NUMERICAL ANALYSIS AND MATHEMATICAL MODELLING
BANACH CENTER PUBLICATIONS, VOLUME 24
PWN-POLISH SCIENTIFIC PUBLISHERS
WARSAW 1990

CONFORMING FINITE ELEMENT APPROXIMATION OF THE STOKES PROBLEM

MICHAL KŘÍŽEK

MÚ ČSAV, Praha, Czechoslovakia

This paper considers a conforming finite element method for the stationary Stokes problem in the plane. An easy way of generating continuous and divergence-free FE-basis functions (e.g. piecewise linear) with small supports is shown. The approximate solution can be obtained solving a system of linear algebraic equations.

1. Introduction

The Stokes problem is usually solved by mixed (non-conforming) FE-methods when the incompressibility condition $\operatorname{div} v = 0$ is satisfied only approximately. Some benefits and also disadvantages of these methods can be found e.g. in [1, 2, 3, 9, 10]. To fulfil the condition $\operatorname{div} v = 0$ exactly, conforming FE-methods have to be employed (see [2, 4, 6, 10, 11]). Here we present a conforming method mentioned in [6].

With the help of a stream function and C^1-elements in $\boldsymbol{R}^2$, we shall construct finite element spaces of continuous and divergence-free vector functions. The method is applicable especially for polygonal domains, since curved C^1-elements are quite complicated [7, 12]. We shall deal with approximation properties of the above-mentioned FE-spaces and apply them to the stationary Stokes problem. However, these spaces may also be used for the Navier–Stokes equations or non-stationary problems.

We denote by Ω a bounded plane domain with a Lipschitz boundary $\partial\Omega$. The outward unit normal $n = (n_1, n_2)$ to $\partial\Omega$ exists almost everywhere (see [8], p. 88). Let $(.,.)_0$ be the inner product in $(L^2(\Omega))^d$, $d \geqslant 1$. By $(H^k(\Omega))^d$, $k = 0, 1, 2, \ldots$ we mean the Cartesian product of the Sobolev spaces $H^k(\Omega)$ with the standard norm $\|\cdot\|_k$ and seminorm $|\cdot|_k$. Further we define the linear operator curl: $H^1(\Omega) \to (L^2(\Omega))^2$ by

$$\operatorname{curl} s = (\partial_2 s, -\partial_1 s), \qquad s \in H^1(\Omega),$$

where $\partial_i = \partial/\partial x_i$, and recall that

$$H_0^1(\Omega) = \{v \in H^1(\Omega) \mid v = 0 \text{ on } \partial\Omega\}$$

and

$$H_0^2(\Omega) = \left\{s \in H^2(\Omega) \mid s = \frac{\partial s}{\partial n} = 0 \text{ on } \partial\Omega\right\}.$$

The homogeneous stationary Stokes problem of the motion of an incompressible viscous fluid in Ω is classically formulated in the following way:

Given $f \in (L^2(\Omega))^2$ (volumic forces per unit mass) and a constant $\nu > 0$ (dynamic viscosity), find the velocity $u = (u_1, u_2)$ and the preasure p such that

$$-\nu\Delta u + \operatorname{grad} p = f \quad \text{in } \Omega, \tag{1}$$

$$\operatorname{div} u = 0 \quad \text{in } \Omega, \tag{2}$$

$$u = 0 \quad \text{on } \partial\Omega, \tag{3}$$

where $\Delta u = (\Delta u_1, \Delta u_2)$.

We shall be not concerned with the way of finding p (for this see e.g. [1]). We roughly outline a variational formulation of (1)–(3) to find the velocity $u = (u_1, u_2) \in V$, where

$$V = \{v \in (H_0^1(\Omega))^2 \mid \operatorname{div} v = 0 \text{ in } \Omega\} \tag{4}$$

is the space of test functions which satisfy the conditions (2) and (3). Multiplying (1) by an arbitrary function $v \in V$ and integrating over Ω, we arrive at

$$-\nu(\Delta u, v)_0 + (\operatorname{grad} p, v)_0 = (f, v)_0.$$

Now the Green formula yields

$$\sum_{q=1}^{2} (\operatorname{grad} u_q, \operatorname{grad} v_q)_0 = (f, v)_0 \qquad \forall v \in V. \tag{5}$$

It follows from the Lax–Milgram lemma that there exists a unique solution to the variational problem (5).

