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ON COMPLETING CONICS

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This article contains a concept of completing varieties in enumerative geometry and its application to the case of conics. A survey on the results about conics is given.

1. Introduction

Hilbert's fifteenth problem [HI] concerning the justification of Schubert's enumerative calculus was solved during the first part of our century in the sense that one has to translate the enumerative problem into an intersection problem on a suitable smooth variety Q. Intersection theory was then well developed. To apply Schubert's calculus is nothing else but working in the intersection ring of the variety Q.

But from the beginning it is not evident on which variety Q the enumerative problem can be well described as an intersection problem. In terms of the classical enumerative geometry this means whether a formula is valid with respect to a "condition" B or not. An element of the family of geometrical objects in the support variety Z, in which the enumerative problem is given, satisfies a "condition" B if the point representing the element in Q lies in a subvariety B of Q. The fulfilment of a number of conditions corresponds to lying in the intersection of the respective subvarieties. The intersection has to be proper(1). This can be reached by moving the subvariety B into an

(1) Applying the modern intersection theory of Fulton or Vogel, one gets another approach to enumerative geometry [FP], [SV].

This paper is in final form and no version of it will be submitted for publication elsewhere.

equivalent subvariety with respect to an equivalence relation $\sim$ that belongs to the intersection theory. But this movement may destroy the geometric significance of B on the support variety Z.

In general, one has a canonical group action $G \times Z \to Z$ on Z, which preserves the properties of B. It leads to an action $G \times Q \to Q$ on Q. We want to make an intersection proper by moving B by an element of G. If this is always possible we shall say that Q is *G-complete* with respect to B.

We shall now explain this concept for the case of conics. Let Z be the projective plane P^2 over an algebraically closed field k (of characteristic $\neq 2$ and 3 (²)) and let $G = \mathrm{Gl}(3)$. The projective space P^5 parametrizes the family of all conics in P^2. We consider the condition $B = B(C, 1)$ of simple contact to a given smooth conic C. P^5 is not G-complete with respect to $B(C, 1)$. This is the background for Steiner's problem. J. Steiner [ST] asked for the number of conics of the plane which have a simple contact to 5 given smooth conics in general position. "General position" here means that one can move each of the given conics by elements of G. He answered himself that the number was $6^5 = 7776$, but this is erroneous. The reason is that he, in fact, worked in the intersection ring of P^5.

The so called "complete" conics solved the problem. The variety M^5 of "complete" conics is G-complete with respect to $B(C, 1)$. It is possible to determine the intersection ring of M^5. Thus we get formulae by which we are able to compute numbers concerning enumerative problems of conics.

Already the geometers of the last century asked whether the formulae are valid in every case [HA]. In fact, M^5 is not G-complete with respect to the condition $B = B(C, 3)$ of superosculating a smooth conic C. One has to find another variety N^5, which has the property of G-completeness.

First, we shall give the exact definition of completeness and mention some general results. Afterwards we shall come back to the case of conics in detail.

2. *G*-Completeness with respect to a condition

Let k be an algebraically closed field. We consider a variety Q over k in a projective space P^N. Further, let G be a subgroup of the linear group $\mathrm{Gl}(N+1)$ over k so that Q is G-stable, the closure $\bar{G}$ of G is a variety over k and each generic point of $\bar{G}$ is in G.

We put $\dim \emptyset = -\infty$ which means that $\dim \emptyset$ is less than each integer. Let A and B be algebraic subsets ($\neq \emptyset$) of Q.

(²) Some of the results are true for any characteristic or at least for characteristic $\neq 2$. But in connection with contact conditions one has to make particular considerations [V].

DEFINITION. We call A a *G-degeneration subset* with respect to B on Q if and only if

$$\dim_{k(g)}(A \cap gB) > \dim A + \dim B - \dim Q \tag{2.1}$$

for all $g \in G$. A variety Q is said to be *G-complete* with respect to B if and only if Q has no G-degeneration subsets with respect to B.

