ZASTOSOWANIA MATEMATYKI
APPLICATIONES MATHEMATICAE
21,2 (1991), pp. 303–326

S. LEWANOWICZ (Wrocław)

A NEW APPROACH TO THE PROBLEM OF CONSTRUCTING RECURRENCE RELATIONS FOR THE JACOBI COEFFICIENTS

Abstract. A new method is presented for obtaining a recurrence relation for the Jacobi series coefficients of a function which satisfies a linear ordinary differential equation with polynomial coefficients.

1. Introduction. Let a function f, defined in the interval $[-1,1]$, be sufficiently regular so that it can be expanded in a uniformly convergent series with respect to the Jacobi polynomials (*Jacobi series*, in short)

$$f = \sum_{k=0}^{\infty} a_k^{(\alpha,\beta)}[f] P_k^{(\alpha,\beta)} \quad (\alpha, \beta > -1), \tag{1.1}$$

where $P_k^{(\alpha,\beta)}$ is the kth Jacobi polynomial defined by

$$P_k^{(\alpha,\beta)}(x) := \frac{(-1)^k}{2^k k!(1-x)^\alpha (1+x)^\beta} \frac{d^k}{dx^k}[(1-x)^{k+\alpha}(1+x)^{k+\beta}], \tag{1.2}$$

and the coefficients $a_k^{(\alpha,\beta)}[f]$ are given by

$$a_k^{(\alpha,\beta)}[f] := \frac{k!(2k+\lambda)\Gamma(k+\lambda)}{2^\lambda \Gamma(k+\alpha+1)\Gamma(k+\beta+1)} \times \int_{-1}^{1} (1-x)^\alpha (1+x)^\beta P_k^{(\alpha,\beta)}(x) f(x)\, dx \quad (k = 0, 1, \ldots), \tag{1.3}$$

where we put $\lambda := \alpha + \beta + 1$ (see, e.g., [1], Vol. 2, Sects. 10.8 and 10.19; or [6], Vol. 1, Sect. 8.3).

1991 *Mathematics Subject Classification*: 42C10, 39A70, 65L05, 65L10.
Key words and phrases: Jacobi series, Jacobi coefficients, recurrence relations, difference operators, differential operators, linear differential equation.

Some alternative forms for the coefficients (1.3), which are available for many elementary and special functions (see, e.g., [6], Vol. 2, Chap. 9), involve other special functions and, therefore, may not be used easily for computations. On the other hand, it is relatively easy to evaluate $a_k^{(\alpha,\beta)}[f]$ for k belonging to some finite set K if the sequence $\{a_k^{(\alpha,\beta)}[f]\}$ is known to satisfy a recurrence relation of the form

$$\sum_{j=0}^{r} \varphi_j(k) a_{k+j}^{(\alpha,\beta)}[f] = \sigma(k)\,, \tag{1.4}$$

where $\varphi_0, \dots, \varphi_r$ and σ are known functions (see [6], Vol. 2, Sect. 12.5; or [7], Sect. 13; or [9], Chap. 11).

There is a class of general methods for obtaining equation (1.4) under the assumption that the function f satisfies a linear differential equation

$$\mathbf{P}_n f \equiv \sum_{i=0}^{n} p_i f^{(i)} = q \tag{1.5}$$

of order n, with suitable initial or boundary conditions. In (1.5), $p_0, \dots, p_n$ are polynomials, and the coefficients $a_k^{(\alpha,\beta)}[q]$ exist and are known. See [5] for some historical comments on the subject.

In [3] and [5] we proposed several methods belonging to the above mentioned class; earlier (see [2]) we discussed the problem of constructing a recurrence relation for the coefficients of the *Gegenbauer series* of a function f, closely related to the Jacobi series of f with $\alpha = \beta$. In particular, we described in [2] and [3] an *optimum method* providing a recurrence relation of minimum order among all such equations which can be obtained from the differential equation (1.5), using basic difference and differential properties of the Jacobi polynomials (see (3.1), (3.2)). This method seems to be of great theoretical value. For instance, it helped us to discover the existence of certain new recurrence relations for a class of hypergeometric functions; explicit forms of those relations were later derived using results from the theory of such functions (see [4]). However, the complexity of the optimum method grows rapidly with n, so the calculations may be very tedious. Also, the order of the recurrence relation cannot be predicted (that is—expressed in terms of the order and coefficients of the differential equation (1.5)) easily. In this connection, let us mention the following problem raised by Paszkowski [8]:

Let $\mathbf{Q}$ *be a linear differential operator with coefficients being rational functions of* x. *Obviously, if the function* f *satisfies the equation* (1.5), *then* f *is also a solution of the differential equation*

$$\mathbf{P}^* f = q^* \tag{1.6}$$

with $\mathbf{P}^* := \mathbf{Q}\mathbf{P}_n$ *and* $q^* := \mathbf{Q}q$. *Find* $\mathbf{Q}$ *which minimizes the order of the recurrence relation for the coefficients* $a_k^{(\alpha,\beta)}[f]$ *obtained using* (1.6).

For example, the function $f(x) = xe^x$ satisfies the first-order equation

$$\mathbf{P}_1 f \equiv xf' - (x+1)f = 0$$

which implies the *fourth-order* recurrence relation

$$t_{k-2}[f] - 2(k-2)t_{k-1}[f] - 2(k+2)t_{k+1}[f] - t_{k+2}[f] = 0$$

for the *Chebyshev coefficients* $t_k[f]$ of the function f (closely related to $a_k^{(-1/2,-1/2)}[f]$; see Section 5). Now, taking

$$\mathbf{Q} := x^{-2}[x(x^2-1)\mathbf{D}^2 + (2x^2+1)\mathbf{D} + (x-1)\mathbf{I}],$$

where $\mathbf{D} := d/dx$, and $\mathbf{I}$ is the identity operator, we obtain

$$\mathbf{P}^* f \equiv (x^2-1)f''' - (x^2-3x-1)f'' - (4x-1)f' - 3f = 0$$

and consequently the *second-order* recurrence relation

$$(k^2+k+1)t_{k-1}[f] - 2k^3 t_k[f] - (k^3-k+1)t_{k+1}[f] = 0$$

(*ibid.*; see also Example 5.1). To be sure that the obtained result is the best one, it is necessary to find a formula expressing the order of the recurrence relation in terms of the coefficients of the operators $\mathbf{P}_1$ and $\mathbf{Q}$.

For the reasons explained above, several simpler methods were proposed (see [2]; [5]; [7], Sect. 13), which, however, are not optimum algorithms, in general.

In the present paper a new method is given, which in the author's belief is equivalent to the optimum method. The method exploits in a substantial way some differential-difference identities satisfied by the coefficients (1.3) (see (3.30)–(3.33)). The main result of the paper is given in Section 4 (see Theorem 4.1). The special cases of Gegenbauer and Chebyshev expansions are discussed in Section 5. Sections 2 and 3 contain the necessary definitions and lemmata.

In the sequel we shall use the notation

$$(1.7) \qquad b_k[f] \equiv b_k^{(\alpha,\beta)}[f] := \frac{\Gamma(k+\alpha+1)}{\Gamma(k+\lambda)(2k+\lambda-1)_3} a_k^{(\alpha,\beta)}[f].$$

Here the symbol $(c)_m$ denotes the shifted factorial: $(c)_0 = 1$, $(c)_m = c(c+1)\ldots(c+m-1)$ $(m > 0)$. We call $b_k[f]$ the *Jacobi coefficients* of the function f.

Sometimes it is convenient to use coefficients with negative indices. We assume that if $\alpha \neq \beta$, or $\alpha = \beta$ but $2\alpha + 1$ is not an integer ≥ 0, then

$$b_{-k}^{(\alpha,\beta)}[f] := 0 \quad (k = 1, 2, \ldots),$$

and if $\alpha = \beta$ and $2\alpha + 1 = m$ is a nonnegative integer, we define

$$b_{-k}^{(\alpha,\alpha)}[f] := \begin{cases} 0 & (k = 0, 1, \ldots, m-1), \\ b_{k-m}^{(\alpha,\alpha)}[f] & (k \geq m) \end{cases}$$

(see [2]).

2. Difference operators. The results given in the following sections are expressed in terms of a certain type of linear operators. Let $\mathcal{S}$ denote the linear space of all "doubly infinite" sequences of complex numbers, with addition of sequences and scalar multiplication defined as usual. Obviously $\mathcal{S}$ is the space of all complex-valued functions defined on the set of all integers. Let $\mathcal{S}_{\text{rat}}$ denote the set of all *rational* functions $s \in \mathcal{S}$.

Consider the set $\mathcal{S}^*$ of all linear operators mapping $\mathcal{S}$ into itself. For $T \in \mathcal{S}^*$ and $\{z_k\} \in \mathcal{S}$, we denote the kth coordinate of the sequence $T\{z_k\} \in \mathcal{S}$ by Tz_k, so that $T\{z_k\} = \{Tz_k\}$. The *zero operator*, the *identity operator* and the mth *shift operator* in $\mathcal{S}^*$ are denoted by Θ, I and E^m, respectively. Then we have

$$\Theta z_k = 0, \quad I z_k = z_k, \quad E^m z_k = z_{k+m} \tag{2.1}$$

for every $\{z_k\} \in \mathcal{S}$. Clearly, $E^0 = I$.

