TOPICS IN ALGEBRA
BANACH CENTER PUBLICATIONS, VOLUME 26, PART 2
PWN POLISH SCIENTIFIC PUBLISHERS
WARSAW 1990

DECOMPOSITION NUMBERS MODULO p OF CERTAIN REPRESENTATIONS OF THE GROUPS $\mathrm{SL}_n(p^k)$, $\mathrm{SU}_n(p^k)$, $\mathrm{Sp}_{2n}(p^k)$

A. ZALESSKII

Institute of Mathematics, Acad. Sci. of Belorussian S.S.R.
Minsk, U.S.S.R.

1. Introduction

Decomposition numbers describe to a great extent the connection between ordinary and modular representations of finite groups. For groups of small orders explicit calculations are possible. Nowadays a computer is often used. Some tables on decomposition numbers of special groups may be found in Parker [18]. For most important groups of infinite series one may hope to obtain some formulas for decomposition matrices. An account of the modular representation theory of symmetric and alternative groups with an emphasis on decomposition matrices is presented in James [13]. As to Chevalley groups the situation is of crucial difference depending on whether p is the characteristic of the definition field or not. Only the former case is discussed here. The first result of this kind is that of Brauer and Nesbitt [2] who calculated the decomposition matrix modulo p for $\mathrm{PSL}_2(p)$, where p is odd. Further essential results were obtained more than 20 years later. Srinivasan [22] constructed the decomposition matrices modulo p for the groups $\mathrm{SL}_2(p^k)$, $p > 2$. The case $p = 2$ was considered by Burkhardt [3]. Moreover Burkhardt determined the decomposition matrix modulo 2 for the groups $\mathrm{PSU}_3(2^k)$, see [4], and $\mathrm{Sz}(2^k)$, see [5]. For other Chevalley groups only fragmentary results are known. The detailed information may be found in Humphreys surveys [9, 10, 12]. Results of a general character [9, 14, 15] do not seem yet to yield any explicit formulas of decomposition numbers for groups of rank > 2.

The aim of this paper is to expose some recent results on explicit calculations of the decomposition numbers for certain representations of the groups $\mathrm{SL}_n(p^k)$, $\mathrm{SU}_n(p^k)$, $(n > 2)$, $\mathrm{Sp}_{2n}(p^k)$ $(n > 1, p > 2)$.

P always denotes an algebraically closed field of characteristic p. It is convenient to parametrize irreducible representations of the groups in question

This paper is in final form and no version of it will be submitted for publication elsewhere.

by their highest weights. It is well known that every irreducible representation φ of a group G is a restriction to G of a certain irreducible representation $\bar{\varphi}$ of the corresponding algebraic group $\bar{G} \supset G$. This $\bar{\varphi}$ is not determined uniquely by φ. We always choose such $\bar{\varphi}$ whose highest weight λ is not of the form $p^k \lambda'$ where λ' is dominant. Under this condition the correspondence $\varphi \to \bar{\varphi}$ is correctly defined and we shall call λ the highest weight of φ.

If ϱ is an irreducible representation of G over $\mathbf{C}$ (the field of complex numbers) and μ is an irreducible representation of G over P, then $D(\varrho, \mu)$ denotes the element of the decomposition matrix D on the position (ϱ, μ).

2. Decomposition numbers for groups $\mathrm{SL}_n(p^k)$, $n > 2$

Let $q = p^k$ and let V be a vector space of dimension n over the field $\mathbf{F}_q$, so the group $G = \mathrm{SL}_n(q)$ acts on V in the usual manner. Denote by Π the permutation representation of G on vectors of V. If F is a field, then let Π_F be the linear representation of G over F associated with Π. It is easily checked that Π_C contains 1_G with multiplicity 2 and $q-1$ other irreducible representations $\varphi_0, \varphi_1, \ldots, \varphi_{q-2}$ whose multiplicities are 1. Note that φ_i $(1 \leqslant i \leqslant q-2)$ is induced by a nonprincipal 1-dimensional representation of the parabolic subgroup $\mathscr{P}$ (the stabilizer of a line of V), and $1_{\mathscr{P}}^G = \varphi_0 \oplus 1_G$. It is known that $\dim \varphi_0 = (q^{n-1}-1)q/(q-1)$ and $\dim \varphi_i = (q^n-1)/(q-1)$ for $i = 1, \ldots, q-2$.

Let $\omega_1, \ldots, \omega_{n-1}$ be the fundamental weights of the algebraic group of type A_{n-1}. For $r \in \mathbf{Z}$, $0 \leqslant r \leqslant (p-1)n$ write $r = (p-1)l+j$ where $l, j \in \mathbf{Z}$, $0 \leqslant j < p-1$. Set $\omega(r) = (p-1-j)\omega_l + j\omega_{l+1}$ where the symbols ω_0, ω_n are interpreted as the zero weight. Let $\Omega_n = \{\omega(r) | 0 \leqslant r < (p-1)n\}$. Furthermore, let $\Omega_n^k = \{a_0 + a_1 p + \ldots + a_{k-1} p^{k-1} | a_0, \ldots, a_{k-1} \in \Omega_n\}$ so $\Omega_n = \Omega_n^1$. If $\lambda \in \Omega_n^k$, then let $\Delta(\lambda) = \{s | a_s = 0\}$. Use the symbol $N_{\mathrm{SL}}(i, \lambda)$ for the number of solutions $\{x_s\}$ with $x_s \in \{0, 1\}$ of the congruence