It is easy to see that if A is a G-degeneration subset with respect to B, then B is a G-degeneration subset with respect to A. If $B \neq Q$ and $A \neq \emptyset$ and $A \subseteqq gB$ for all $g \in G$, then A is a G-degeneration subset with respect to B. In particular, if $B \neq \emptyset$ and $\bigcap_{g \in G} gB \neq \emptyset$ then $\bigcap_{g \in G} gB$ is a G-stable G-degeneration subset with respect to B. In the case of $\operatorname{codim}^Q B = 1$ the condition is also necessary: A is a G-degeneration variety with respect to B if and only if $A \subseteqq gB$ for all $g \in G$. Therefore each G-degeneration variety with respect to B is contained in $\bigcap_{g \in G} gB$ and Q is G-complete with respect to B if and only if $\bigcap_{g \in G} gB = \emptyset$. If this is not so, then Q has a G-stable G-degeneration subset.
The last assertion is true also in the general case:

THEOREM 1. *For any G-degeneration subset A with respect to B on Q there exists a G-stable G-degeneration subset T with respect to B on Q so that $A \cap T$ is a G-degeneration subset with respect to B.*

Sketch of proof (for details see [DS8]): Take a generic point g_G of $\bar{G}$. Because A is a G-degeneration subset with respect to B, there exists a component V_{g_G} of $A \cap g_G B$ so that

$$\dim_{k(g_G)} V_{g_G} > \dim A + \dim B - \dim Q. \tag{2.2}$$

Let P_V be a generic point of V_{g_G} over the algebraic closure of $k(g_G)$ and let g'_G be another generic point of $\bar{G}$, and choose all these points sufficiently independently. $P_T = g'_G P_V$ defines a variety T over k. T is G-stable. We shall see that T is a G-degeneration variety with respect to B. Because of the assumed independence, we have

$$\dim T = \dim_k g'_G P_V = \dim_k g'_G g_G + \dim_{k(g'_G g_G)} g'_G P_V - \dim_{k(g'_G P_V)} g'_G g_G, \tag{2.3}$$

$$\dim_k g'_G g_G \leqslant \dim_k g_G, \tag{2.4}$$

$$\dim_{k(P_V)} g_G = \dim_{k(P_V, g'_G)} g'_G g_G \leqslant \dim_{k(g'_G P_V)} g'_G g_G, \tag{2.5}$$

$$\dim_k g_G - \dim_{k(P_V)} g_G = \dim_k P_V - \dim_{k(g_G)} P_V. \tag{2.6}$$

Combining these we see that

$$\dim_k T \leqslant \dim_k P_V - \dim_{k(g_G)} P_V + \dim_{k(g'_G g_G)} g'_G P_V. \tag{2.7}$$

Then it follows by (2.2) that

$$\dim_{k(g'_G g_G)} g'_G P_V > \dim T + \dim B - \dim Q. \tag{2.8}$$

The element $\tilde{g}_G = g'_G g_G$ of G is also a generic point of $\bar{G}$.
Furthermore, $P_T = g'_G P_V \in T \cap \tilde{g}_G B$. Thus

$$\dim_{k(\tilde{g}_G)}(T \cap \tilde{g}_G B) > \dim T + \dim B - \dim Q. \tag{2.9}$$

Because of $P_V \in T$ and $P_V \in A \cap \tilde{g}_G B$, we also have

$$\begin{aligned} \dim_{k(\tilde{g}_G)}(A \cap T \cap \tilde{g}_G B) \geqslant \dim_{k(\tilde{g}_G)} P_V &> \dim A + \dim B - \dim Q \\ &\geqslant \dim (A \cap T) + \dim B - \dim Q. \end{aligned} \tag{2.10}$$

The assertions of the Theorem follow from (2.9) and (2.10).

COROLLARY. *Grassmannians and flag spaces are G-complete with respect to any condition if $G = \mathrm{Gl}(n)$ is the linear group of the underlying vector space.*

A way to complete a variety could be to blow up it in a smallest G-stable G-degeneration subvariety T, because one then has a possibility to control the completeness and the intersection ring under the blowing-up $\varphi: \tilde{Q} \to Q$. (Assume that T is not singular on Q, otherwise start with a desingularization.) If no component of B is in T, then the total image $\tilde{T}$ of T is not a G-degeneration subvariety with respect to $\tilde{B} = \varphi^{-1} B$ of Q.

Is it always possible to get $\tilde{Q}$ G-complete with respect to $\tilde{B}$ by a sequence of blowing-ups of this kind?

Is it possible to replace an arbitrary condition B by a codimension-one condition so that a completion with respect to it is a completion with respect to B?

We have no answers to these questions in general.

3. Contact conditions

From now on we consider the family of all conics in P^2, and we specify B to be contact conditions. Firstly, we define a condition $B(c, r)$ of r-fold contact to a branch c (of a curve) in P^2, $r \in N$. Choose a coordinate triangle in P^2 so that the support point of c is a vertex and the tangent line of c is a face, and describe c in the coordinates by power series in t. The intersection of c with a conic p leads to an equation in t. The conic p has an r-fold contact to the branch c if and only if the multiplicity of the solution $t = 0$ of that equation is greater than r.