Let $\mathcal{L}$ be the set of all operators $L \in \mathcal{S}^*$ such that

$$L = \sum_{i=0}^{r} \lambda_i(k) E^{u+i}, \tag{2.2}$$

where $r \geq 0$ and u are integers, and $\lambda_0, \ldots, \lambda_r \in \mathcal{S}_{\text{rat}}$. Every nonzero operator $L \in \mathcal{L}$ can be expressed in the form (2.2) with $\lambda_0 \not\equiv 0$ and $\lambda_r \not\equiv 0$. The number $r = r(L)$ is referred to as the *order* of the operator L, while λ_i are called the *coefficients* of L. The elements of the set $\mathcal{L}$ are known as *difference operators*.

Let $L \in \mathcal{L}$ be defined by (2.2) and let $M \in \mathcal{L}$ be such that

$$M := \sum_{j=0}^{t} \mu_j(k) E^{v+j}.$$

We define the *product* of the operators L and M to be the operator

$$LM := \sum_{i=0}^{r} \lambda_i(k) \sum_{j=0}^{t} \mu_j(k+u+i) E^{u+v+i+j}.$$

It can be seen that under this definition of multiplication, with addition of operators defined in a natural manner, $\mathcal{L}$ forms a (noncommutative) ring with the identity I.

Let $L \in \mathcal{L}$ and $\sigma \in \mathcal{S}$. The equation

$$Lz_k = \sigma(k) \tag{2.3}$$

is the *recurrence relation* for the sequence $\{z_k\} \in \mathcal{S}$; the *order of the recurrence relation* (2.3) is the order of the difference operator L.

3. Difference and differential-difference properties of the Jacobi coefficients. Let us recall the basic identities satisfied by the Jacobi coefficients ([3], [5]):

$$b_k[xf(x)] = Xb_k[f]\,, \tag{3.1}$$

$$Db_k[\mathbf{D}f] = b_k[f]\,, \tag{3.2}$$

where $\mathbf{D} := d/dx$, and X, D are the following difference operators of the second order:

$$X := \sum_{j=0}^{2} \xi_j(k)E^{j-1}\,, \tag{3.3}$$

$$D := \sum_{j=0}^{2} \delta_j(k)E^{j-1}\,, \tag{3.4}$$

where in turn

$$\begin{cases} \delta_0(k) := 2(k+\alpha)(2k+\lambda-3)/\gamma(k)\,, \\ \delta_1(k) := 2(\alpha-\beta)(2k+\lambda)/\gamma(k)\,, \\ \delta_2(k) := -2(k+\beta+1)(2k+\lambda+3)/\gamma(k),\ \gamma(k) := (2k+\lambda-1)_3\,, \\ \xi_0(k) := k\delta_0(k)\,, \quad \xi_1(k) := (1-\lambda)\delta_1(k)/2\,, \\ \xi_2(k) := -(k+\lambda)\delta_2(k)\,. \end{cases} \tag{3.5}$$

Identity (3.1) follows from the three-term recurrence relation obeyed by the Jacobi polynomials (see, e.g., [1], Vol. 2, Sect. 10.8; or [6], Vol. 1, Sect. 8.3) while in the proof of (3.2) one can exploit the representation of the polynomial $(1-x^2)\mathbf{D}P_k^{(\alpha,\beta)}(x)$ as a linear combination of the polynomials $P_j^{(\alpha,\beta)}$ $(j = k-1, k, k+1)$ and use the second-order differential equation satisfied by $P_k^{(\alpha,\beta)}$ (*ibid.*; see also (3.27)).

The following identities are the generalized forms of (3.1), (3.2), respectively:

$$b_k[pf] = p(X)b_k[f] \quad (p \text{ a polynomial})\,, \tag{3.6}$$

$$D^r b_k[\mathbf{D}^r f] = b_k[f] \quad (r = 0, 1, \ldots)\,. \tag{3.7}$$

Now, assume that f is a solution of the differential equation (1.5) and that $f^{(n)}$ has a uniformly convergent expansion in a Jacobi series. The kth Jacobi coefficients of both sides of (1.5) are equal, hence $b_k[\mathbf{P}_n f] = b_k[q]$. Using (3.6) we get

$$\sum_{i=0}^{n} p_i(X)b_k[\mathbf{D}^i f] = b_k[q]\,. \tag{3.8}$$

A recurrence relation for the Jacobi coefficients of f may be obtained by eliminating, by means of (3.7), the Jacobi coefficients $b_k[\mathbf{D}^i f]$ $(i = 1, \ldots, n)$ from the identity (3.8). To be strict, one should rather speak about a *class of methods characterized by the set $\Pi(\mathbf{P}_n)$ of pairs $\langle P, L\rangle$ of nonzero difference operators $P, L \in \mathcal{L}$ obeying the identity*

$$Pb_k[\mathbf{P}_n f] = Lb_k[f]. \tag{3.9}$$

Given an arbitrary pair $\langle P, L\rangle \in \Pi(\mathbf{P}_n)$, the recurrence relation

$$Lb_k[f] = \sigma(k) \tag{3.10}$$

holds, where $\sigma(k) := Pb_k[q]$; the order of (3.10) equals

$$r(L) = r(P) + 2 \max_{0 \le j \le n, p_j \not\equiv 0} (\partial p_j - j), \tag{3.11}$$

where ∂p_j is the degree of the polynomial p_j (see [3], Sect. 4.2). In the cited paper, it was shown, giving a generalization of an earlier result [2], that the set $\Pi(\mathbf{P}_n)$ is *generated* by a "minimal" pair $\langle P^*, L^*\rangle$ in the sense that for every pair $\langle P, L\rangle$ in this set there exists a nonzero difference operator C such that $P = CP^*$ and $L = CL^*$. A specific feature of the *generating pair* (or the *generator*) $\langle P^*, L^*\rangle$ is, therefore, that the operators P^* and L^* have no common left divisor of positive order. In virtue of (3.11), the recurrence (3.10) is of the lowest order when C is an operator of zeroth order, i.e. when the operators P and P^* as well as L and L^* are equivalent. An *algorithm* for construction of a generating pair for $\Pi(\mathbf{P}_n)$ is given in [3]. However, in some cases this pair can be given *explicitly*. For instance, we have the following

LEMMA 3.1. *Let $\mathbf{P}_n$ be the differential operator*

$$\mathbf{P}_n := \sum_{i=0}^{n} p_i \mathbf{D}^i, \tag{3.12}$$

$p_0, \ldots, p_n$ being polynomials. Assume that $p_n(-1) \neq 0$, $p_n(1) \neq 0$. Let

$$P^* := D^n, \quad L^* := \sum_{i=0}^{n} D^{n-i} q_i(X), \tag{3.13}$$

where

$$q_i := \sum_{j=i}^{n} (-1)^{j-i} \binom{j}{i} p_j^{(j-i)} \quad (i = 0, \ldots, n). \tag{3.14}$$

Then $\langle P^*, L^*\rangle$ is a generator of $\Pi(\mathbf{P}_n)$.

Proof. It is known that the operator (3.12) can be written in the equivalent form

$$\mathbf{P}_n f = \sum_{i=0}^{n} \mathbf{D}^i(q_i f), \tag{3.15}$$

$q_0, \ldots, q_n$ being the polynomials given by (3.14) (see, e.g., [7], p. 232). Hence, in view of (3.7), it follows that the pair $\langle P^*, L^* \rangle$, defined in (3.13), belongs to $\Pi(\mathbf{P}_n)$.

We will prove the lemma using induction on n.

When $n = 1$, the result can be obtained easily with the aid of Lemma 3.5 of [3].

