ZASTOSOWANIA MATEMATYKI
APPLICATIONES MATHEMATICAE
21,3 (1992), pp. 335–341

H. BRZESKWINIEWICZ (Poznań)

# ON THE BASIC CONTRASTS IN PBIB DESIGNS

*Abstract.* The notion of an association scheme for the basic contrasts is introduced. A necessary and sufficient condition is given for the basic contrasts to determine a certain association scheme.

**1. Introduction.** This paper presents a new approach to the problem of basic contrasts in a PBIB design. The novelty consists in defining an association scheme by the basic contrasts. In the literature the basic contrasts have been discussed only in the analysis of variance corresponding to the design.

The main results are contained in Section 3. They are: Definition 1 of an association scheme of the basic contrasts, and Theorem 1. This theorem provides a necessary and sufficient condition for given basic contrasts to determine a certain association scheme. This yields very interesting corollaries, given in Sections 3 and 4.

**2. Preliminaries.** Let $v$, $n_i$, $p^i_{jk}$, $i,j,k = 1,\ldots,m$, be the parameters of an association scheme with association matrices $\mathbf{A}_0, \mathbf{A}_1, \ldots, \mathbf{A}_m$, where $m$ is the number of associate classes. The above matrices are all symmetric, linearly independent and satisfy the following conditions:

$$(1)\qquad \mathbf{A}_0 = \mathbf{I}, \quad \sum_{i=0}^{m} \mathbf{A}_i = \mathbf{1}\mathbf{1}', \quad \mathbf{A}_i\mathbf{1} = n_i\mathbf{1}, \quad \mathbf{A}_j\mathbf{A}_k = \sum_{i=0}^{m} P^i_{jk}\mathbf{A}_i,$$

where $\mathbf{I}$ and $\mathbf{1}$ are the unit $v \times v$ matrix and the column vector of $v$ ones, respectively.

Let $\mathbf{N} = (n_{ij})$ be the incidence matrix of a PBIB design in which each of $v$ treatments $(i = 1,\ldots,v)$ occurs $r$ times and each of $b$ blocks $(j = 1,\ldots,b)$

---

1991 *Mathematics Subject Classification*: Primary 62K10.

*Key words and phrases*: basic contrasts, PBIB design, association scheme, $F_p$ type association scheme, $C_p$ type association scheme.

is of size $k$. Then the matrix $\mathbf{NN}'$ can be expressed as

$$\mathbf{NN}' = \sum_{i=0}^{m} \lambda_i \mathbf{A}_i, \tag{2}$$

where the $\lambda_i$ are called the *coincidence numbers* of the PBIB design.

It is known (see, for example, [1]) that $\mathbf{NN}'$ can be equivalently expressed as

$$\mathbf{NN}' = \sum_{i=0}^{m} \varrho_i \mathbf{A}_i^{\#}, \tag{3}$$

where $\mathbf{A}_i^{\#}$ and $\varrho_i$ are orthogonal idempotent matrices and the latent roots of $\mathbf{NN}'$ with multiplicities $\alpha_i = \mathrm{tr}(\mathbf{A}_i^{\#})$, respectively.

It is also known that there is a relation between $\mathbf{A}_i^{\#}$ and $\mathbf{A}_i$ given by

$$\mathbf{A}_i^{\#} = \sum_{j=0}^{m} z^{ji} \mathbf{A}_j, \quad i = 0, 1, \ldots, m, \tag{4}$$

where the $(m+1) \times (m+1)$ matrix of real coefficients $z^{ji}$ is nonsingular.

**3. Main results.** Let

$$\mathbf{p}_{ij}, \quad i = 1, \ldots, m;\ j = 1, \ldots, \alpha_i, \tag{5}$$

be $v - 1 = \sum_{i=1}^{m} \alpha_i$ orthonormal column vectors satisfying the condition $\mathbf{p}_{ij}'\mathbf{1} = 0$ for every $i$, $j$. We introduce

DEFINITION 1. We say that the vectors $\mathbf{p}_{ij}$ in (5) *determine an association scheme with $m$ associate classes* if there exist symmetric binary matrices $\widetilde{\mathbf{A}}_j$, $j = 0, \ldots, m$, which satisfy (1) and

$$\sum_{j=1}^{\alpha_i} \mathbf{p}_{ij}\mathbf{p}_{ij}' = \sum_{j=0}^{m} \widetilde{z}^{ji} \widetilde{\mathbf{A}}_j, \quad i = 1, \ldots, m, \tag{6}$$

where $\widetilde{z}^{ji}$ are scalar constants.