Disregarding moving by elements of G one gets finitely many linear subspaces $B(c, r)$ of P^5 of two kinds: Either $B(c, r) \subset P_{(2)} = \{p \in P^5: rkp \leqslant 2\}$ (and we have no chance to get completeness), or there exists an ordinary branch c' and an $r' \in N$ so that $B(c, r) = B(c', r')$.

Secondly, we define the condition $B(C, r)$ of r-fold contact to an irreducible reduced curve C ([3]) in P^2 ($\deg C \geqslant 2$ if $r > 1$) to be the closure of the union of all $B(c, r)$ where c is an ordinary branch of C. $Q = P^5$ is G-complete neither with respect to $B = B(c, 1)$ nor to $B = B(C, 1)$ even if C is a line or a conic. Namely, $\dim(P_{(1)} \cap B(c, 1)) = 1$ and $\dim(P_{(1)} \cap B(C, 1)) = 2$, in particular $P_{(1)} \subset B(C, 1)$ and $\operatorname{codim} B(C, 1) = 1$. Therefore $P_{(1)} = \{p \in P^5: rkp = 1\}$ is a G-stable G-degeneration variety with respect to B. A first step to get completeness is to blow up $Q = P^5$ in $T = P_{(1)}$. Then $\tilde{Q} = M^5$ is the thoroughly investigated variety of "complete" conics [W]. We view it as a subvariety of $P^5 \times L^5$ where L^5 parametrizes the line conics. M^5 is G-complete with respect to $\tilde{B}(c, r)$ and $\tilde{B}(C, r)$, $r = 1, 2$.

Therefore, in particular, Steiner's problem is solvable on M^5 if the intersection ring $H^* M^5$ is known together with a representation of $\tilde{B}(C, 1)$ in it where C is a conic in P^2.

Let P be the cycle given in M^5 by the image of a hyperplane of P^5 and $L \sim \tilde{B}(C, 1)$ where C is a line in P^2. Then a description of $H^* M^5$ is

$$H^* M^5 \cong Q[P, L]/(2P^3 - 3P^2 L + 3PL^2 - 2L^3,\ 2PL^3 - 3P^2 L^2 + 2P^3 L), \tag{3.1}$$

and

$$\tilde{B}(C, 1) \sim 2P + 2L \quad \text{in } H^* M^5 \tag{3.2}$$

(see [DS6], [DS4], (4.31)). A calculation gives us

$$\tilde{B}^5(C, 1) \sim 3264\, P^5, \tag{3.3}$$

and this is now really the number of conics touching fixed five conics in a general position.

4. A problem similar to Steiner's

Take a regular conic C_i, $i = 1, \ldots, 5$ and let $S(C_i)$ be the closure of the union of all $B(C, 3)$ for all regular C in $B(C_i, 3)$. $S(C_i)$ is one of the so-called Halphen conditions [SE]. It is of codimension one. We called it in [DS4] a condition of "mediate superosculation" and asked in [DS5] for the number of all conics which superosculate five given smooth conics C_i, $i = 1, \ldots, 5$, in general position in this mediate way. By a calculation which was done in [DS4], (4.32), we have in $H^* M^5$

$$\tilde{S}(C_i) \sim \tilde{B}(C_i, 1) \sim 2P + 2L, \tag{4.1}$$

and the number would be the same as above in (3.3).

But this is incorrect. The Halphen locus

$$P_{(1)} L_{(1)} = \{(p, l) \in M^5 \subset P^5 \times L^5:\ rkp = rkl = 1\} \tag{4.2}$$

([3]) The definition is possible for reducible curves, too (see [DS9]).

is contained in $\tilde{S}(C_i)$ for any C_i. $P_{(1)}L_{(1)}$ is a G-stable G-degeneration subvariety of M^5 with respect to $\tilde{S}(C_i)$ and, by the way, with respect to $\tilde{B}(C, 3)$ and $\tilde{B}(c, 3)$, too. The variety M^5 of point-line conics is therefore not G-complete with respect to these conditions.

5. The variety of point-line superosculant conics

We blow up $Q = M^5$ in $T = P_{(1)}L_{(1)}$ and call $\tilde{Q} = N^5$ the variety of point-line superosculant conics. Consider first an embedding of N^5 in $P^5 \times L^5 \times S^{251}$ where S^{251} is aprojective space of dimension 251. This embedding has a geometrical interpretation in terms of mediate superosculation (see [DS7]). Start with a k-vector space V of dimension 3.