Now, let $n = m > 1$. Assume that the lemma is true for all $n < m$. Let $\langle P, L \rangle$ be an arbitrary pair in $\Pi(\mathbf{P}_m)$. Then

$$Pb_k[\mathbf{P}_m f] = Lb_k[f]. \tag{3.16}$$

Let $\widehat{\mathbf{P}}_{m-1} := \sum_{i=0}^{m-1} p_{i+1}\mathbf{D}^i$, so that $\mathbf{P}_m - p_0\mathbf{I} = \widehat{\mathbf{P}}_{m-1}\mathbf{D}$. It is easy to see that

$$Pb_k[\widehat{\mathbf{P}}_{m-1}\mathbf{D}f] = \{L - Pp_0(X)\}b_k[f].$$

Letting $g := \mathbf{D}f$ and using (3.2) we get

$$Pb_k[\widehat{\mathbf{P}}_{m-1}g] = L'b_k[g],$$

where $L' := \{L - Pp_0(X)\}D$. Thus $\langle P, L' \rangle \in \Pi(\widehat{\mathbf{P}}_{m-1})$ and by assumption we obtain, in particular,

$$P = CD^{m-1} \tag{3.17}$$

for some nonzero difference operator C. Hence, in view of (3.13), (3.15) and (3.7),

$$Cb_k[\mathbf{D}(q_m f)] = L''b_k[f], \tag{3.18}$$

where $L'' := L - C\sum_{i=0}^{m-1} D^{m-i-1}q_i(X)$. Equation (3.18) means that $\langle C, L'' \rangle \in \Pi(\mathbf{P}_1^*)$, where $\mathbf{P}_1^* f := \mathbf{D}(q_m f)$. According to the first part of the proof we have $C = C'D$, where C' is a nonzero difference operator. This, in view of (3.17), implies that $P = C'D^m$. Using (3.2) in (3.18) yields

$$L = C' \sum_{i=0}^{m} D^{m-i}q_i(X).$$

Thus the lemma is true for $n = 1, 2, \ldots$ ∎

The following differential operators will play an important role in the sequel:

$$\mathbf{U} := (x^2 - 1)\mathbf{D} + [(\lambda + 1)x + \alpha - \beta]\mathbf{I}, \quad \mathbf{J} := \mathbf{UD}, \tag{3.19}$$

$$\mathbf{V}_\varepsilon := (x + \varepsilon)\mathbf{D} + \nu_\varepsilon\mathbf{I}, \quad \mathbf{K}_\varepsilon := \mathbf{V}_\varepsilon\mathbf{D} \quad (\varepsilon = \pm 1). \tag{3.20}$$

Here $\mathbf{I}$ denotes the identity operator, and

$$\nu_\varepsilon := [\lambda + 1 + \varepsilon(\beta - \alpha)]/2 \quad (\varepsilon = \pm 1). \tag{3.21}$$

Recall the following definition.

DEFINITION 3.1 ([5]). Given $\varepsilon \in \{-1, 1\}$ we define the sequence $\{A_m^{(\varepsilon)}\}$ of difference operators by

$$A_m^{(\varepsilon)} := I + \tau_m^{(\varepsilon)}(k)E \quad (m = 0, 1, \ldots), \tag{3.22}$$

where

$$\tau_m^{(\varepsilon)}(k) := -\frac{2\varepsilon(k + \nu_\varepsilon)}{\delta_0(k+1)(2k + \lambda + m + 1)_2} \quad (m \geq 0) \tag{3.23}$$

(δ_0 is given in (3.5)). Further, let

$$S_{ij}^{(\varepsilon)} := \begin{cases} I & (i < j), \\ A_i^{(\varepsilon)} S_{i-1,j}^{(\varepsilon)} & (i \geq j \geq 0), \end{cases} \tag{3.24}$$

$$P_h^{(\varepsilon)} := S_{h-1,0}^{(\varepsilon)} \quad (h = 0, 1, \ldots). \tag{3.25}$$

Finally, let

$$\begin{aligned} \mu_{2j}^{(\varepsilon)}(k) &:= (k + \nu_\varepsilon)_j \Big/ \prod_{h=1}^{j} \delta_0(k+h) \\ \mu_{2j+1}^{(\varepsilon)}(k) &:= (k + j + \nu_\varepsilon)\mu_{2j}(k) \end{aligned} \quad (j = 0, 1, \ldots). \tag{3.26}$$

Observe that $\mathbf{J}$ and $\mathbf{K}_\varepsilon$ occur in the following equations satisfied by the Jacobi polynomials:

$$[\mathbf{J} - \varkappa(k)\mathbf{I}]P_k^{(\alpha,\beta)} = 0, \tag{3.27}$$

$$\mathbf{K}_\varepsilon P_k^{(\alpha,\beta)}(x) = (x - \varepsilon)^{-2} B^{(\varepsilon)}[h_k P_k^{(\alpha,\beta)}(x)] \quad (\varepsilon = \pm 1), \tag{3.28}$$

where

$$\begin{aligned} \varkappa(k) &:= k(k + \lambda), \\ B^{(\varepsilon)} &:= \frac{1}{\mu_2^{(\varepsilon)}(k)} P_2^{(\varepsilon)}, \\ h_k &:= 2^\lambda[(2k + \lambda)^2 - 1]\Gamma(k + \beta + 1)/k!. \end{aligned} \tag{3.29}$$

Obviously, (3.27) is the well-known second-order differential equation satisfied by the kth Jacobi polynomial. Equation (3.28) seems to be a less standard result.

From (3.19), (3.20), the following identities can be deduced, using (3.6) and (3.7):

$$b_k[\mathbf{U}f] = \varkappa(k)Db_k[f], \tag{3.30}$$

$$b_k[\mathbf{J}f] = \varkappa(k)b_k[f], \tag{3.31}$$

$$\left.\begin{aligned} &(3.32) \qquad P_1^{(\varepsilon)} b_k[\mathbf{V}_\varepsilon f] = \mu_1^{(\varepsilon)}(k) P_1^{(-\varepsilon)} b_k[f] \\ &(3.33) \qquad P_2^{(\varepsilon)} b_k[\mathbf{K}_\varepsilon f] = \mu_2^{(\varepsilon)}(k) E b_k[f] \end{aligned}\right\} \quad (\varepsilon = \pm 1).$$

As an immediate consequence we obtain

LEMMA 3.2. *The pairs* $\langle I, \varkappa(k) D\rangle$, $\langle I, \varkappa(k) I\rangle$, $\langle P_1^{(\varepsilon)}, \mu_1^{(\varepsilon)}(k) P_1^{(-\varepsilon)}\rangle$ *and* $\langle P_2^{(\varepsilon)}, \mu_2^{(\varepsilon)}(k) E\rangle$ *are generators of the sets* $\Pi(\mathbf{U})$, $\Pi(\mathbf{J})$, $\Pi(\mathbf{V}_\varepsilon)$ *and* $\Pi(\mathbf{K}_\varepsilon)$ $(\varepsilon = \pm 1)$, *respectively.*

The above results can be generalized to the case of powers of the operators $\mathbf{J}$ and $\mathbf{K}_\varepsilon$ and their products. Obviously we have

$$(3.34) \qquad b_k[\mathbf{J}^t f] = \varkappa^t(k) b_k[f]$$

for any $t = 0, 1, \ldots$ By induction on s $(s = 0, 1, \ldots)$ one can prove the equalities

$$(B^{(\varepsilon)})^s = E^{-s} \frac{1}{\mu_{2s}^{(\varepsilon)}(k)} P_{2s}^{(\varepsilon)}, \quad P_1^{(\varepsilon)} (B^{(\varepsilon)})^s = E^{-s} \frac{1}{\mu_{2s}^{(\varepsilon)}(k)} P_{2s+1}^{(\varepsilon)},$$

which lead to the following generalization of the identities (3.32) and (3.33):

$$(3.35) \qquad P_{2s+\delta}^{(\varepsilon)} b_k[\mathbf{K}_\varepsilon^s \, \mathbf{V}_\varepsilon^\delta f] = \mu_{2s+\delta}^{(\varepsilon)}(k) E^s P_\delta^{(-\varepsilon)} b_k[f] \qquad (s = 0, 1, \ldots;\ \delta = 0, 1;\ \varepsilon = \pm 1).$$

In the next section, we will discuss differential operators of the form

$$(3.36) \qquad \mathbf{Q} := \mathbf{D}^r \mathbf{K}_\varepsilon^s \mathbf{V}_\varepsilon^\delta \mathbf{J}^t \mathbf{U}^{\delta'},$$

where r, s, t are nonnegative integers, δ, $\delta' \in \{0, 1\}$, and $\varepsilon \in \{-1, 1\}$. Using (3.7), (3.35), (3.34) and (3.30), we obtain

$$(3.37) \qquad Q b_k[\mathbf{Q} f] = M b_k[f],$$

where

$$(3.38) \qquad Q := P_{2s+\delta}^{(\varepsilon)} D^r,$$

$$(3.39) \qquad M := \mu_{2s+\delta}^{(\varepsilon)}(k) E^s P_\delta^{(-\varepsilon)} \varkappa^{t+\delta'}(k) D^{\delta'}.$$

It can be seen that the operators Q and M have no common left divisor of positive order. Consequently, we have

LEMMA 3.3. *The pair* $\langle Q, M\rangle$ *of difference operators defined by* (3.38), (3.39) *is a generator of the set* $\Pi(\mathbf{Q})$, *where* $\mathbf{Q}$ *is the differential operator* (3.36).

The next lemmata give some properties of operators of the type (3.38) which will be needed in the next section. It will be useful to introduce a new family of difference operators.