$$n(p-1) \sum_{s \in \Delta(\lambda)} x_s p^{s-1} + \sum_{s \notin \Delta(\lambda)} r_s p^{s-1} \equiv i \pmod{q-1}$$

where r_s is determined by the equality $a_s = \omega(r_s)$. If $\Delta(\lambda) = \emptyset$ then we put $N_{\mathrm{SL}}(i, \lambda)$ to be equal to 1 if the congruence above holds and to 0 otherwise.

2.1. THEOREM [28]. *Let λ be the highest weight of an irreducible representation μ of $G = \mathrm{SL}_n(p^k)$, $n > 2$ over P.* (i) *If $\lambda \notin \Omega_n^k$ then $D(\varphi_i, \mu) = 0$* $(i = 0, \ldots, q-2)$. (ii) *Let $\lambda \in \Omega_n^k$. Then $D(\varphi_0, 1_G) = N_{\mathrm{SL}}(0, 0) - 2$ and $D(\varphi_i, \mu)$ $= N_{\mathrm{SL}}(i, \lambda)$ unless $i = 0$, $\mu = 1_G$. Furthermore $\sum_i D(\varphi_i, \mu) = 2^t$ where $t = |\Delta(\lambda)|$ and $\mu \neq 1_G$.*

2.2. COROLLARY. *The fragment of the decomposition matrix D of the group $G = \mathrm{SL}_n(p^k)$, $n > 2$ corresponding to its q ordinary representations 1_G, $\varphi_0, \ldots, \varphi_{q-2}$ contains exactly $((p-1)n)^k$ nonzero columns.*

2.3. COROLLARY. *If* $n(p-1) \equiv 0 \pmod{q-1}$, *then* $D(\varphi_0, 1_G) = 2^k - 2$ *and the other numbers* $D(\varphi_i, \mu)$ *are equal to* 0 *or* 2^s $(0 \leqslant s \leqslant k-1)$.

2.4. COROLLARY. *If* $k = 1$, *then the decomposition numbers* $D(\varphi_i, \mu)$ *are equal to* 0 *or* 1.

3. Decomposition numbers for the groups $\mathrm{Sp}_{2m}(p^k)$, $p > 2$, $m > 1$

Set $n = 2m$, $q = p^k$. In this section we describe the fragment of the decomposition matrix of the group $H = \mathrm{Sp}_{2m}(q)$, which corresponds to its irreducible representations involved in $\Pi_{\mathbf{C}}|_H$ (see § 2). It is convenient to identify the parametrization of these representations with that of G in § 2. Set $\psi_i = \varphi_i|_H$ $(i = 0, \ldots, q-2)$.

3.1. PROPOSITION [28]. (i) ψ_i *is irreducible for* $i \neq 0$, $(q-1)/2$ *and is equivalent to* ψ_{q-i-1}. (ii) ψ_0 *is a direct sum of two irreducible representations* ψ_0', ψ_0'' *with*

$$\dim \psi_0' = q(q^m - 1)(q^{m-1} + 1)/2(q-1) \quad \text{and}$$
$$\dim \psi_0'' = q(q^m + 1)(q^{m-1} - 1)/2(q-1).$$

(iii) $\psi_{(q-1)/2}$ *is a direct sum of two irreducible representations* $\psi'_{(q-1)/2}$, $\psi''_{(q-1)/2}$ *of equal dimensions.*

Denote by Ψ_n the set of representations

$$\{1_H, \psi_0', \psi_0'', \psi_1, \ldots, \psi_{(q-3)/2}, \psi'_{(q-1)/2}, \psi''_{(q-1)/2}\}.$$

In this section we describe the fragment of the decomposition matrix whose rows correspond to Ψ_n. Note that ψ_0' is just the reflection representation introduced by Curtis, Iwahori and Kilmoyer [6]. Note also that the assertion (i) of Proposition 3.1 is proved independently by Seitz [21].

Let $\bar{\omega}_1, \ldots, \bar{\omega}_m$ be the fundamental weights of the algebraic group of type $\mathbf{C}_m$. The symbol $\bar{\omega}_0$ which appears under certain values of parameters has to be interpreted as the zero weight.

3.2. PROPOSITION. *Let* μ *be an irreducible representation of* $\mathrm{SL}_{2m}(p)$ *with* $p > 2$, $m > 1$ *over* P *and let* λ *be the highest weight of* μ. *Suppose that* $\lambda \in \Omega_{2m}$. (i) $\nu = \mu|_{\mathrm{Sp}_{2m}(p)}$ *is irreducible unless* $\lambda = (p-1)\,\omega_m$. (ii) *If* $\lambda = (p-1)\,\omega_m$, *then* ν *is a direct sum of two irreducible representations* ν_1 *and* ν_2 *whose highest weights are* $(p-1)\,\bar{\omega}_m$ *and* $\bar{\omega}_{m-2} + (p-2)\,\bar{\omega}_m$ *respectively.*

(i) follows immediately from results obtained independently by Suprunenko [24] and Seitz [20]. (ii) is established in [28].