2. The case of simply connected domains

In this section we assume that Ω is simply connected.

THEOREM 2.1. *The linear mapping*

$$\operatorname{curl}\colon H_0^2(\Omega) \to V \tag{6}$$

is bijective.

Proof. For $s \in H_0^2(\Omega)$ evidently $\operatorname{curl} s \in (H_0^1(\Omega))^2$ and $\operatorname{div}\operatorname{curl} s = 0$ in Ω, i.e., $\operatorname{curl} s \in V$ (cf. (4)).

Injectivity. Let $s \in H_0^2(\Omega)$ be in the kernel of the mapping (6), i.e., $\operatorname{curl} s = 0$. Since $\partial_1 s = \partial_2 s = 0$, the function s is constant in Ω, and due to the boundary condition $s = 0$ on $\partial\Omega$, we see that $s = 0$ in the whole domain Ω.

Surjectivity. Let $v \in V$ be arbitrary. Then by [3], p. 22, there exists the so-called stream function $s \in H^1(\Omega)$ unique apart from an additive constant (this constant will be chosen later) such that

$$v = \operatorname{curl} s. \tag{7}$$

Since $v \in V$, we find that $\partial_1 s$, $\partial_2 s \in H^1(\Omega)$, i.e., $s \in H^2(\Omega)$. However, $\partial_1 s = \partial_2 s = 0$ on $\partial\Omega$ which implies that

$$\frac{\partial s}{\partial t} = \frac{\partial s}{\partial n} = 0 \quad \text{on } \partial\Omega,$$

where $t = (n_2, -n_1)$ is the unit tangent vector to $\partial\Omega$. Therefore, s is constant on $\partial\Omega$ (as $\partial\Omega$ is connected). Choosing s in (7) so that $s = 0$ on $\partial\Omega$, we get that $s \in H_0^2(\Omega)$. ∎

COROLLARY 2.2. *It is*

$$V = \operatorname{curl} H_0^2(\Omega), \tag{8}$$

where the symbol $\operatorname{curl} H_0^2(\Omega)$ *represents the space of the rotations of all functions from* $H_0^2(\Omega)$. ∎

Now, let $S_h \subset H_0^2(\Omega)$ be an arbitrary finite element space and let us define

$$V_h = \operatorname{curl} S_h. \tag{9}$$

From (8) we immediately see that $V_h \subset V$ (i.e., $\operatorname{div} v_h = 0$ whenever $v_h \in V_h$) and thus V_h is called the space of divergence-free (solenoidal) finite elements.

COROLLARY 2.3. *We have*

$$\dim V_h = \dim S_h.$$

If $\{s^i\}_{i=1}^m$ *is a basis in* S_h *and if we set*

$$v^i = \operatorname{curl} s^i, \quad i = 1, \ldots, m, \tag{10}$$

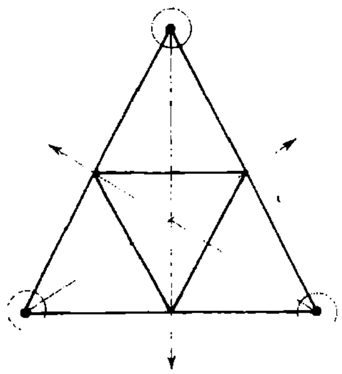
then $\{v^i\}_{i=1}^m$ *is a basis in* V_h. ∎

The proof follows directly from (9) and Theorem 2.1. Moreover, from (10) we find that

$$\operatorname{supp} v^i \subseteq \operatorname{supp} s^i, \quad i = 1, \ldots, m, \tag{11}$$

where supp denotes a support. Consequently, if the basis $\{s^i\}_{i=1}^m$ is generated by the standard C^1-elements, then thanks to the definition formula (10), the basis functions v^i are continuous, exactly divergence-free and by (11) they have small supports (if $\operatorname{supp} s^i$ are small).