DEFINITION 3.2. Given $\varepsilon \in \{-1,1\}$ we define the sequence $\{R_m^{(\varepsilon)}\} \subset \mathcal{L}$ by

$$R_m^{(\varepsilon)} := \delta_0(k)E^{-1} + \varrho_m^{(\varepsilon)}(k)I \quad (m = 0,1,\ldots), \tag{3.40}$$

where

$$\varrho_m^{(\varepsilon)}(k) := 2\varepsilon \frac{k+m+\nu_{-\varepsilon}}{(2k+\lambda+m)_2} \quad (m = 0,1,\ldots), \tag{3.41}$$

δ_0 is given in (3.5), and $\nu_{-\varepsilon}$ in (3.21). Further, let $T_{i,j}^{(\varepsilon)} \in \mathcal{L}$ and $U_h^{(\varepsilon)} \in \mathcal{L}$ be defined by

$$T_{ij}^{(\varepsilon)} := \begin{cases} I & (i > j), \\ R_i^{(\varepsilon)} T_{i+1,j}^{(\varepsilon)} & (0 \le i \le j), \end{cases} \tag{3.42}$$

$$U_h^{(\varepsilon)} := T_{0,h-1}^{(\varepsilon)} \quad (h = 0,1,\ldots). \tag{3.43}$$

The two families of difference operators introduced by Definitions 3.1 and 3.2 are closely related. First, we have

$$R_0^{(\varepsilon)} A_0^{(\varepsilon)} = D, \quad R_m^{(\varepsilon)} A_m^{(\varepsilon)} = A_{m-1}^{(\varepsilon)} R_{m-1}^{(\varepsilon)} \quad (m = 1,2,\ldots) \tag{3.44}$$

for $\varepsilon \in \{-1,1\}$. Using the above relations, it can be shown that

$$S_{ij}^{(\varepsilon)} T_{jl}^{(\varepsilon)} = T_{i+1,h}^{(\varepsilon)} S_{h,l+1}^{(\varepsilon)} \quad (i+l+1 = j+h; h > i \ge j; h > l \ge j), \tag{3.45}$$

which for $l := 0$, $i := h-1$, $j := 0$ reduces to

$$P_h^{(\varepsilon)} R_0^{(\varepsilon)} = R_h^{(\varepsilon)} S_{h1}^{(\varepsilon)}.$$

Multiplying this on the right by $A_0^{(\varepsilon)}$ and using the first equality of (3.44) and $S_{h1}^{(\varepsilon)} A_0^{(\varepsilon)} = P_{h+1}^{(\varepsilon)}$ (cf. Definition 3.1) yields

$$P_h^{(\varepsilon)} D = R_h^{(\varepsilon)} P_{h+1}^{(\varepsilon)},$$

which leads to the more general equation

$$P_v^{(\varepsilon)} D^r = T_{v,v+r-1}^{(\varepsilon)} P_{v+r}^{(\varepsilon)} \quad (v, r = 0,1,\ldots). \tag{3.46}$$

LEMMA 3.4. *Let the difference operators $Q_1, \ldots, Q_h$ $(h > 1)$ be given by*

$$Q_i := P_{v_i}^{(\varepsilon)} D^{r_i} \quad (i = 1,\ldots,h), \tag{3.47}$$

where $\varepsilon \in \{-1,1\}$, and v_i, r_i are nonnegative integers. Further, let the operator Q be defined by

$$Q := P_v D^r, \tag{3.48}$$

where

$$r := \max_{1\le i\le h} r_i, \quad v := \max_{1\le i\le h}(r_i + v_i) - r. \tag{3.49}$$

Then Q is the least common multiple of $Q_1, \ldots, Q_h$. Moreover,

$$Q = Y_i Q_i \quad (i = 1,\ldots,h), \tag{3.50}$$

where

$$(3.51)\qquad Y_i := S^{(\varepsilon)}_{v-1,v-\gamma_i} T^{(\varepsilon)}_{v-\gamma_i,v_i-1} \quad (i = 1,\dots,h),$$

where, in turn,

$$\gamma_i := v + r - (v_i + r_i) \quad (i = 1,\dots,h).$$

Proof. First, we will show that equation (3.50) is valid, i.e. that Q is a common multiple of $Q_1,\dots,Q_h$. Let us transform the right-hand side of this equation by substituting the expressions for Y_i and Q_i ((3.51) and (3.47)), and using (3.46), (3.24) and (3.25). We obtain consecutively

$$\begin{aligned} Y_i Q_i &= S^{(\varepsilon)}_{v-1,v-\gamma_i} T^{(\varepsilon)}_{v-\gamma_i,v_i-1} P^{(\varepsilon)}_{v_i} D^{r_i} \\ &= S^{(\varepsilon)}_{v-1,v-\gamma_i} P^{(\varepsilon)}_{v-\gamma_i} D^{r-r_i} D^{r_i} = P^{(\varepsilon)}_v D^r = Q. \end{aligned}$$

Now we will show that Q is the least common multiple of $Q_1,\dots,Q_h$.

1° If there exists $i_0 \in \{1,\dots,h\}$ such that $r = r_{i_0}$ and $v + r = v_{i_0} + r_{i_0}$ then also $v = v_{i_0}$ and, according to (3.51), (3.24) and (3.42), we have

$$Y_{i_0} = S^{(\varepsilon)}_{v-1,v} T^{(\varepsilon)}_{v,v-1} = I.$$

Thus Q is identical with Q_{i_0} and obviously has the property in question.

2° If no index has the above property, there exist $i_1, i_2 \in \{1,\dots,h\}$ such that $i_1 \neq i_2$, $r = r_{i_1} > r_{i_2}$ and $v + r = v_{i_2} + r_{i_2} > v_{i_1} + r_{i_1}$ (hence $v_{i_2} > v > v_{i_1}$). Equation (3.50) yields, after some simplification,

$$Y_{i_1} = S^{(\varepsilon)}_{v-1,v_{i_1}}, \qquad Y_{i_2} = T^{(\varepsilon)}_{v,v_{i_2}-1}.$$

Suppose that Y_{i_1} and Y_{i_2} have a common left-hand factor of positive order. Then there exists either an operator $W \in \mathcal{L}\backslash\{\Theta\}$ such that $Y_{i_1} = R^{(\varepsilon)}_v W$ or an operator $Z \in \mathcal{L}\backslash\{\Theta\}$ such that $Y_{i_2} = A^{(\varepsilon)}_{v-1} Z$. Consider the first case (the second can be treated in an analogous way). The operator

$$Q' := P^{(\varepsilon)}_{v+1} D^{r-1}$$

is then a common multiple of Q_{i_1} and Q_{i_2}, i.e.

$$Q' = W Q_{i_1} = T^{(\varepsilon)}_{v+1,v_{i_2}-1} Q_{i_2}.$$

(Note that $Q = R^{(\varepsilon)}_v Q'$.) Substituting in the first equation the expression for Q_{i_1} obtained from (3.47), we get

$$P^{(\varepsilon)}_{v+1} D^{r-1} = W P^{(\varepsilon)}_u D^r \quad (u := v_{i_1}),$$

which yields $P^{(\varepsilon)}_{v+1} = W P^{(\varepsilon)}_u D$. As $P^{(\varepsilon)}_u D = R^{(\varepsilon)}_u P^{(\varepsilon)}_{u+1}$ (cf. (3.46)), we obtain

$$(3.52)\qquad S^{(\varepsilon)}_{v,u+1} = W R^{(\varepsilon)}_u.$$

Multiplying (3.52) on the left by $R_v^{(\varepsilon)}$ and making use of

$$R_v^{(\varepsilon)} S_{v,u+1}^{(\varepsilon)} = S_{v-1,u}^{(\varepsilon)} R_u^{(\varepsilon)}$$

(cf. (3.45)), we obtain, after some simple algebra,

$$S_{v-1,u}^{(\varepsilon)} = R_v^{(\varepsilon)} W . \tag{3.53}$$

The equations which are obtained by multiplication of (3.52) on the right by $A_v^{(\varepsilon)}$, and of (3.53) on the left by $A_v^{(\varepsilon)}$, have identical left-hand sides, hence the equality of their right-hand sides follows:

$$W R_u^{(\varepsilon)} A_u^{(\varepsilon)} = A_v^{(\varepsilon)} R_v^{(\varepsilon)} W . \tag{3.54}$$

Taking into consideration the orders and the form of the operators $S_{v,u+1}^{(\varepsilon)}$ and $R_u^{(\varepsilon)}$ it is easy to deduce from (3.52) that

$$W = \sum_{i=0}^{d} \omega_i(k) E^{i+1} ,$$

where $d := v - u - 1$, and where ω_i are rational functions. Putting the above expression, as well as the expressions for $A_u^{(\varepsilon)}$, $A_v^{(\varepsilon)}$, $R_v^{(\varepsilon)}$ and $R_u^{(\varepsilon)}$ obtained from Definitions 3.1 and 3.2, into (3.54), performing the multiplications and equating the coefficients of E^{d+1} on both sides, we obtain

$$\varphi_u(k+d+1)\omega_d(k) = \varphi_v(k)\omega_d(k+1) ,$$

where $\varphi_i(k) := \varrho_i^{(\varepsilon)}(k)\tau_i^{(\varepsilon)}(k)$ ($i \geq 0$; cf. (3.41) and (3.23)). Hence

$$\omega_d(k) = \text{const.} \prod_{i=1}^{k-1} \frac{\varphi_u(i+d+1)}{\varphi_v(i)} .$$

However, the above formula means that ω_d is not a rational function of k, thus W is not in $\mathcal{L}$. This contradiction ends the proof. ∎

LEMMA 3.5. *Let Q_1, Q_2 be the difference operators*

$$Q_1 := P_v^{(\varepsilon)} D^r , \tag{3.55}$$

$$Q_2 := P_u^{(-\varepsilon)} D^s , \tag{3.56}$$

where $\varepsilon \in \{-1, 1\}$ and v, r, u, s are nonnegative integers such that $v + r \geq u + s$. Then the operator

$$Q := P_w^{(\varepsilon)} D^t , \tag{3.57}$$

where

$$t := \max\{u+s, r\} , \qquad w := v + r - t , \tag{3.58}$$

is the least common multiple of Q_1, Q_2. Moreover,

$$Q = W_i Q_i \quad (i = 1, 2) , \tag{3.59}$$

where

$$W_1 := T^{(\varepsilon)}_{w,v-1}\,, \tag{3.60}$$

$$W_2 := P^{(\varepsilon)}_w D^{t-u-s} U^{(-\varepsilon)}_u\,. \tag{3.61}$$

Proof. Equation (3.59) can be verified by putting in it the expressions for Q, W_i, Q_i given by (3.57), (3.60), (3.55), (3.56), and using (3.46).