Set $\bar{\omega}(i) = (p-1-j)\bar{\omega}_l + j\bar{\omega}_{l+1}$ where $0 \leqslant j < p-1$, $i = (p-1)l + j$, $0 \leqslant i \leqslant (p-1)m$. Let $\bar{\Omega}_m = \{\bar{\omega}_{m-2} + (p-2)\bar{\omega}_m,\ \bar{\omega}(i) | 0 \leqslant i \leqslant (p-1)m\}$. Note that if $\lambda = \omega(i)$ with $i < (p-1)m$ in Proposition 3.2, then the highest weight of $\mu|_{\mathrm{Sp}_{2m}(p)}$ is $\bar{\omega}(i)$. It follows that the highest weights of irreducible components of representations $\mu|_{\mathrm{Sp}_{2m}(p)}$ with $\lambda \in \Omega_{2m}$ belong to $\bar{\Omega}_m$. Set

$$\bar{\Omega}^k_m = \{a_0 + a_1 p + \dots + a_{k-1} p^{k-1} | a_0, \dots, a_{k-1} \in \bar{\Omega}_m\}, \quad \text{so } \bar{\Omega}^1_m = \bar{\Omega}_m.$$

3.3. Proposition. *Let μ be an irreducible representation of $\mathrm{SL}_n(p^k)$ over P where $n = 2m > 2$, q is odd. Suppose that the highest weight $\lambda = a_0 + a_1 p + \dots + a_{k-1} p^{k-1}$ of μ belongs to Ω^k_n. Then $v = \mu|_{\mathrm{Sp}_{2m}(p^k)}$ is completely reducible. The number of irreducible components of v is equal to 2^d where d is the number of s with $a_s = (p-1)\omega_m$.*

3.4. Proposition. *If $\psi \in \Psi_m$ and $\lambda \notin \bar{\Omega}^k_m$ then $D(\psi, \mu) = 0$ where λ is the highest weight of μ.*

This follows from § 2 and the definition of Ψ_m and $\bar{\Omega}^k_m$.

Let $\gamma(\bar{\omega}(i)) = i$ and $\gamma(\bar{\omega}_{m-2} + (p-2)\bar{\omega}_m) = m(p-1)$ so γ is a map $\bar{\Omega}_m \to \mathbf{N}$. If

$$\lambda = a_0 + a_1 p + \dots + a_{k-1} p^{k-1} \in \bar{\Omega}^k_m,$$

then $\bar{\Delta}(\lambda) = \{s | \gamma(a_s) \neq m(p-1)\}$. Denote by $N_{\mathrm{Sp}}(i, \lambda)$ the number of solutions $x_s \in \{1, -1\}$ of the congruence

$$\sum_{s \in \bar{\Delta}(\lambda)} x_s p^{s-1} (m(p-1) - \gamma(a_s)) \equiv i (\mathrm{mod}\, q-1).$$

If $\bar{\Delta}(\lambda) = \varnothing$ then we put $N_{\mathrm{Sp}}(i, \lambda) = 1$ provided $i = 0$ and 0 otherwise.

3.5. Theorem [28]. *Let μ be an irreducible representation of $H = \mathrm{Sp}_{2m}(p^k)$ $(p > 2, m > 1)$ over P and let λ be the highest weight of μ. Suppose that $\lambda \in \bar{\Omega}^k_m$. Then the following assertions hold:*

(i) $D(\psi'_{(q-1)/2}, \mu) = D(\psi''_{(q-1)/2}, \mu) = \frac{1}{2} N_{\mathrm{Sp}}((q-1)/2, \lambda)$.

(ii) $D(\psi_i, \mu) = N_{\mathrm{Sp}}(i, \lambda)$ *for* $0 < i < (q-1)/2$.

(iii) $D(\psi'_0, 1_H) = D(\psi''_0, 1_H) = \frac{1}{2} N_{\mathrm{Sp}}(0, 0) - 1$.

(vi) *Let* $\mu \neq 1_H$. *Then* $D(\psi'_0, \mu) = D(\psi''_0, \mu) = \frac{1}{2} N_{\mathrm{Sp}}(0, \lambda)$ *unless* $\gamma(a_s) \in \{0, m(p-1)\}$ *for all* $s = 0, \dots, k-1$ *and* $\Delta(\lambda) = \varnothing$.