Note that the matrices $\widetilde{\mathbf{A}}_j$ in (6) are the association matrices of the above scheme.

Let $\boldsymbol{\gamma}$ be the column vector of the treatment parameters; then the functions $\mathbf{p}_{ij}'\boldsymbol{\gamma}$ will be called the *basic contrasts*.

It can be seen that if $c_i = r - \varrho_i/k \neq 0$, then the contrasts $\mathbf{p}_{ij}'\boldsymbol{\gamma}$ are estimable with the same variance $\mathrm{Var}(\widehat{\mathbf{p}_{ij}'\boldsymbol{\gamma}}) = \sigma^2/c_i$ for $j = 1, \ldots, \alpha_i$, where $\sigma^2$ denotes the error variance of the intra-block analysis.

Now we are going to give a method for finding matrices $\widetilde{\mathbf{A}}_j$ satisfying (6) for given $\mathbf{p}_{ij}$, which is useful in practical applications of Definition 1.

From (4) and (6) we have

$$\mathbf{A}_i^{\#} = \sum_{j=1}^{\alpha_i} \mathbf{p}_{ij}\mathbf{p}'_{ij}\,, \quad i = 1, \ldots, m. \tag{7}$$

Of the matrices $\mathbf{A}_i^{\#}$, let $\mathbf{A}_l^{\#}$ be one with the greatest number, say $l^*$, of different elements.

If $l^* > m+1$ then the vectors $\mathbf{p}_{ij}$ cannot determine an association scheme with $m$ associate classes.

If $l^* = m+1$ then the matrix $\mathbf{A}_l^{\#}$ has $m+1$ different elements $\widetilde{z}^{0l}, \ldots, \widetilde{z}^{ml}$. We construct matrices $\widetilde{\mathbf{A}}_j$ by putting one in the place where $z^{jl}$ occurs, and zero in the remaining places.

If $l^* < m+1$ then we first construct $l^*$ matrices $\widetilde{\mathbf{A}}_j^1$ as above. If one of the matrices $\mathbf{A}_i^{\#}$ has $s \geq 2$ different elements in places of ones in $\widetilde{\mathbf{A}}_j^1$, then we divide $\widetilde{\mathbf{A}}_j^1$ into $s$ matrices corresponding to those $s$ elements. If it is not possible to construct $m+1$ matrices $\widetilde{\mathbf{A}}_j$ in this way, then the vectors $\mathbf{p}_{ij}$ cannot determine an association scheme with $m$ associate classes.

After determining the matrices $\widetilde{\mathbf{A}}_j$ we must check the conditions (1), according to Definition 1. It turns out that satisfying the first two conditions implies the fulfilment of the other ones, which is expressed in Theorem 1.

THEOREM 1. *The vectors* (5) *determine an association scheme with $m$ associate classes if and only if there exist binary matrices $\widetilde{\mathbf{A}}_i$, $i = 0, \ldots, m$, which satisfy*

(a) $\widetilde{\mathbf{A}}_0 = \mathbf{I}, \quad \sum_{i=0}^{m} \widetilde{\mathbf{A}}_i = \mathbf{1}\mathbf{1}',$

(b) $\mathbf{A}_i^{\#} = \sum_{j=0}^{m} \widetilde{z}^{ji}\widetilde{\mathbf{A}}_j\,, \quad i = 1, \ldots, m,$

*where $\mathbf{A}_i^{\#}$ are defined by* (7) *and $\widetilde{z}^{ji}$ are scalar constants. The matrices $\widetilde{\mathbf{A}}_i$ are the association matrices of the above scheme.*

Proof. The *necessity* is obvious: take $\mathbf{A}_i = \widetilde{\mathbf{A}}_i$.

*Sufficiency.* According to Definition 1 we must prove that the matrices $\widetilde{\mathbf{A}}_i$ are symmetric and satisfy the third and fourth conditions of (1). We first represent the matrices $\widetilde{\mathbf{A}}_i$ as linear combinations of the matrices $\mathbf{A}_i^{\#}$ and $\mathbf{A}_0^{\#} = \mathbf{1}\mathbf{1}'/v$. Since these matrices are linearly independent, the matrix $\widetilde{\mathbf{Z}}^{-1} = (\widetilde{z}^{ji})$, where $\widetilde{z}^{ji}$ denotes the $j$th element of $\mathbf{A}_i^{\#}$, is nonsingular. Hence $\widetilde{\mathbf{A}}_i = \sum_{l=0}^{m} \widetilde{z}_{li}\mathbf{A}_l^{\#}$, where $\widetilde{z}_{li}$ is the $(l,i)$th element of $\widetilde{\mathbf{Z}}$. It follows that