If $r \geq u+s$ then, according to (3.58), we have $t = r$, $w = v$, so that Q and Q_1 are identical in this case.

If $r < u+s$ then $t = u+s$, $w = v+r-u-s$ (cf. (3.58)), and equations (3.60), (3.61) yield

$$W_1 = T^{(\varepsilon)}_{w,r-1}\,, \quad W_2 = P^{(\varepsilon)}_w U^{(-\varepsilon)}_u\,.$$

It can be shown, using an argument similar to that of the second part of the proof of Lemma 3.4, that W_1 and W_2 have no common left factor of positive order. ∎

4. Recurrence relation for the Jacobi coefficients. In this section we give a description of a method for constructing a recurrence relation for the Jacobi coefficients (1.6) of the function f satisfying the differential equation (1.5). The main result of this work is contained in Theorem 4.1.

Let $\mathbf{P}_n$ be the differential operator of order n,

$$\mathbf{P}_n := \sum_{i=0}^{n} p_{ni}\mathbf{D}^i\,, \tag{4.1}$$

where $p_{n0}, p_{n1}, \ldots, p_{nn}$ are polynomials, $p_{nn} \not\equiv 0$. Define the differential operators $\mathbf{P}_i$ $(i = 0, \ldots, n-1)$, $\mathbf{Q}_j$ $(j = 1, \ldots, n)$, and the polynomials q_m $(m = 0, \ldots, n)$ in the following recursive way.

For $i = n, n-1, \ldots, 1$, given the operator $\mathbf{P}_i$,

$$\mathbf{P}_i = \sum_{j=0}^{i} p_{ij}\mathbf{D}^j\,, \tag{4.2}$$

where p_{ij} $(j = 0, \ldots, i)$ are polynomials, and given $\varepsilon \in \{-1, 1\}$, nonnegative integers ϱ, σ, and a polynomial w_i such that $w_i(-1) \neq 0$, $w_i(1) \neq 0$, and

$$p_{ii}(x) = (x^2-1)^{\varrho}(x+\varepsilon)^{\sigma} w_i(x)\,, \tag{4.3}$$

1° define the differential operator $\mathbf{Q}_i$ by

$$\mathbf{Q}_i := \begin{cases} \mathbf{J}^{m-\omega}\mathbf{U}^{\omega} & \text{if } \varrho \geq m \quad \text{(case A)},\\ \mathbf{K}^{m-\varrho-\omega}\mathbf{V}^{\omega}_{\varepsilon}\mathbf{J}^{\varrho} & \text{if } \varrho+\sigma \geq m > \varrho \quad \text{(case B)},\\ \mathbf{D}^{i-2\varrho-2\sigma}\mathbf{K}^{\sigma}_{\varepsilon}\mathbf{J}^{\varrho} & \text{if } m > \varrho+\sigma \quad \text{(case C)}, \end{cases} \tag{4.4}$$

where $m := \lfloor (i+1)/2 \rfloor$, $\omega := i - 2m$, and $\mathbf{U}$, $\mathbf{J}$, $\mathbf{V}_{\varepsilon}$, $\mathbf{K}_{\varepsilon}$ are introduced in (3.19), (3.20), 2° define the polynomial q_i by

$$q_i(x) := w_i(x) \begin{cases} (x^2-1)^{\varrho-m}(x+\varepsilon)^{\sigma} & \text{(case A)}, \\ (x+\varepsilon)^{\varrho+\sigma-m} & \text{(case B)}, \\ 1 & \text{(case C)}, \end{cases} \tag{4.5}$$

3° define the operator $\mathbf{P}_{i-1}$ by

$$\mathbf{P}_{i-1}f := \mathbf{P}_i f - \mathbf{Q}_i(q_i f)\,. \tag{4.6}$$

Remark 4.1. It follows readily from (4.2)–(4.5) that (4.6) actually defines a linear differential operator of order $i-1$.

Remark 4.2. It is convenient to introduce the notation

$$\mathbf{Q}_0 := \mathbf{I}\,, \quad q_0 := p_{00}\,.$$

Here p_{00} is the only coefficient of the operator $\mathbf{P}_0$.

Remark 4.3. It is easy to observe that the differential operator (4.1) may be written in the form

$$\mathbf{P}_n f = \sum_{j=i}^{n} \mathbf{Q}_j(q_j f) + \mathbf{P}_{i-1} f \tag{4.7}$$

for any $i = 1, \ldots, n$. In particular,

$$\mathbf{P}_n f = \sum_{i=0}^{n} \mathbf{Q}_i(q_i f)\,. \tag{4.1'}$$

Remark 4.4. Note that (4.4) is equivalent to

$$\mathbf{Q}_i = \mathbf{D}^{r_i}\mathbf{K}_{\varepsilon_i}^{s_i}\mathbf{V}_{\varepsilon_i}^{\delta_i}\mathbf{J}^{t_i}\mathbf{U}^{\delta_i'}\,, \tag{4.8}$$

where

$$\begin{cases} r_i := i - 2s_i - 2t_i - \delta_i - \delta_i'\,, \\ s_i := \min\{\sigma, m - t_i - \omega\}\,, \\ t_i := \min\{\varrho, m-\omega\}\,, \\ \delta_i := \begin{cases} \omega & \text{(case B)}, \\ 0 & \text{(case A or C)}, \end{cases} \\ \delta_i' := \begin{cases} \omega & \text{(case A)}, \\ 0 & \text{(case B or C)}, \end{cases} \\ \varepsilon_i := \begin{cases} \varepsilon & (\sigma > 0), \\ 1 & (\sigma = 0), \end{cases} \end{cases} \tag{4.9}$$

and where m, ω, ϱ, σ, ε are as in (4.4).

Also, notice that (4.4) simplifies considerably for small values of i. In particular,

$$\mathbf{Q}_1 = \begin{cases} \mathbf{U} & (\varrho > 0), \\ \mathbf{V}_\varepsilon & (\varrho = 0,\ \sigma > 0), \\ \mathbf{D} & (\varrho = \sigma = 0), \end{cases} \tag{4.10}$$

$$\mathbf{Q}_2 = \begin{cases} \mathbf{J} & (\varrho > 0), \\ \mathbf{K}_\varepsilon & (\varrho = 0,\ \sigma > 0), \\ \mathbf{D}^2 & (\varrho = \sigma = 0), \end{cases} \tag{4.11}$$

$$\mathbf{Q}_3 = \begin{cases} \mathbf{JU} & (\varrho > 1), \\ \mathbf{V}_\varepsilon\mathbf{J} & (\varrho = 1,\ \sigma > 0), \\ \mathbf{K}_\varepsilon\mathbf{V}_\varepsilon & (\varrho = 0,\ \sigma > 1), \\ \mathbf{DJ} & (\varrho = 1,\ \sigma = 0), \\ \mathbf{DK}_\varepsilon & (\varrho = 0,\ \sigma = 1), \\ \mathbf{D}^3 & (\varrho = \sigma = 0), \end{cases} \tag{4.12}$$

where ϱ, σ are as in (4.3) with $i = 1, 2, 3$, respectively.

Let us introduce the notation

$$\Omega := \{1, \dots, n\}, \quad \Omega_\eta := \{i \in \Omega : \varepsilon_i = \eta\} \quad (\eta = \pm 1), \tag{4.13}$$

$$v_i := 2s_i + \delta_i \quad (i \in \Omega), \tag{4.14}$$

$$e_\eta := \max_{i \in \Omega_\eta} r_i, \quad d_\eta := \max_{i \in \Omega_\eta}(v_i + r_i) - e_\eta \quad (\eta = \pm 1), \tag{4.15}$$

$$\varepsilon := \begin{cases} 1 & (d_1 + e_1 \geq d_{-1} + e_{-1}), \\ -1 & (d_1 + e_1 < d_{-1} + e_{-1}), \end{cases} \tag{4.16}$$

$$e := \max\{e_\varepsilon, d_{-\varepsilon} + e_{-\varepsilon}\}, \quad d := d_\varepsilon + e_\varepsilon - e. \tag{4.17}$$

Here ε_i, s_i, δ_i, r_i are the constants defined by (4.9). If the set Ω_η is empty for some η, we assume that both maxima in (4.15) are zero.