(v) *Let* $\gamma(a_s) = m(p-1)$ *for all s. Then* $D(\psi'_0, \mu) + D(\psi''_0, \mu) = 1$. *Moreover* $D(\psi'_0, \mu) = 1$ *if and only if the number of* $\bar{\omega}_{m-2} + (p-1)\bar{\omega}_m$ *among* a_s $(s = 0, \dots, k-1)$ *is even.*

3.6. Corollary. *Let $H = \mathrm{Sp}_{2m}(p^k)$, $p > 2$, $m > 1$. The fragment of the decomposition matrix D of H corresponding to its $(q+7)/2$ ordinary representations Ψ_m contains exactly $(m(p-1)+2)^k$ nonzero columns.*

3.7. Corollary. *Let $k = 1$.* (i) *The decomposition numbers of the representations $\psi \in \Psi_m$ are equal to 0 or 1.* (ii) *The highest weights of irreducible components of the representation $\psi'_0 (\mathrm{mod}\, p)$ are $(p-1)\bar{\omega}_i$, $i = 1, \dots, m$.*

4. Weil representations

In the papers [27, 29] the decomposition numbers of certain representations of symplectic and unitary groups are determined. These representations are irreducible components of Weil representations. The latter were introduced by Weil [26] for classical groups over local fields. Weil [26] mentioned that the finite field case may be considered analogously. This was developed in detail by Howe [8] and Gérardin [7]. The same representations were introduced independently by Ward [25] for symplectic group and Seitz [19] for unitary group. In this section we recall the construction of Weil representation.

Let p be a prime integer and $\varepsilon \in \mathbf{C}$ a pth root of 1. Let e_{ij} be the matrix units and E_p the unit matrix. Consider the following $p \times p$ matrices: $a_1' = e_{11} + \varepsilon e_{22} + \ldots + \varepsilon^{p-1} e_{pp}$, $b_1' = e_{12} + e_{23} + \ldots + e_{p-1,p} + e_{p1}$. Set

$$a_i = E_p \otimes \ldots \otimes E_p \otimes a_1' \otimes E_p \otimes \ldots \otimes E_p,$$

$$b_i = E_p \otimes \ldots \otimes E_p \otimes b_1' \otimes E_p \otimes \ldots \otimes E_p$$

where a_1', b_1' are in ith position. Let r be the number of factors in the expressions for a_i, b_i, so a_i, b_i are $p^r \times p^r$ matrices over $\mathbf{C}$. The sign $\otimes$ denotes here the Kronecker product of matrices. Let $\mathscr{E}_r$ be the group generated by a_i, b_i $(1 \leqslant i \leqslant r)$ unless $p = 2$, and by a_i, b_i, $\sqrt{-1} \cdot E_{2^r}$ if $p = 2$. Note that $\mathscr{E}_r$ is an extraspecial group realized by its irreducible representation of degree p^r. Let $\mathscr{N}$ be the subgroup of all matrices with determinant ∓ 1 of the normalizer of $\mathscr{E}_r$ in $\mathrm{GL}(p^r, \mathbf{C})$. Let Z be the center of $\mathscr{N}$. It is well known that $\mathscr{N}/Z\mathscr{E}_r \cong \mathrm{Sp}_{2r}(p)$ and the action of $\mathscr{N}$ on $Z\mathscr{E}_r$ by conjugation induces the structure of the natural $\mathbf{F}_p - \mathrm{Sp}_{2r}(p)$-module on $Z\mathscr{E}_r/Z$. Let $\Gamma \subset \mathrm{Sp}_{2r}(p)$ be a subgroup and Γ_1 its preimage in $\mathscr{N}$. Suppose that $\Gamma_1 = Z\mathscr{E}_r \leftthreetimes \Gamma_2$ is a semi-direct product with $\Gamma_2 \cong \Gamma$. Then $\Gamma \to \Gamma_2 \subset \mathrm{GL}(p^r, \mathbf{C})$ is a linear representation which we shall call the Weil representation of Γ. Usually very special groups are taken for Γ. The following cases are of interest for us.

(1) $\Gamma \cong \mathrm{Sp}_{2r}(p)$, $p > 2$ (note that for $p = 2$ $\Gamma_1 \cong N$ is not splittable). There are two Weil representations of Γ up to equivalence depending on the choice of the isomorphism $\Gamma \to \Gamma_2$. We shall denote them by the symbols $\theta_{r,p}$ and $\bar{\theta}_{r,p}$.

(2) Let $r = kn$, $p > 2$ and $\Gamma = \mathrm{Sp}_{2n}(p^k) \subset \mathrm{Sp}_{2r}(p)$. This embedding is realized by the change of matrix elements of $\mathrm{Sp}_{2n}(p^k)$ by $k \times k$ matrices from a regular representation of the field $\mathbf{F}_{p^k}$ over $\mathbf{F}_p$. The Weil representation of Γ is independent of the choice of a basis of $\mathbf{F}_{p^k}$ over $\mathbf{F}_p$ but the restrictions $\theta_{n,q} = \theta_{r,p}|_\Gamma$ and $\bar{\theta}_{n,q} = \bar{\theta}_{r,p}|_\Gamma$ are non-equivalent representations of Γ.

(3) $\Gamma \cong \mathrm{U}_n(q) \subset \mathrm{Sp}_{2n}(q)$, $q = p^k$. Here $\mathrm{U}_n(q) \cong \mathrm{U}_n(F_{q^2})$ is the unitary group. Its embedding in $\mathrm{Sp}_{2n}(q)$ is realized in the same manner as in (2). The case $p = 2$ is not excluded since Γ_1 is splittable in this case. The Weil representation of $\mathrm{U}_n(q)$ does not depend on the choice of the basis of $\mathbf{F}_{q^2}$

over $\mathbf{F}_q$. Moreover $\theta_{n,q}|_\Gamma \cong \bar{\theta}_{n,q}|_\Gamma$. Thus there is a unique Weil representation of $\mathrm{U}_n(q)$ up to equivalence. We shall denote it by the symbol $\zeta_{n,q}$ omitting q if this implies no confusion.