$\widetilde{\mathbf{A}}_i' = \widetilde{\mathbf{A}}_i$, $\widetilde{\mathbf{A}}_i\mathbf{1} = \widetilde{n}_i\mathbf{1}$ and $\widetilde{\mathbf{A}}_j\widetilde{\mathbf{A}}_k = \sum_{i=0}^m \widetilde{p}_{jk}^i\widetilde{\mathbf{A}}_i$, where $\widetilde{n}_i = \widetilde{z}_{0i}$ and

$$\widetilde{p}_{jk}^i = \sum_{l=0}^m \widetilde{z}_{lj}\widetilde{z}_{lk}\widetilde{z}^{il}, \quad i,j,k = 0,1,\ldots,m. \tag{8}$$

This completes the proof.

This useful theorem gives criteria for the vectors (5) to determine an association scheme. The following corollaries are also helpful:

COROLLARY 1. *Let* $\mathbf{p}_{ij} = (f_{ij}^1,\ldots,f_{ij}^v)'$ *for* $i = 1,\ldots,m$; $j = 1,\ldots,\alpha_i$. *If there exist* $i$, $s$ *and* $s'$ *such that either*

(i) $f_{ij}^s = 0$ *for* $j = 1,\ldots,\alpha_i$, *or*

(ii) $|f_{ij}^s| \geq |f_{ij}^{s'}|$ *for* $j = 1,\ldots,\alpha_i$ *and* $|f_{ij}^s| > |f_{ij_0}^{s'}|$ *for some* $j_0 \in \{1,\ldots,\alpha_i\}$, *or*

(iii) $\alpha_i = 1$ *and* $|f_{i1}^s| \neq |f_{i1}^{s'}|$,

*then the vectors* (5) *do not determine an association scheme with* $m$ *associate classes.*

Proof. Assume that (5) determines an association scheme. Then from (7) we have $\mathbf{A}_i^{\#} = (z_i^{ss'})$, where $z_i^{ss'} = \sum_{j=1}^{\alpha_i} f_{ij}^s f_{ij}^{s'}$ and, in particular,

$$z_i^{ss} = \sum_{j=1}^{\alpha_i}(f_{ij}^s)^2. \tag{9}$$

Conditions (a) and (b) of Theorem 1 imply that $z_i^{ss}$ are all equal, for $s = 1,\ldots,v$. From this and (i) it follows that $\mathbf{p}_{ij}$, $j = 1,\ldots,\alpha_i$, are the null vectors, which is impossible. In cases (ii) and (iii) the values $z_i^{ss}$ defined by (9) are not equal for fixed $i$, which is also impossible. This completes the proof.

From condition (iii) we get a very interesting conclusion:

COROLLARY 2. *If, in a PBIB design,* $\alpha_i = 1$ *for some* $i \in \{1,\ldots,m\}$, *then the number of treatments* $v$ *must be even.*

## 4. Examples

**4.1.** Let

$$\mathbf{w}_0^{(i)} = \mathbf{1}, \quad \mathbf{w}_1^{(i)} = (s_i - 1, -1,\ldots,-1)',$$
$$\mathbf{w}_2^{(i)} = (0, s_i - 2, -1,\ldots,-1)', \ \ldots, \ \mathbf{w}_{s_i-1}^{(i)} = (0,\ldots,0,1,-1)'$$

be $s_i \times 1$ vectors, where $s_i \geq 2$ for $i = 1,\ldots,p$. Assume there are $m = 2^p - 1$ associate classes, which are denoted by $(c_1,\ldots,c_p)$, where $c_i$ is 0 or 1 and $\sum_{i=1}^p c_i > 0$.

We define $W_{c_1 \dots c_p}$ as the set of all vectors of the form

$$\mathbf{w} = (\mathbf{w}_{e_1}^{(1)} \otimes \dots \otimes \mathbf{w}_{e_p}^{(p)})/\|\mathbf{w}_{e_p}^{(1)} \otimes \dots \otimes \mathbf{w}_{e_p}^{(p)}\|\,, \tag{10}$$

where $\|\mathbf{w}\| = \sqrt{\mathbf{w}'\mathbf{w}}$, $e_i = 0$ if $c_i = 0$ and $e_i$ is one of $1, \dots, s_i - 1$ if $c_i = 1$. The operator "$\otimes$" always denotes the Kronecker product of some matrices or vectors. Let

$$W = \Big\{\mathbf{w} \in W_{c_1 \dots c_p} : c_i = 0 \text{ or } 1,\ i = 1, \dots, p,\ \sum_{i=1}^{p} c_i > 0\Big\}$$

be the same vectors as in (5) with $m = 2^p - 1$.