Further, define the difference operators P, $W^{(\eta)}$ ($\eta = \pm 1$), Y_i ($i \in \Omega$), Z_i ($i \in \Omega \cup \{0\}$) by

$$P := P_d^{(\varepsilon)} D^e, \tag{4.18}$$

$$W^{(\eta)} := \begin{cases} T^{(\varepsilon)}_{d, d_\varepsilon - 1} & (\eta = \varepsilon), \\ P_d^{(\varepsilon)} D^{e - e_{-\varepsilon} - d_{-\varepsilon}} U^{(-\varepsilon)}_{d_{-\varepsilon}} & (\eta = -\varepsilon), \end{cases} \tag{4.19}$$

$$\left.\begin{aligned} Y_i &:= S^{(\eta)}_{d_\eta - 1, d_\eta - \gamma_i} T^{(\eta)}_{d_\eta - \gamma_i, v_i - 1} && (4.20) \\ Z_i &:= W^{(\eta)} Y_i && (4.21) \end{aligned}\right\} \quad (i \in \Omega_\eta; \eta = \pm 1),$$

$$Z_0 := P, \tag{4.22}$$

where

$$\gamma_i := d_\eta + e_\eta - v_i - r_i \quad (i \in \Omega_\eta; \eta = \pm 1). \tag{4.23}$$

Finally, define the operators M_i by

$$M_i := \mu_{v_i}^{(\varepsilon_i)}(k) E^{s_i} P_{\delta_i}^{(-\varepsilon_i)} \varkappa^{t_i+\delta'_i}(k) D^{\delta'_i} \quad (i \in \Omega), \tag{4.24}$$

where $\varkappa(k) := k(k+\lambda)$, and $\mu_{v_i}^{(\varepsilon_i)}(k)$ is given in (3.25).

THEOREM 4.1. *Let f be a solution of*

$$\mathbf{P}_n f = q, \tag{4.25}$$

where $\mathbf{P}_n$ is the differential operator (4.1). *Suppose the derivative $f^{(n)}$ can be expanded in a uniformly convergent Jacobi series. Then the Jacobi coefficients of f obey the recurrence relation*

$$L b_k[f] = \omega(k), \tag{4.26}$$

where $L \in \mathcal{L}$, $\omega \in \mathcal{S}$,

$$L := \sum_{i=0}^{n} Z_i M_i q_i(X), \tag{4.27}$$

$$\omega(k) := P b_k[q]. \tag{4.28}$$

The order of the recurrence (4.26) *is*

$$r = d + 2e + 2 \max_{0 \le i \le n, p_{ni} \not\equiv 0} (\partial p_{ni} - i). \tag{4.29}$$

Proof. It suffices to show that the pair of the difference operators $\langle P, L\rangle$, given by (4.18) and (4.27), belongs to the set $\Pi(\mathbf{P}_n)$. To this end, we will use the equation (4.1′),

$$\mathbf{P}_n f = \sum_{i=0}^{n} \mathbf{Q}_i(q_i f),$$

and will show that the operator P is a common multiple of the first elements of the pairs $\langle Q_i, M_i\rangle$ which generate the sets $\Pi(\mathbf{Q}_i)$ $(i = 0, \ldots, n)$.

More specifically, we will show that P is the least common multiple of the operators

$$Q_i := P_{v_i}^{(\varepsilon_i)} D^{r_i} \quad (i = 1, \ldots, n) \tag{4.30}$$

and that

$$P = Z_i Q_i \quad (i = 1, \ldots, n), \tag{4.31}$$

Z_i being the operators (4.21). By Lemma 3.4, the operator

$$N^{(\eta)} := P_{d_\eta}^{(\eta)} D^{\eta} \quad (\eta = \pm 1),$$

where d_η, e_η are the numbers defined in (4.15), is the least common multiple of all the operators in $\{Q_i : i \in \Omega_\eta\}$; moreover,

$$N^{(\eta)} = Y_i Q_i \quad (i \in \Omega_\eta; \eta = \pm 1), \tag{4.32}$$

where Y_i is given by (4.20) for any $i \in \Omega$. Now, using Lemma 3.5 we see that P is the least common multiple of $N^{(1)}$ and $N^{(-1)}$, and that

$$P = W^{(\eta)} N^{(\eta)} \quad (\eta = \pm 1),$$

$W^{(\eta)}$ being given by (4.19). Hence, in view of (4.32) and (4.21), equation (4.31) follows.

By Lemma 3.3, the pair $\langle Q_i, M_i \rangle$, where M_i is the operator defined in (4.24), generates the set $\Pi(\mathbf{Q}_i)$ $(i = 1, \ldots, n)$; hence

$$Q_i b_k[\mathbf{Q}_i f] = M_i b_k[f] \quad (i = 0, \ldots, n).$$

Here we set $Q_0 := I$, $M_0 := I$, for the sake of symmetry.

Applying P to both sides of (4.1′), using (4.31), (3.6) and the above equation, we get

$$\begin{aligned} P b_k[\mathbf{P}_n f] &= \sum_{i=0}^{n} Z_i Q_i b_k[\mathbf{Q}_i(q_i f)] = \sum_{i=0}^{n} Z_i M_i b_k[q_i f] \\ &= \Big\{ \sum_{i=0}^{n} Z_i M_i q_i(X) \Big\} b_k[f] = L b_k[f], \end{aligned}$$

where $L \in \mathcal{L}$ is given by (4.27). Thus $\langle P, L \rangle \in \Pi(\mathbf{P}_n)$.

Now, in view of the remarks given in Section 3 (see the fragment containing the formulae (3.9)–(3.11)), we readily obtain the recurrence relation (4.26) as well as the expression (4.29) for its order. (Obviously, we have $r(P) = r(P_d^{(\epsilon)}) + r(D^e) = d + 2e$.) ■

Remark 4.5. Lemma 3.1 may be used to obtain the following simplification of the method. Recall that the differential operator (4.1) can be written in the form

$$\mathbf{P}_n f = \sum_{j=i}^{n} \mathbf{Q}_j(q_j f) + \mathbf{P}_{i-1} f$$

for any $i \in \Omega$ (cf. (4.7)). Now, let i be the greatest (i.e. the first one met in the course of the computation) index such that the leading coefficient p_{ii} of the differential operator $\mathbf{P}_i$ (see (4.2)) satisfies $p_{ii}(-1) \neq 0$ and $p_{ii}(1) \neq 0$. Then, according to Lemma 3.1, the pair $\langle Q_i, L_i \rangle$,

$$Q_i := D^i, \quad L_i := \sum_{j=0}^{i} D^{i-j} q_j(X),$$

where

$$q_j := \sum_{h=j}^{i} (-1)^{h-j} \binom{h}{j} \mathbf{D}^{h-j} p_{ih} \quad (j = 0, \ldots, i), \tag{4.33}$$

generates $\Pi(\mathbf{P}_i)$. Without affecting its validity, we may put in Theorem 4.1 $\mathbf{Q}_j := \mathbf{D}^j$ $(j = 0, \ldots, i)$, and assume that the polynomials $q_0, \ldots, q_i$ are defined by (4.33). (Of course, we should assume that $r_j = j$, $s_j = \delta_j = t_j = \delta'_j = 0$ for $j = 1, \ldots, i$ in (4.13)–(4.24)).

Some partial results as well as results of numerical experiments led us to the following conjecture.

CONJECTURE 4.1. The pair $\langle P, L\rangle$ given by (4.18) and (4.26) generates $\Pi(\mathbf{P}_n)$. In other words, among all recurrence relations with coefficients from $\mathcal{S}_{\mathrm{rat}}$ satisfied by the Jacobi coefficients of the function f which may be obtained from the differential equation (4.25) using the identities (3.1), (3.2), the relation (4.26) has the lowest order.

Remark 4.6. Theorem 4.2 of [2] implies that the above conjecture is true in the case of $n = 1$.