(4) $\Gamma \cong \mathrm{GL}_n(q) \subset \mathrm{Sp}_{2n}(q)$. We mean here the diagonal embedding $g \to \to \operatorname{diag}(g, {}^t g^{-1}) \subset \mathrm{Sp}_{2n}(q)$ $(g \in \mathrm{GL}_n(q))$, where the group $\mathrm{Sp}_{2n}(q)$ is written by matrices with respect to a Witt basis. The case $p = 2$ is involved. The Weil representation of $\mathrm{GL}_n(q)$ is unique up to equivalence. We shall denote this by $\tau_{n,q}$.

4.1. Proposition (Ward [25], see also Gérardin [7]). *Let c be the central involution. Let W be the space of the representation $\theta_{n,q}$ or $\bar{\theta}_{n,q}$. Set $W^+ = \{w \in W | cw = w\}$, $W^- = \{w \in W | cw = -w\}$. Then W^+ and W^- are irreducible modules. Furthermore* $\dim W^+$, $\dim W^- \in (q^n \pm 1)/2$ *and* $\dim W^+$ *is odd.*

We denote by $\theta^1_{n,q}$ (resp. $\theta^2_{n,q}$) the component of dimension $(q^n - 1)/2$ (resp. $(q^n + 1)/2$) and analogously $\bar{\theta}^1_{n,q}$, $\bar{\theta}^2_{n,q}$ for $\bar{\theta}_{n,q}$.

4.2. Proposition (Seitz [19], see also Gérardin [7]). *Let c be a generator of the center of $\mathrm{U}_n(q)$. Let W be the space of $\zeta_{n,q}$. Set $W_i = \{w \in W | cw = \xi^i w\}$ where ξ is a primitive $(q+1)$-th root of 1 and $i = 0, 1, \ldots, q$. Let $\zeta^i_{n,q}$ be the restriction of $\zeta_{n,q}(\mathrm{SU}_n(q))$ on W_i. Then $\zeta^i_{n,q}$ are irreducible pairwise nonequivalent representations of the special unitary group $\mathrm{SU}_n(q)$. Furthermore* $\dim \zeta^0_{n,q} = ((q^n + q(-1)^n)/(q+1)$ *and* $\dim \zeta^i_{n,q} = ((q^n - (-1)^n)/(q+1)$ *for* $i > 0$.

5. Decomposition numbers for irreducible parts of the Weil representation of $\mathrm{Sp}_{2n}(q)$, q odd

Let $q = p^k$, $p > 2$, $\Gamma = \mathrm{Sp}_{2n}(q)$. The following assertion shows that it suffices to consider only one of two Weil representations.

5.1. Proposition [29]. *Let μ be an irreducible representation of Γ over P. Then $D(\theta^i_{n,q}, \mu) = D(\bar{\theta}^i_{n,q}, \mu)$, $i = 1, 2$.*

Let τ^1_n (resp. τ^2_n) be the irreducible representation of $\mathrm{Sp}_{2n}(p)$ over P whose highest weight is $\bar{\omega}_{n-1} + ((p-3)/2)\bar{\omega}_n$ (resp. $((p-1)/2)\omega_n$).

5.2. Proposition [27]. $\theta^i_{n,p} \pmod p \cong \tau^i_n$ $(i = 1, 2)$.

Notice that there is a minor misprint in the formulation of the main result in [27] where the weights $\bar{\omega}_{n-1} + ((p-3)/2)\bar{\omega}_n$ and $((p-2)/2)\bar{\omega}_n$ turned out to be interchanged.

5.3. Corollary. *The representations $\theta^1_{n,p}$ and $\bar{\theta}^1_{n,p}$ remain irreducible under reduction modulo an arbitrary prime p.*

Corollary 5.3 follows directly from 5.2 and Ward's results [25]. The phenomenon described in 5.3 seems to be very rare. Another example of this kind is the group $\mathcal{N}$ in § 4 for $p = 2$.

5.4. PROPOSITION [29]. *For $p = 2$ the group $\mathcal{N}\,(\mathrm{mod}\,2)$ is irreducible and isomorphic to $\mathrm{Sp}_{2r}(2)$. The highest weight of this representation of $\mathrm{Sp}_{2r}(2)$ is $\bar{\omega}_r$.*

It is obvious that $\mathcal{N}\,(\mathrm{mod}\,p)$ is irreducible for $p > 2$.