Let

$$P^2 = \operatorname{Proj} V, \qquad P^5 = \operatorname{Proj} \operatorname{Sym}^2 V, \qquad L^5 = \operatorname{Proj} \operatorname{Sym}^2 \textstyle\bigwedge^2 V.$$

If $f \in \operatorname{Sym}^2 V$, then $f\colon V \to V^*$ is a map into the dual space V^*. The *adjoint* f^* to f is defined by

$$f^*\colon \textstyle\bigwedge^2 V \to \bigwedge^2 V^*, \qquad f^*\colon (v \wedge w) \mapsto (fv \wedge fw). \tag{5.1}$$

This leads to the *adjoint map*

$$\operatorname{ad}\colon \operatorname{Sym}^2 V \to \operatorname{Sym}^2 \textstyle\bigwedge^2 V, \qquad \operatorname{ad}\colon f \mapsto f^*. \tag{5.2}$$

Then $P_{(1)} = \operatorname{Proj} \operatorname{Ker} \operatorname{ad}$, $U = P^5 \backslash P_{(1)}$, $\varphi_1^{-1} = (\mathrm{id}, \operatorname{ad})\colon U \to P^5 \times L^5$, and the closure $\overline{\varphi_1^{-1} U}$ is the variety M^5 of point-line conics. Next define

$$\begin{aligned} d\colon & \operatorname{Sym}^2 V \times \operatorname{Sym}^2 V^* \\ & \to \operatorname{Sym}^2 \operatorname{Sym}^2 V^* \otimes \operatorname{Sym}^2 \textstyle\bigwedge^2 V \oplus \operatorname{Sym}^2 \operatorname{Sym}^2 V \otimes \operatorname{Sym}^2 \textstyle\bigwedge^2 V^* = W \end{aligned} \tag{5.3}$$

by

$$d\colon (f, f^*) \mapsto \operatorname{seg}(\operatorname{ver} f^*, \operatorname{ad} f) + \operatorname{seg}(\operatorname{ver} f, \operatorname{ad} f^*) \tag{5.4}$$

where seg and ver are the Segre and the Veronese embeddings. $\operatorname{Ker} d$ cuts out the Halphen locus $P_{(1)}L_{(1)}$ on M^5. Let $Y = M^5 \backslash P_{(1)}L_{(1)}$, φ_2^{-1}: $Y \xrightarrow{(\mathrm{id}, d)} M^5 \times S^{251}$ and, finally,

$$N^5 = \overline{\varphi_1^{-1} Y}. \tag{5.5}$$

A point (p, l, s) of N^5 is a point-line superosculant conic with $s = [s_\cdot + s^\cdot] \in S^{251}$ where $s_\cdot$ and $s^\cdot$ are in the first and the second summand of W, respectively. There exist five G-stable subvarieties on N^5: the proper images of $P_{(1)}$ and $L_{(1)}$ of dimension 4 given by $s_\cdot = 0$ and by $s^\cdot = 0$, the total image E of $P_{(1)}L_{(1)}$ of dimension 4 given by $\operatorname{ad} f = 0$, $\operatorname{ad} f^* = 0$ and their two 3-dimensional intersections $E_\cdot$ and $E^\cdot$ given by $\operatorname{ad} f^* = 0$, $s_\cdot = 0$ and $\operatorname{ad} f = 0$, $s^\cdot = 0$, respectively.

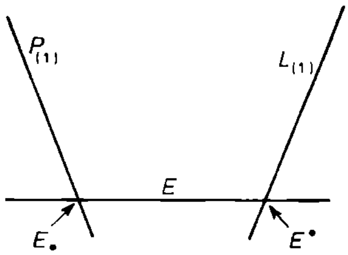

Fig. 1. G-stable subvarieties of N^5

In [DS1] we presented a more intuitive construction of N^5. We started with a regular conic p and a point X of it. The pencil of conics superosculating p at X is a line in P^5 (called a *superosculant* of p) and therefore a point of the Grassmannian $\Gamma = \mathrm{Grass}(2,6)$. If we move X along p we get a rational curve s of order 4 on $\Gamma \subset P^{14}$. We considered its Chow form in [DS1]. Then N^5 was defined to be the closure of the set of all triples (p, l, s). The main point was to study the Chow forms associated with a point (p, l) of $P_{(1)} L_{(1)}$ (a fibre of φ_2). In this case (p, l) is geometrically a double line w with a double point x on it, which is the support point of a double pencil of lines (the corresponding line conic). Let g_0 be the line in Γ representing the set of singular conics in P^2 with support point x, and let g_T be the line representing the set of singular conics in P^2 with fixed part w. They have a common point in Γ and determine an involution i in the pencil given by them with fixed elements g_0 and g_T.