EXAMPLE 4.1. Let

$$f(x) = {}_3F_2\left(\begin{matrix}\varphi_1, \varphi_2, \varphi_3\\ \psi_1, \psi_2\end{matrix}\,\middle|\,\frac{1+x}{2}\right), \tag{4.34}$$

where ${}_3F_2$ denotes the generalized hypergeometric function, and where φ_1, φ_2, φ_3, ψ_1, ψ_2 are complex parameters such that

$$\mathrm{re}(\varphi_1 + \varphi_2 + \varphi_3 - \psi_1 - \psi_2) < 0\,. \tag{4.35}$$

It is known that f can be expanded in a Jacobi series (1.1) with coefficients

$$\begin{aligned} a_k^{(\alpha,\beta)}[f] \\ = \frac{(\varphi_1)_k(\varphi_2)_k(\varphi_3)_k}{(k+\lambda)_k(\psi_1)_k(\psi_2)_k}\, {}_4F_3\left(\begin{matrix}k+\varphi_1, k+\varphi_2, k+\varphi_3, k+\beta+1\\ k+\psi_1, k+\psi_2, 2k+\lambda+1\end{matrix}\,\middle|\,1\right)\end{aligned} \tag{4.36}$$

(see [6], Vol. 2, Sect. 9.3). The following differential equation can be obtained using a linear differential equation satisfied by ${}_3F_2(t)$ (see, e.g., [1], Vol. 1, Sect. 4.2; or [6], Vol. 1, Sect. 5.1):

$$\mathbf{P}_3 f \equiv (x^2-1)(x+1)f''' + (x+1)(c_{32}x + d_{32})f'' + (c_{31}x + d_{31})f' + c_{30}f = 0\,,$$

where

$$\begin{aligned} &c_{30} := \varphi_1\varphi_2\varphi_3\,, \quad c_{31} := \varphi_1\varphi_2 + \varphi_2\varphi_3 + \varphi_1\varphi_3 + \varphi_1 + \varphi_2 + \varphi_3 + 1\,,\\ &c_{32} := \varphi_1 + \varphi_2 + \varphi_3\,, \quad d_{31} := c_{31} - 2\psi_1\psi_2\,, \quad d_{32} := c_{32} - 2\psi_1 - 2\psi_2 - 2\,. \end{aligned}$$

Thus we have

$$p_{33}(x) = (x^2-1)(x+1)\,, \quad \varrho = \sigma = \varepsilon = 1, \quad w_3 \equiv 1$$

(cf. (4.3)), and

$$\begin{gathered} \mathbf{Q}_3 = \mathbf{V}_1\mathbf{J}\,, \quad q_3 \equiv 1\,,\\ r_3 = s_3 = \delta'_3 = 0\,, \quad \varepsilon_3 = \delta_3 = t_3 = 1 \end{gathered}$$

(cf. (4.8), (4.9)). Further, we get

$$\mathbf{P}_2 f = \mathbf{P}_3 f - \mathbf{Q}_3(q_3 f) = (x+1)(c_{22}x + d_{22})f'' + (c_{21}x + d_{21})f' + c_{30}f\,,$$

where

$$c_{22} := c_{32} - \lambda - \beta - 4\,, \qquad d_{22} := d_{32} - \alpha + 2\beta + 1\,,$$
$$c_{21} := c_{31} - (\lambda+1)(\beta+2)\,,\ d_{21} := d_{31} - \lambda - 1 - (\beta+1)(\alpha-\beta)\,.$$

The form of the coefficient

$$p_{22}(x) = (x+1)(c_{22}x + d_{22})$$

of the derivative f'' implies

$$r_2 = \delta_2 = t_2 = \delta_2' = 0\,, \quad s_2 = \varepsilon_2 = 1\,,$$
$$\mathbf{Q}_2 = \mathbf{K}_1\,, \quad q_2(x) = w_2(x) = c_{22}x + d_{22}\,.$$

(Note that $w_2(1) = 0$ or $w_2(-1) = 0$ for some values of the parameters φ_i, ψ_i. However, in the former case, nothing essentially new is brought in, while in the latter case, the third derivative of f, which is a constant multiple of

$${}_3F_2\left(\begin{matrix}\varphi_1+3, \varphi_2+3, \varphi_3+3\\ \psi_1+3, \psi_2+3\end{matrix}\middle|\frac{1+x}{2}\right),$$

does not fulfil a condition analogous to (4.35), which precludes the uniform convergence of the Jacobi series for this function.)

Using again the formulae (4.2)–(4.6), we get

$$\mathbf{P}_1 f = \mathbf{P}_2 f - \mathbf{Q}_2(q_2 f) = (c_{11}x + d_{11})f' + c_{10}f\,,$$

where

$$c_{11} := c_{21} - (\beta+3)c_{22}\,,\ d_{11} := d_{21} - (\beta+1)d_{22} - 2c_{22}\,,\ c_{10} := c_{30} - (\beta+1)c_{22}\,,$$

and

$$q_1(x) = p_{11}(x) = c_{11}x + d_{11}, \quad r_1 = \varepsilon_1 = 1\,,$$
$$s_1 = \delta_1 = t_1 = \delta_1' = 0\,, \quad \mathbf{Q}_1 = \mathbf{D}\,.$$

Finally, we have

$$p_{00} := c_{10} - c_{11}\,, \quad \mathbf{Q}_0 := \mathbf{I}\,, \quad q_0 := p_{00}\,.$$

Making use of the formulae (4.13)–(4.24), we obtain:

$$\Omega = \Omega_1 = \{1,2,3\}\,, \quad \Omega_{-1} = \emptyset; \quad v_1 = 0\,, \quad v_2 = 2\,, \quad v_3 = 1\,;$$
$$r_1 = 1\,, \quad r_2 = r_3 = 0\,; \quad e_{-1} = d_{-1} = 0\,, \quad e_1 = d_1 = 1\,;$$
$$\varepsilon = 1\,; \quad e = d = 1\,; \quad P = P_1^{(1)}D\,, \quad W^{(1)} = I \quad (W^{(-1)} = P)\,;$$
$$\gamma_1 = \gamma_3 = 1\,, \quad \gamma_2 = 0\,;$$
$$Z_1 = Y_1 = A_0^{(1)}\,, \quad Z_2 = Y_2 = A_1^{(1)}\,, \quad Z_3 = Y_3 = A_0^{(1)}R_0^{(1)}\,;$$
$$M_0 = M_1 = I\,, \quad M_2 = \mu_2^{(1)}(k)E\,, \quad M_3 = \mu_1^{(1)}(k)A_0^{(-1)}\varkappa(k)I\,.$$

By Theorem 4.1, the Jacobi coefficients of the function (4.34) obey the recurrence relation of the third order:

$$Lb_k[f] = 0\,,$$

where

$$L = A_0^{(1)} R_0^{(1)} M_3 + R_1^{(1)} M_2 q_2(X) + A_0^{(1)} q_1(X) + q_0 P\,.$$

The obtained result is equivalent to the one obtained in [4], using another approach.

5. Recurrence relation for the Gegenbauer coefficients. In the special case $\alpha = \beta$ of the Jacobi series (1.1), it is more convenient to deal with the *Gegenbauer series* expansion

$$f = \sum_{k=0}^{\infty} g_k^{(\nu)}[f] C_k^{(\nu)} \quad (\nu > -1/2) \tag{5.1}$$

of f, where $C_k^{(\nu)}$ is the kth Gegenbauer polynomial,

$$\begin{aligned} C_k^{(\nu)} &:= \frac{(2\nu)_k}{(\nu+1/2)_k} P_k^{(\nu-1/2,\nu-1/2)} \quad (\nu \neq 0)\,, \\ C_k^{(0)} &:= \lim_{\nu\to 0} \nu^{-1} C_k^{(\nu)}\,. \end{aligned} \tag{5.2}$$

The coefficients $g_k^{(\nu)}[f]$ are

$$g_k^{(\nu)}[f] = \frac{(k+\nu)k!\Gamma(\nu)}{\sqrt{\pi}(2\nu)_k \Gamma(\nu+1/2)} \int_{-1}^{1} (1-x^2)^{\nu-1/2} C_k^{(\nu)}(x) f(x)\, dx\,. \tag{5.3}$$

The case $\nu = 0$, distinguished in (5.2), is closely connected with the *Chebyshev series* expansion

$$f = t_0[f]T_0/2 + \sum_{k=1}^{\infty} t_k[f] T_k \tag{5.4}$$

of f, where T_k is the kth Chebyshev polynomial of the first kind,

$$T_k = \frac{k!}{(1/2)_k} P_k^{(-1/2,-1/2)} = \frac{2}{k} C_k^{(0)}\,, \tag{5.5}$$

and where the *Chebyshev coefficients* $t_k[f]$ of f are

$$t_k[f] = \frac{2}{\pi} \int_{-1}^{1} (1-x^2)^{-1/2} T_k(x) f(x)\, dx \quad (k = 0, 1, \ldots)\,. \tag{5.6}$$

We will call the quantities

$$c_k[f] \equiv c_k^{(\nu)}[f] := 2(k+\nu-\tfrac{1}{2}) b_k^{(\nu-1/2,\nu-1/2)}[f] \tag{5.7}$$

the *Gegenbauer coefficients* of the function f. They are related to the coefficients (5.3) and (5.6) in the following way:

$$\text{(5.8)} \qquad \begin{aligned} c_k^{(\nu)}[f] &= \frac{\sqrt{\pi}}{\Gamma(\nu)(k+\nu)} g_k^{(\nu)}[f] \quad (\nu \neq 0), \\ c_k^{(0)}[f] &= \frac{\sqrt{\pi}}{4} t_k[f]. \end{aligned}$$

A recurrence relation for the Gegenbauer coefficients can be constructed by a method analogous to that used in Section 4. However, in the present section we obtain neater looking results.