Let $\mathcal{M}_n^q$ be the set of irreducible representations of Γ which are obtained from $\{\tau_n^1, \tau_n^2\}$ with the aid of Steinberg's (or Curtis') construction. More precisely retain the notation τ_n^1, τ_n^2 for the representations of Γ which extend τ_n^1, τ_n^2 (this is possible in accordance with Steinberg's general theory, see [23], Theorem 43). Let Fr denote a Frobenius automorphism of Γ; this is induced by a Galois automorphism of the field extension $\mathbf{F}_q/\mathbf{F}_p$ and has order k. The set $\mathcal{M}_n^q$ consists of representations of the form

$$\tau_n^{i_0} \otimes \mathrm{Fr} \circ \tau_n^{i_1} \otimes \ldots \otimes \mathrm{Fr}^{k-1} \circ \tau_n^{i_{k-1}} \tag{1}$$

where $i_s \in \{1, 2\}$, $s = 0, \ldots, k-1$.

5.5. THEOREM [29]. *Let μ be as in (5.1). (i) If $\mu \notin \mathcal{M}_n^q$, then $D(\theta_{n,q}^i, \mu) = 0$ ($i = 1, 2$). (ii) If $\mu \in \mathcal{M}_n^q$ then $D(\theta_{n,q}^1, \mu) + D(\theta_{n,q}^2, \mu) = 1$. Moreover $D(\theta_{n,q}^1, \mu) = 1$ if and only if the sum $i_0 + i_1 + \ldots + i_{k-1}$ in the expression (1) for μ is odd.*

5.6. COROLLARY. *The fragment of the decomposition matrix modulo p for $\Gamma = \mathrm{Sp}_{2n}(q)$, $q = p^k$, $p > 2$, $n > 1$ corresponding to the ordinary representations $\theta_{n,q}^1$, $\theta_{n,q}^2$, $\bar{\theta}_{n,q}^1$, $\bar{\theta}_{n,q}^2$ contains exactly 2^k nonzero columns. The number of nonzero elements of every such row is equal to 2^{k-1}.*

6. Decomposition numbers for components of the Weil representation of $\mathrm{SU}_n(p^k)$, $n > 2$

Let Ω_n^k be as in § 2, $\zeta^i = \zeta_{n,q}^i$ (see § 4), $q = p^k$. For $\lambda \in \Omega_n^k$ write $\lambda = a_0 + a_1 p + \ldots + a_{k-1} p^{k-1}$ with $a_0, \ldots, a_{k-1} \in \Omega_n$. Let $\Delta(\lambda) = \{s \mid 0 \leqslant s \leqslant k-1,\ a_s = 0\}$ and $N_{\mathrm{SU}}(i, \lambda)$ be the number of solutions $x_s \in \{0, 1\}$ of the congruence

$$n(p-1) \sum_{s \in \Delta(\lambda)} x_s p^s + \sum_{s \notin \Delta(\lambda)} a_s p^s \equiv i + \frac{n(q-1)}{2} \pmod{q+1}$$

unless $p = 2$, and of the congruence

$$n \sum_{s \in \Delta(\lambda)} x_s p^s + \sum_{s \notin \Delta(\lambda)} a_s p^s \equiv i + n \pmod{q+1} \quad \text{for } p = 2.$$

If $\Delta(\lambda) = \emptyset$ we put $N_{\mathrm{SU}}(i, \lambda) = 1$, if this congruence is true, and 0 otherwise.

6.1. THEOREM [29]. *Let μ be an irreducible representation of $\mathrm{SU}_n(p^k)$, $n > 2$, over P and let λ be the highest weight of μ. (i) If $\lambda \notin \Omega_n^k$, then $D(\zeta^i, \mu) = 0$. (ii) Let $\lambda \in \Omega_n^k$. Then $D(\zeta^i, \mu) = N_{\mathrm{SU}}(i, \lambda)$. Furthermore $\sum_i D(\zeta^i, \mu) = 2^t$ where $t = |\Delta(\lambda)|$.*

6.2. Corollary. *The fragment of the decomposition matrix D modulo p corresponding to $(q+1)$ irreducible representations $\zeta^0, \zeta^1, \ldots, \zeta^q$ of* $\mathrm{SU}_n(q)$ *contains exactly $(n(p-1))^k$ nonzero columns.*

6.3. Corollary. *If $n \equiv 0 \pmod{q+1}$, then the decomposition numbers of ζ^i $(i = 0, \ldots, q)$ are 0 or 2^s $(0 \leqslant s \leqslant k-1)$.*

6.4. Corollary. *The decomposition numbers of the representations $\zeta^0, \ldots, \zeta^p$ of the group $\mathrm{SU}_n(p)$, $n > 2$, are equal to 0 or 1 except the number $D(\zeta^i, 1) = 2$ for $i = (p-1)/2$, $p > 2$, $2n \equiv 0 \pmod{p+1}$ or $p = 2$, $i = 0$, $n \equiv 0 \pmod 3$.*

7. Composition factors of modular permutation representations of $\mathrm{SL}_n(q)$ and $\mathrm{Sp}_n(q)$ associated with the action of these groups over vectors or lines of the definition space

Let V be a vector space of dimension n over $\mathbf{F}_q$ in which the group $\Gamma = \mathrm{SL}_n(q)$ or $\mathrm{Sp}_n(q)$ acts naturally (n is even for Sp). Let $P(V)$ denote the set of lines of V. Denote by Π_P and L_P the $P\Gamma$-modules associated with the action of Γ on the vectors and the lines of V respectively.