Remark 2.4. In [5], Heindel has presented a triangular composed piecewise quadratic C^1-element (see fig.) with only 12 degrees of freedom (like the Hsieh–Clough–Tocher element [1]). Hence, the corresponding divergence-free basis functions $v^i = (v_1^i, v_2^i)$ satisfying (11) are piecewise linear (cf. [2]) as follows from (10). ∎



A conforming FE-approximation of the problem (5) will consist in finding $u_h = (u_{h1}, u_{h2}) \in V_h \subset V$ such that

$$(12) \qquad \sum_{q=1}^{2} (\operatorname{grad} u_{hq}, \operatorname{grad} v_{hq})_0 = (f, v_h)_0 \qquad \forall v_h \in V_h.$$

Seeking u_h in the form

$$u_h = \sum_{i=1}^{m} c^i v^i,$$

we obtain from (12) a system of linear algebraic equations

$$\sum_{q=1}^{2} \sum_{j=1}^{m} (\operatorname{grad} v_q^i, \operatorname{grad} v_q^j)_0 \, c^j = (f, v^i)_0, \qquad i = 1, \dots, m,$$

for the unknowns $c^1, \dots, c^m$. The corresponding matrix is clearly symmetric positive definite and by (11) it can be band.

The next theorem states the convergence of u_h defined by (12) to the solution $u \in V$ of the variational problem (5) without any regularity assumptions upon u. However, to derive some rate of convergence, we shall later assume that u is smooth enough.

THEOREM 2.5. *Let $\{S_h\}$ be a system of finite element subspaces of $H_0^2(\Omega)$ such that the union $\bigcup_h S_h$ is dense in $H_0^2(\Omega)$ (with the topology of $H^2(\Omega)$). Then*

$$\|u - u_h\|_1 \to 0 \quad \text{as } h \to 0.$$

Proof. By Theorem 2.1 there exists $z \in H_0^2(\Omega)$ such that

$$u = \operatorname{curl} z \quad \text{in } \Omega.$$

Since the bilinear form corresponding to (5) is evidently continuous and V-elliptic, i.e.,

$$\nu \sum_{q=1}^{2} (\operatorname{grad} v_q, \operatorname{grad} v_q)_0 \geqslant c \, \|v\|_1^2 \qquad \forall v \in V,$$

we may apply Céa's Lemma (see [1], p. 104). Thus there exists a constant $C > 0$ independent of V_h such that

$$(13) \quad \frac{1}{C}\|u-u_h\|_1 \leqslant \inf_{v_h \in V_h} \|u-v_h\|_1 = \inf_{s_h \in S_h} \|\operatorname{curl} z - \operatorname{curl} s_h\|_1$$
$$= \inf_{s_h \in S_h} \|\operatorname{grad}(z-s_h)\|_1 \leqslant \inf_{s_h \in S_h} \|z-s_h\|_2 \to 0 \quad \text{when } h \to 0. \;\blacksquare$$

Remark 2.6. A sufficient condition for the density assumption in Theorem 2.5 can be found in [1], p. 354. Roughly speaking, this condition requires the regularity of a family $\{\mathcal{T}_h\}$ of triangulations of a polygonal domain, the existence of a reference C^1-element to which all elements are almost-affine equivalent, and the validity of the inclusions

$$P_2(K) \subset P_K \subset H^2(K) \qquad \forall K \in \mathcal{T}_h,$$

where $P_2(K)$ is the space of quadratic polynomials defined on K, and P_K is the space of ansatz-functions of each element K (with appropriate degrees of freedom). The foregoing inclusions are valid e.g. for the Heindel element mentioned in Remark 2.4. $\blacksquare$

Remark 2.7. (The rate of convergence.) Suppose that for some integer $k \geqslant 1$ and for all $s \in H_0^2(\Omega) \cap H^{k+2}(\Omega)$, we can define an S_h-interpolant $\pi_h s \in S_h$ such that

$$(14) \qquad \|s-\pi_h s\|_2 \leqslant ch^k |s|_{k+2},$$

where c is independent of h. Then for any $v \in V \cap (H^{k+1}(\Omega))^2$ we may define the V_h-interpolant $\Pi_h v \in V_h$ by

$$(15) \qquad \Pi_h v = \operatorname{curl}(\pi_h s),$$

where s corresponds to v by Theorem 2.1 and $s \in H^{k+2}(\Omega)$ as $\partial_1 s$, $\partial_2 s \in H^{k+1}(\Omega)$.