First of all, the basic identities (3.1), (3.2) may be replaced by

$$\text{(5.9)} \qquad \widehat{X} c_k[f] = c_k[xf(x)],$$

$$\text{(5.10)} \qquad \widehat{D} c_k[\mathbf{D}f] = c_k[f],$$

where $\widehat{X}$ and $\widehat{D}$ are the following difference operators:

$$\text{(5.11)} \qquad \widehat{X} := \frac{1}{2(k+\nu)}(kE^{-1} + (k+2\nu)E),$$

$$\text{(5.12)} \qquad \widehat{D} := \frac{1}{2(k+\nu)}(E^{-1} - E).$$

Further, the differential operators (3.19), (3.20) take the following somewhat simpler forms:

$$\text{(5.13)} \qquad \mathbf{U} := (x^2-1)\mathbf{D} + (2\nu+1)x\mathbf{I}, \quad \mathbf{J} := \mathbf{U}\mathbf{D},$$

$$\text{(5.14)} \qquad \mathbf{V}_\varepsilon := (x+\varepsilon)\mathbf{D} + (\nu+1/2)\mathbf{I}, \quad \mathbf{K}_\varepsilon := \mathbf{V}_\varepsilon\mathbf{D} \quad (\varepsilon = \pm 1).$$

DEFINITION 5.1 ([5]). For any $m = 0, 1, \ldots$ and $\varepsilon \in \{-1, 1\}$ we define the difference operator $\widehat{A}_m^{(\varepsilon)} \in \mathcal{L}$ by

$$\widehat{A}_m^{(\varepsilon)} := I - \varepsilon\tau_m(k)E, \quad \text{where} \quad \tau_m(k) := \frac{(2k+2\nu+1)_2}{(2k+2\nu+m+1)_2}.$$

Further, let

$$\widehat{S}_{ij}^{(\varepsilon)} := \begin{cases} I & (i < j), \\ \widehat{A}_i^{(\varepsilon)} \widehat{S}_{i-1,j}^{(\varepsilon)} & (i \geq j \geq 0), \end{cases}$$

$$\widehat{P}_h^{(\varepsilon)} := \widehat{S}_{h-1,0}^{(\varepsilon)} \quad (h = 0, 1, \ldots).$$

Finally, introduce the sequence $\{\mu_i\} \subset \mathcal{S}_{\text{rat}}$ by

$$\mu_i(k) := 2^{-\omega}(2k+2\nu+1)_i \quad (i = 0, 1, \ldots),$$

where $\omega := \lfloor (i+1)/2 \rfloor$.

It can be verified that

$$
(5.15)\qquad
\begin{aligned}
&c_k[\mathbf{U}f] = \varkappa(k)Dc_k[f] \\
&c_k[\mathbf{J}f] = \varkappa(k)c_k[f]
\end{aligned}
\qquad (\varkappa(k) := k(k+2\nu)),
$$

$$
\begin{aligned}
&P_1^{(\varepsilon)}c_k[\mathbf{V}_\varepsilon f] = \mu_1(k)P_1^{(-\varepsilon)}c_k[f] \\
&P_2^{(\varepsilon)}c_k[\mathbf{K}_\varepsilon f] = \mu_2(k)Ec_k[f]
\end{aligned}
\qquad (\varepsilon = \pm 1)
$$

(cf. (3.30)–(3.33)).

DEFINITION 5.2. For any $m = 0, 1, \ldots$ and $\varepsilon \in \{-1, 1\}$ we define the difference operator $\widehat{R}_m^{(\varepsilon)} \in \mathcal{L}$ by

$$\widehat{R}_m^{(\varepsilon)} := (2k+2\nu)^{-1}E^{-1} + \varepsilon\varrho_m(k)I,$$

where

$$\varrho_m(k) := (2k+2\nu+2m+1)/(2k+2\nu+m)_2 .$$

Further, let

$$\widehat{T}_{ij}^{(\varepsilon)} := \begin{cases} I & (i > j), \\ \widehat{R}_i^{(\varepsilon)}\widehat{T}_{i+1,j}^{(\varepsilon)} & (0 \le i \le j), \end{cases}$$

$$\widehat{U}_h^{(\varepsilon)} := \widehat{T}_{0,h-1}^{(\varepsilon)} \quad (h = 0, 1, \ldots).$$

It can be checked that all the identities and lemmata given in Section 3 may be repeated here with obvious modifications. Also, the results of Section 4 remain true for the Gegenbauer coefficients provided 1° the symbols $\mathbf{U}$, $\mathbf{J}$, $\mathbf{V}_\varepsilon$, $\mathbf{K}_\varepsilon$ in (4.4), (4.8), (4.10)–(4.12) are defined by (5.13), (5.14), 2° the symbols $P_d^{(\varepsilon)}$, D, $T_{ij}^{(\varepsilon)}$, $U_l^{(\varepsilon)}$, $S_{jh}^{(\varepsilon)}$ in (4.18)–(4.22), (4.24) are replaced by the appropriate symbols with hats, and 3° in (4.24), the factor $\mu_{v_i}^{(\varepsilon_i)}(k)$ is replaced by $\mu_{v_i}(k)$. In particular, the following theorem is true.

THEOREM 5.1. *Let f be a solution of the equation* (4.25), *and suppose $f^{(n)}$ can be expanded in a Gegenbauer series which is uniformly convergent in $[-1, 1]$. Then the Gegenbauer coefficients of f obey the recurrence relation*

$$Lc_k[f] = \omega(k),$$

where $L \in \mathcal{L}$, $\omega \in \mathcal{S}$,

$$L := \sum_{i=0}^{n} Z_iM_iq_i(X), \quad \omega(k) := Pc_k[q],$$

and the operators $P, Z_i, M_i \in \mathcal{L}$ and the polynomials q_i are defined as in Section 4, *with the modifications indicated above.*

Notice that Remark 4.5 and Conjecture 4.1, suitably modified, remain valid in the considered case. The latter can be proved in the case $n = 2$, using the apparatus introduced in [2]; the proof is based on the equation

$$\mathbf{P}_2 f = \mathbf{Q}_2(q_2 f) + \mathbf{Q}_1(q_1 f) + \mathbf{Q}_0(q_0 f)$$

(cf. (4.1′)), in which $\mathbf{Q}_2$ is either $\mathbf{J}$, $\mathbf{K}_\varepsilon$, or $\mathbf{D}^2$ (cf. (4.11)). The first and the last case can be treated easily (it suffices then to use (5.15) and [an equivalent of] Lemma 3.1), while the case where $\mathbf{Q}_2 = \mathbf{K}_\varepsilon$ for $\varepsilon = 1$ or $\varepsilon = -1$ needs longer considerations.

EXAMPLE 5.1. We construct a recurrence relation for the Chebyshev coefficients $t_k[f]$ of the function $f(x) = xe^x$, using the fact that f is a solution of

$$\mathbf{P}_3 f \equiv (x^2-1)f''' - (x^2-3x-1)f'' - (4x-1)f' - 3f = 0\,.$$

This example was first studied by Paszkowski in connection with the problem of minimizing the order of the recurrence relation for the Jacobi coefficients of f via a preliminary transformation of the differential equation (1.5) (see Section 1).

Now, according to the second part of (5.8) one should put $\nu = 0$ in the used formulae. Thus we have, in particular,

$$\widehat{X} = \frac{1}{2}(E^{-1}+E)\,, \quad \widehat{D} = \frac{1}{2k}(E^{-1}-E)\,, \quad \mathbf{J} = (x^2-1)\mathbf{D}^2 + x\mathbf{D}\,.$$

It can be checked that $\mathbf{P}_3 f = \mathbf{Q}_3 f - \mathbf{Q}_2(f) - \mathbf{Q}_1(3xf)$, where $\mathbf{Q}_3 := \mathbf{DJ}$, $\mathbf{Q}_2 := \mathbf{J}$, $\mathbf{Q}_1 := \mathbf{D}$. We may easily deduce from (5.10), (5.15) and (5.9) that the pair $\langle P, L\rangle$ given by

$$P := \widehat{D}\,, \quad L := k^2 I - \widehat{D}k^2 I - 3\widehat{X}$$

belongs to $\Pi(\mathbf{P}_3)$. Thus we obtain the second-order recurrence relation $Lt_k[f] = 0$, or, in scalar form,

$$(k^2+k+1)t_{k-1}[f] - 2k^3 t_k[f] - (k^2-k+1)t_{k+1}[f] = 0\,,$$

which is identical with Paszkowski's result [8].

Acknowledgement. The author would like to thank Professor S. Paszkowski for his stimulating comments and discussions on this problem.

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STANISŁAW LEWANOWICZ
INSTITUTE OF COMPUTER SCIENCE
UNIVERSITY OF WROCŁAW
UL. PRZESMYCKIEGO 20
51-151 WROCŁAW, POLAND

Received on 16.2.1990