7.1. Theorem [28]. *Let $\Gamma = \mathrm{SL}_n(p^k)$, $n > 2$. Then the highest weights of the composition factors of Π_P belong to Ω_n^k. If $\lambda \in \Omega_n^k$, then the multiplicity of the representation with the highest weight λ in Π_P is 2^t, where $t = |\Delta(\lambda)|$, see § 2. The length of the composition series of Π_P is $(n(p-1)+1)^k$.*

7.2. Theorem [28]. *Let Γ be as in (7.1) and suppose that λ is the highest weight of a composition factor μ of the $P\Gamma$-module L_P. Then $\lambda \in \Omega_n^k$ and the multiplicity of μ in L_P is equal to $N_{\mathrm{SL}}(0, \lambda)$ (see § 2) if $\lambda \neq 0$, and $N_{\mathrm{SL}}(0, 0)-1$ if $\lambda = 0$.*

7.3. Theorem [28]. *Let $\Gamma = \mathrm{Sp}_n(p^k)$, $p > 2$, $n = 2m > 2$. Then the highest weights of composition factors of $\Pi_P|_\Gamma$ belong to $\bar{\Omega}_m^k$. For $\lambda \in \Omega_m^k$ the multiplicity of the representation with the highest weight λ is equal to 2^l where l is the number of the weights a_s in the expression $\lambda = a_0 + a_1 p + \ldots + a_{k-1} p^{k-1}$ which differ from $(p-1)\,\bar{\omega}_m$ and $\bar{\omega}_{m-2} + (p-2)\,\bar{\omega}_m$. The length of the composition series of $\Pi_P|_\Gamma$ is equal to $(2m(p-1)+2)^k$.*

7.4. Theorem [28]. *Let Γ be as in (7.3) and let λ be the highest weight of a composition factor μ in $L_P|_\Gamma$. Then $\lambda \in \bar{\Omega}_m^k$ and the multiplicity of μ in $L_P|_\Gamma$ is equal to $N_{\mathrm{Sp}}(0, \lambda)$ if $\lambda \neq 0$ and $N_{\mathrm{Sp}}(0, 0)-1$ for $\lambda = 0$.*

Earlier Bhattacharya [1] proved that the length of the composition series in (7.1) for $k = 1$ is equal to $1+n(p-1)$. Mortimer [17] showed that the length of the composition series in (7.2) is at least 3. Note also that Liebeck [16] described the structure of submodules of L_F for $\mathrm{Sp}_n(p^k)$ and Mortimer [17] did this for $\mathrm{SL}_n(p^k)$ provided F is an algebraically closed field of characteristic $f \neq p$.

8. Restriction of certain representations of algebraic groups of type $\mathbf{A}_n$ and $\mathbf{C}_n$ on some subgroups

In this section I expose certain results on representations of algebraic groups which were obtained in the process of the proofs of results of the previous sections. They seem to be of independent interest.

Let X be a set of dominant weights of an algebraic group Y. Then $R(X)$ denotes the set of the irreducible representations of Y whose highest weights belong to X.

8.1. THEOREM [28]. *Let* $\mathbf{A}_{n-2} \to \mathbf{A}_{n-1}$ *be the natural embedding of algebraic groups. Let* $\varphi \in R(\Omega_n)$ *be a representation of* $\mathbf{A}_{n-1}$. *Then* (i) *the restriction* $\varphi|_{\mathbf{A}_{n-2}}$ *is completely reducible*; (ii) *the composition factors of* $\varphi|_{\mathbf{A}_{n-2}}$ *belong to* $R(\Omega_{n-1})$.

8.2. THEOREM [27]. *Let* $\mathbf{C}_{m-1} \to \mathbf{C}_m$ *be the natural embedding and* $\varphi \in R(\bar{\Omega}_m)$. *Then* (i) *the restriction* $\varphi|_{\mathbf{C}_{m-1}}$ *is completely reducible and* (ii) *the composition factors of* $\varphi|_{\mathbf{C}_{m-1}}$ *belong to* $R(\bar{\Omega}_{m-1})$.

8.3. THEOREM [28]. *Let* $\mathbf{C}_m \to \mathbf{A}_{2m-1}$ *be the standard embedding and* $\varphi \in R(\Omega_{2m})$. *Then* $\varphi|_{\mathbf{C}_m}$ *is completely reducible and its factors belong to* $R(\bar{\Omega}_m)$.

For $p > 2$ let Ξ_m be the set of irreducible representations of $\mathbf{C}_m$ with the highest weights $((p-1)/2)\bar{\omega}_m$ and $\bar{\omega}_{m-1} + ((p-3)/2)\bar{\omega}_m$.

8.4. THEOREM [27]. *For* $\varphi \in \Xi_m$ *the restriction* $\varphi|_{\mathbf{C}_{m-1}}$ *is completely reducible and its composition factors belong to* Ξ_{m-1}.