Let us suppose that the solution of (5) belongs to $V \cap (H^{k+1}(\Omega))^2$, and let $z \in H_0^2(\Omega) \cap H^{k+2}(\Omega)$ be the corresponding stream function, i.e.,

$$(16) \qquad u = \operatorname{curl} z.$$

Then by Céa's Lemma (cf. (13)), (16), (15) and (14), we obtain the following a priori error estimate

$$\frac{1}{C}\|u-u_h\|_1 \leqslant \inf_{v_h \in V_h} \|u-v_h\|_1 \leqslant \|u-\Pi_h u\|_1 = \|\operatorname{curl}(z-\pi_h z)\|_1$$
$$\leqslant \|z-\pi_h z\|_2 \leqslant ch^k |z|_{k+2} = ch^k |\operatorname{curl} z|_{k+1} = ch^k |u|_{k+1}.$$

Thus the rate of convergence is k and we get the same rate in the L^2-norm for the so-called vorticity $\operatorname{rot} u = \partial_1 u_2 - \partial_2 u_1$. ∎

3. The case of multiply connected domains

Let $\Omega \subset \boldsymbol{R}^2$ be a multiply connected domain with a Lipschitz boundary, let $\Omega_1, \ldots, \Omega_r$ $(1 \leqslant r < \infty)$ be all bounded components of the set $\boldsymbol{R}^2 - \bar{\Omega}$ and let

$$\Omega_0 = \Omega \cup \bigcup_{j=1}^{r} \bar{\Omega}_j,$$

i.e., $\partial\Omega = \partial\Omega_0 \cup \partial\Omega_1 \cup \ldots \cup \partial\Omega_r$, where r is the number of holes in Ω.
First of all we present an analogue of Theorem 2.1.

THEOREM 3.1. *There exist functions* $z^1, \ldots, z^r \in H^2(\Omega) - H_0^2(\Omega)$ *such that the mapping*

$$\operatorname{curl}\colon \mathscr{L}\big(H_0^2(\Omega) \cup \{z^1, \ldots, z^r\}\big) \to V, \tag{17}$$

where $\mathscr{L}$ *denotes the linear span, is bijective.*

Proof. Let $z^j \in H^2(\Omega)$, $j = 1, \ldots, r$, be arbitrary functions satisfying

$$z^j = \delta_{ij} \quad \text{on } \partial\Omega_i,\ i = 0, \ldots, r,\ j = 1, \ldots, r, \tag{18}$$

(δ_{ij} is Kronecker's symbol) and

$$\partial_1 z^j = \partial_2 z^j = 0 \quad \text{on } \partial\Omega,\ j = 1, \ldots, r. \tag{19}$$

Note that the distances of the boundaries $\partial\Omega_j$ are positive because $\partial\Omega$ is Lipschitz. By Theorem 2.1 we already know that $\operatorname{curl} H_0^2(\Omega) \subset V$ and due to (19), $\operatorname{curl} z^j \in V$, too.

Injectivity. According to (18), any z^j vanishes on $\partial\Omega_0$ and thus we may proceed as in Theorem 2.1.

Surjectivity. Let $v \in V$ be arbitrary. Since $v = 0$ on each component $\partial\Omega_i$, there exists (by [3], p. 22) a stream function $s \in H^1(\Omega)$ (unique apart from an additive constant) such that

$$v = \operatorname{curl} s. \tag{20}$$

As $\partial_1 s, \partial_2 s \in H_0^1(\Omega)$, we observe again that $s \in H^2(\Omega)$, $\partial s/\partial n = 0$ on $\partial\Omega$ and that the tangential derivative of s vanishes on the boundary, i.e., $\partial s/\partial t = 0$ on $\partial\Omega$. This implies that s equals to a constant c_j $(j = 0, 1, \ldots, r)$ on each part $\partial\Omega_j$. Let s in (20) be chosen so that $c_0 = 0$, i.e., $s|_{\partial\Omega_0} = 0$. Putting

$$z^0 = s - \sum_{j=1}^{r} c_j z^j, \tag{21}$$

we find that $z^0 \in H^2(\Omega)$ and by (18) and (19) it holds that $z^0 = \partial_1 z^0 = \partial_2 z^0$ on $\partial\Omega$. Hence, $z^0 \in H_0^2(\Omega)$ and the mapping (17) is due to (21) surjective. ∎

