structure of R_r as an algebra with straightening law over B, cf. [DEP], Section 12. The underlying poset is formed by the s-pfaffians of x, $s \leqslant r$, ordered by $[a_1, \ldots, a_u] \leqslant [b_1, \ldots, b_v] \Leftrightarrow u \geqslant v$, $a_i \leqslant b_i$, $i = 1, \ldots, v$. This poset is obviously a distributive lattice, in particular it is wonderful. As one sees easily

$$\bar{R}/I \cong R_r/J,$$

J being generated by all the pfaffians in the set $\Psi = \{\pi\colon \pi \not\geqslant [1, \ldots, r-1, r+2]\}$. Ψ is a poset ideal, and by [DEP], Lemma 8.2 or [BV], (5.13) $\bar{R}/I$ is an ASL on a wonderful poset again. This implies that $\bar{R}/I$ is a Cohen–Macaulay ring. Using [DEP], Section 6 or [BV], (5.10) to determine dimensions one sees that (under the hypothesis of (d)) $\dim \bar{R}/I = \dim R_r - 2 = \dim \bar{R} - 1$. Together with the Cohen–Macaulay property of $\bar{R}$ and $\bar{R}/I$ the equation $\dim \bar{R}/I = \dim \bar{R} - 1$ implies (d).

In order to prove (c) one starts with the natural presentation

$$\bar{R}^n \xrightarrow{(x|r)} \bar{R}^r \to M \to 0.$$

Then one defines an epimorphism $\varphi\colon \bar{R}^r \to I$ by $\varphi(e_i) = (-1)^i \times [1, \ldots, \hat{i}, \ldots, r+1]$. The expansion of the pfaffian $[1, \ldots, r+1, k]$ along its $(r+2)$-th row,

$$[1, \ldots, r+1, k] = -\sum_{i=1}^{r+1} (-1)^i X_{ki}[1, \ldots, \hat{i}, \ldots, r+1]$$

together with $[1, \ldots, r+1, k] = [1, \ldots, r] = 0$ in $\bar{R}$ shows that the rows of $(x|r)$ are relations of the generators of I, the map φ factors through M. It remains to show that the kernel of φ is generated by the rows of $(x|r)$. By virtue of [BV], (5.6) it is enough to find linear combinations of the rows of $(x|r)$ of the form

$$\pi e_i - \sum_{j>i} a_j e_j, \qquad a_j \in \bar{R},$$

for all $i = 1, \ldots, r$ and all pfaffians $\pi \not\geq [1, \ldots, \hat{i}, \ldots, r+1]$. Let π be such a pfaffian. Then

$$\pi = [1, \ldots, i, b_1, \ldots, b_q], \qquad i < b_1 < \ldots < b_q, \qquad i+q \equiv 0 \ (2).$$

Let x_j be the jth row of $(x|r)$ and $[a_1, \ldots, a_{i+q}] = [1, \ldots, i, b_1, \ldots, b_q]$. The element

$$y = \sum_{j=1}^{i+q} (-1)^{i+j} \sigma(i,j) [a_1, \ldots, \hat{a}_i, \ldots, \hat{a}_j, \ldots, a_{i+q}] x_j$$

has the desired form: Its component with respect to e_k, $k \leqslant i$, is $[1, \ldots, k, \ldots, i-1, k, b_1, \ldots, b_q]$, hence 0 for $k < i$, and $[1, \ldots, i, b_1, \ldots, b_q]$ for $k = i$. ■

Remarks. (a) Both Theorems 1 and 2 can be generalized along the lines of [Br], Theorems 1 and 2, and supplemented by statements analogous to [Br], Theorems 4 and 5. Furthermore it is not difficult to show that $\operatorname{Ker} x$ is generated by the natural determinantal resp. pfaffian relations of the rows of x.

(b) Under the hypotheses of Theorem 2 let n be an odd integer and $r+2 = n-1$. As remarked above, $I = \mathrm{Pf}_{r+1}(X)$ is a (Gorenstein) ideal of grade 3, and obviously $M \cong I/I^2$. Thus Theorem 2 may be considered a generalization of Herzog's result [He], Satz 2.8 saying essentially that I/I^2 is a perfect module over $B[X]$.

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