8.5. THEOREM [29]. *Let* $\mathbf{A}_{n-1} \to \mathbf{C}_n$ *be a Witt embedding* (*that is* $g \to \operatorname{diag}(g, {}^t g^{-1}) \in \mathrm{Sp}_{2n}(P)$ *where g runs over* $\mathrm{SL}_n(P)$ *and t denotes the transpose. Here* $\mathrm{Sp}_{2n}(P)$ *is written in a Witt basis*). *If* $\varphi \in \Xi_n$, *then* $\varphi|_{\mathbf{A}_{n-1}}$ *is completely reducible and its factors belong to* Ω_n.

References

[1] P. Bhattacharya, *On the structure of the group ring of an elementary abelian group and a theorem of Brauer and Nesbitt*, Acad. Roy. Belg. Bull. Cl. Sci. (5) 67 (1981), 260–267.

[2] R. Brauer and C. Nesbitt, *On the modular characters of groups*, Ann. of Math. 42 (1941), 556–590.

[3] R. Burkhardt, *Die Zerlegungsmatrizen der Gruppen* PSL$(2, p^f)$, J. Algebra 40 (1976), 75–96.

[4] —, *Über die Zerlegungszahlen der unitären Gruppen* PSU$(3, 2^{2f})$, ibid. 61 (1979), 548–581.

[5] —, *Über die Zerlegungszahlen der Suzukigruppen* Sz(q), ibid. 59 (1979), 421–433.

[6] C. W. Curtis, N. Iwahori and R. Kilmoyer, *Hecke algebras and characters of parabolic type of finite groups with* (B, N)*-pairs*, Inst. Hautes Études Sci. Publ. Math. 40 (1972), 81–116.

[7] P. Gérardin, *Weil representations associated to finite fields*, J. Algebra 46 (1977), 54–101.

[8] R. E. Howe, *On the characters of Weil's representations*, Trans. Amer. Math. Soc. 177 (1973), 287–298.

[9] J. E. Humphreys, *Ordinary and Modular Representations of Chevalley Groups*, Lecture Notes in Math. 528, Springer, Berlin 1976.

[10] J. E. Humphreys, *Cartan invariants and decomposition numbers of Chevalley groups* in: The Santa Cruz conference of finite groups, Proc. Symp. Pure Math. 37, Amer. Math. Soc., Providence 1980, 347–351.
[11] —, *Ordinary and modular characters of* SL(3, p), J. Algebra 72 (1981), 8–16.
[12] —, *Cartan invariants*, Bull. London Math. Soc. 17 (1985), 1–14.
[13] G. D. James, *The Representation Theory of the Symmetric Groups*, Lecture Notes in Math. 682, Springer, Berlin 1978.
[14] J. C. Jantzen, *Zur Reduktion modulo p der Charaktere von Deligne und Lustig*, J. Algebra 70 (1981), 452–471.
[15] —, *Zur Reduktion modulo p unipotenter Charaktere endlicher Chevalley-Gruppen*, Math. Z. 181 (1982), 97–128.
[16] M. W. Liebeck, *Permutation modules of rank 3 for the groups* $\mathrm{Sp}_n(q)$ *and* $\mathrm{SO}_n(q)$, J. Algebra 92 (1985), 9–16.
[17] B. Mortimer, *The modular permutation representations of the known doubly transitive groups*, Proc. London Math. Soc. 41 (1980), 1–20.
[18] R. A. Parker, *Modular character tables*, Cambridge Univ. 1986 (preprint).
[19] G. M. Seitz, *Some representations of classical groups*, J. London Math. Soc. 10 (1975), 115–120.
[20] —, *The maximal subgroups of classical groups*, Mem. Amer. Math. Soc. vol. 67, 1987.
[21] —, *Cross characteristic embeddings of finite groups of Lie type*, Oregon Univ. (USA) 1987 (preprint).
[22] B. Srinivasan, *On the modular characters of the special linear group* SL(2, p^n), Proc. London Math. Soc. 14 (1964), 101–114.
[23] R. Steinberg, *Lectures of Chevalley Groups*, Yale Univ. 1967.
[24] I. D. Suprunenko, *On restrictions of irreducible representations of the group* SL(m, K) *on classical groups*, Inst. of Mathematics Acad. Sci. BSSR, preprint N 16 (201), 1984 (in Russian).
[25] H. N. Ward, *Representations of symplectic groups*, J. Algebra 20 (1972), 182–195.
[26] A. Weil, *Sur certains groupes d'opérateurs unitaires*, Acta Math. 111 (1964), 143–211.
[27] A. E. Zalesskii and I. D. Suprunenko, *Representations of degree* $(p^n-1)/2$ *of the symplectic group of degree* $2n$ *over a field of characteristic p*, Vestnik Akad. Nauk. BSSR, Ser. Fiz-Mat. Navuk 6 (1987), 9–15 (in Russian).
[28] —, —, *Modular permutation representations and a fragment of the decomposition matrix of symplectic and special linear group over a finite field*, Inst. of Mathematics Acad. Sci. BSSR, preprint N 16 (286), 1987 (in Russian).
[29] A. E. Zalesskii, *A fragment of the decomposition matrix of the special unitary group over a finite field*, Inst. of Mathematics Acad. Sci. BSSR, preprint N 2 (312), 1988 (in Russian).