COROLLARY 3.2. *According to Theorem* 3.1, *it is*

$$V = \operatorname{curl} Z,$$

where

$$Z = \mathscr{L}(H_0^2(\Omega) \cup \{z^1, \ldots, z^r\})$$
$$= \left\{ z \in H^2(\Omega) \,\middle|\, \frac{\partial z}{\partial n} = 0 \text{ on } \partial\Omega,\ z|_{\partial\Omega_0} = 0,\ \exists c_1, \ldots, c_r \in R^1; \right.$$
$$\left. z|_{\partial\Omega_j} = c_j,\ j = 1, \ldots, r \right\}. \ \blacksquare$$

We may therefore define the space of divergence-free finite elements as follows

$$V_h = \operatorname{curl} Z_h,$$

where Z_h is an arbitrary finite element subspace of Z.

Remark 3.3. Let us set

$$Z_h = \mathscr{L}(S_h \cup \{z^1, \ldots, z^r\}),$$

where z^j belong to a fixed finite element space $X_{h_0} \subset H^2(\Omega)$ and satisfy (18) and (19),

$$S_h \supset X_{h_0} \cap H_0^2(\Omega),$$

and let the union $\bigcup_h S_h$ be dense in $H_0^2(\Omega)$ (with respect to the $\|\cdot\|_2$-norm). Then we may again easily prove that

$$\|u - u_h\|_1 \to 0 \quad \text{as } h \to 0.$$

Assuming further (14), we can derive that the rate of convergence is k when u is sufficiently smooth. If $\{s^i\}_{i=1}^m$ is a basis of S_h then

$$\{\operatorname{curl} s^i\}_{i=1}^m \cup \{\operatorname{curl} z^j\}_{j=1}^r \tag{22}$$

is a basis of V_h. The supports of the basis functions $\operatorname{curl} z^j$ may have, for instance, a circular shape around any hole Ω_j, $j = 1, \ldots, r$. Hence, to save computer memory, we should store only non-zero entries of the Gram matrix corresponding to the basis (22), and then use some iterative method for finding the discrete solution. For r fixed merely $O(m)$ memory cells are needed. ■

References

[1] P. G. Ciarlet, *The Finite Element Method for Elliptic Problems*, North-Holland, Amsterdam 1978.

[2] M. Fortin, *Approximation des fonctions à divergence nulle par la méthode des éléments finis*, Proc. of the 3rd Int. Conf. Numer. Meth. in Fluid Mech., Paris 1972, 99–103.

[3] V. Girault and P. A. Raviart, *Finite Element Approximation of the Navier-Stokes Equations*, Springer, Berlin 1979.
[4] D. F. Griffiths, *Finite elements for compressible flow*, Math. Methods Appl. Sci. 1 (1979), 16–31.
[5] G. Heindl, *Interpolation and approximation by piecewise quadratic C^1-functions of two variables*, in: Multivariate approximation theory, ed. by W. Schempp and K. Zeller, Internat. Schriftenreihe Numer. Math. 51 (1979), 146–161.
[6] M. Křížek, *Conforming equilibrium finite element methods for some elliptic plane problems*, RAIRO Anal. Numér. 17 (1982), 35–65.
[7] L. Mansfield, *A Clough-Tocher type element useful for fourth order problems over non-polygonal domains*, Math. Comp. 32 (1978), 135–142.
[8] J. Nečas, *Les Méthodes Directes en Théorie des Equations Elliptiques*, Academia, Prague 1967.
[9] J. C. Nedelec, *Eléments finis mixtes incompressibles pour l'équation de Stokes dans R^3*, Numer. Math. 39 (1982), 97–112.
[10] R. Temam, *Navier-Stokes Equations*, North-Holland, Amsterdam 1979.
[11] F. Thomasset, *Implementation of Finite Element Methods for Navier-Stokes Equations*, Springer Ser. Comp. Phys. Springer, New York 1981.
[12] A. Ženíšek, *Curved triangular finite C^m-elements*, Apl. Mat. 23 (1978), 346–377.

Presented to the Semester
Numerical Analysis and Mathematical Modelling
February 25 — May 29, 1987