Then it turned out that $2g_0 + g^{(1)} + g^{(2)}$, where $(g^{(1)}, g^{(2)})$ is a pair of i, give us the 1-dimensional set of all possible Chow forms s.

In particular, $(g^{(1)}, g^{(2)})$ may be equal to (g_0, g_0) or (g_T, g_T), i.e., $s = 4g_0$ or $s = 2g_0 + 2g_T$. These Chow forms describe the points of the fibre belonging to $E^\bullet$ and $E_\bullet$, respectively. In general, a line $g^{(j)}$, $j = 1, 2$, represents a two-dimensional set in P^2 of the following kind: Fix a smooth conic π touching w at x and consider the set of all conics which osculate it at x. The choice of π determines a point in the fibre of φ_2. Each of $g^{(1)}$, $g^{(2)}$, g_0 and g_T can be viewed as a line of pencils of conics. The description becomes symmetric if one considers s together with its dual version (see [DS1]).

6. Completeness of N^5

Let now $\tilde{B}(c, r)$ and $\tilde{B}(C, r)$ be the proper images of $B(c, r)$ and $B(C, r)$ on N^5.

THEOREM 2. *N^5 is G-complete with respect to*

(i) *$\tilde{B}(c, r)$ for any ordinary c and for each $r \in N$,*
(ii) *$\tilde{B}(C, r)$ for each $r \in N$ if C is a smooth conic,*
(iii) *$\tilde{B}(C, r)$ for all C(4) and $r = 1, 2, 3$,*
(iv) *$\tilde{S}(C)$ if C is a smooth conic.*

(4) $\deg C \geqslant 2$ if $r > 1$.

Proofs. For (i) see [DS3] and [DS9]. For (iv) see [DS5] and [SE]. For (ii) and (iii) we shall give in [DS9] a suitable description of the points of $\tilde{B}(C, r)$ belonging to branches c of C which are not ordinary and check the completeness by the intersection with $E_{\bullet}$ and $E^{\bullet}$.

Now we are able to solve the problem of § 4. Because of Theorem 2 (iv), the computation of S^5 in $H^* N^5$, where S is the cycle corresponding to $\tilde{S}(C)$, with C a smooth conic, gives us the correct number. In [DS6] we summarized the results of [DS2] and obtained the following description of the intersection ring of N^5:

$$(6.1) \qquad H^* N^5 = Q[P, L, S]/(2P^3 - 3P^2 L + 3PL^2 - 2L^3, \\ 2P^3 + 2L^3 - SP^2 - SL^2 + SPL, S^2 + 9PL - 3SP - 3SL).$$

By a rather lengthy but elementary calculation one gets

$$(6.2) \qquad S^5 \sim 1296 P^5$$

(see [DS5], [DS6]). Semple got in [SE] the same number basing on Schubert's methods.

We remark with a look at Theorem 2 that if c is ordinary then $\tilde{B}(c, 4)$ is a single smooth conic, and $\tilde{B}(c, r) = \emptyset$ for $r > 4$. Moreover, $\tilde{B}(C, r) = \emptyset$ for $r > 4$. Thus $\tilde{B}(C, 4)$, where C is not a conic, is the only remaining case. For a long time we thought that N^5 is G-complete with respect to $\tilde{B}(C, 4)$, too. But our research in [DS9] showed that this is not true. The cusps or inflection points of C give rise to the fact that $E_{\bullet}$ and $E^{\bullet}$ are G-degeneration varieties of N^5 with respect to $\tilde{B}(C, 4)$. Moreover, one can choose C so that one needs further and further blowing-ups to complete the variety of conics with respect to $\tilde{B}(C, 4)$. In [DS10] we shall describe this in detail.

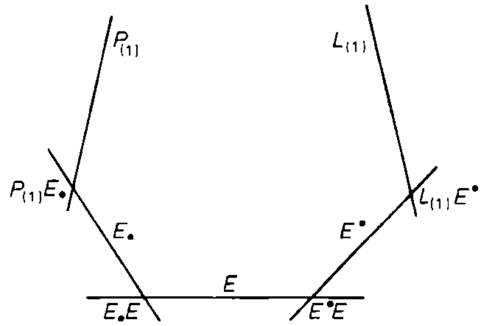

Fig. 2. G-stable subvarieties of the blowing-up in the minimal G-stable subvarieties of N^5

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