COROLLARY 3. *The vectors $\mathbf{w} \in W$ determine an association scheme with $2^p - 1$ associate classes. The association matrices of the above scheme are of the form*

$$\mathbf{A}_{a_1 \dots a_p} = \bigotimes_{i=1}^{p} \mathbf{A}_{a_i}^{(i)}\,, \tag{11}$$

*where $a_i = 0$ or $1$, $i = 1, \dots, p$, $\mathbf{A}_0^{(i)} = \mathbf{I}$ and $\mathbf{A}_1^{(i)} = \mathbf{1}\mathbf{1}' - \mathbf{I}$ are $s_i \times s_i$ matrices.*

Proof. Substituting $\mathbf{w} \in W_{c_1 \dots c_p}$ in place of $\mathbf{p}_{ij}$ on the left hand side of formula (6) we get

$$\mathbf{A}_{c_1 \dots c_p}^{\#} = \sum_{\mathbf{w} \in W_{c_1 \dots c_p}} \mathbf{w}\mathbf{w}' = \bigotimes_{i=1}^{p} \mathbf{A}_{c_i}^{\#(i)}\,, \tag{12}$$

where $\mathbf{A}_0^{\#(i)} = \mathbf{1}\mathbf{1}'/s_i$ and $\mathbf{A}_1^{\#(i)} = \mathbf{I} - \mathbf{1}\mathbf{1}'/s_i$ are $s_i \times s_i$ matrices. Note that $\mathbf{A}_0^{\#(i)} = (\mathbf{A}_0^{(i)} + \mathbf{A}_1^{(i)})/s_i$ and $\mathbf{A}_1^{\#(i)} = ((s_i - 1)\mathbf{A}_0^{(i)} - \mathbf{A}_1^{(i)})/s_i$, thus

$$\mathbf{A}_{c_1 \dots c_p}^{\#} = \sum_{a_1, \dots, a_p} z^{a_1 \dots a_p, c_1 \dots c_p} \mathbf{A}_{a_1 \dots a_p}\,,$$

where

$$z^{a_1 \dots a_p, c_1 \dots c_p} = v^{-1} \prod_{i : c_i = 1} \{(1 - a_i)(s_i - 1) - a_i\}\,, \qquad v = \prod_{i=1}^{p} s_i\,;$$

on the other hand, the matrices $\mathbf{A}_{a_1 \dots a_p}$ are defined by (11). In this way we have proven that each of the matrices (12) is a linear combination of the matrices (11). Thus the matrices (11) satisfy condition (b) of Theorem 1. Since (a) is clear, this ends the proof.

Assume now that there are $v = \prod_{i=1}^{p} s_i$ treatments, and $p$ factors at $s_1, \dots, s_p$ levels, respectively. The treatments are denoted by $\phi(\beta_1, \dots, \beta_p)$, where $\beta_i = 0, 1, \dots, s_i - 1$; $i = 1, \dots, p$. In the literature (see [4], p. 197) the following notion of association scheme is called an $F_p$ *type association scheme* with $2^p - 1$ associate classes:

DEFINITION 2. Two treatments $\phi(\beta_1, \ldots, \beta_p)$ and $\phi(\beta'_1, \ldots, \beta'_p)$ are $(a_1, \ldots, a_p)$*-th associates* when $\beta_i = \beta'_i$ if $a_i = 0$, and $\beta_i \neq \beta'_i$ if $a_i = 1$.

After numbering the $v$ treatments in lexicographical order, we find that the association matrices occurring in the above definition of association scheme are the same as in (11). In this way Corollary 3 is a new definition of an $F_p$ type association scheme. It is worth noticing that $\mathbf{w} \in W_{c_1 \ldots c_p}$ are the contrasts belonging to the interaction between the factors $i_1, \ldots, i_q$, where $c_{i_1} = \ldots = c_{i_q} = 1$ and $\sum_{i=1}^{p} c_i = q$. If in the above $q = 1$ then we call these vectors the contrasts of the main effects of the factor $i_1$.

**4.2.** Let $W_i = \bigcup_i W_{c_1 \ldots c_p}$, where $\bigcup_i$ denotes the sum over all $c_1, \ldots, c_p$ with $c_1 + \ldots + c_p = i$, and $W_{c_1 \ldots c_p}$ is defined by (10). Let $W^*$ be the set $\{\mathbf{w} \in W_i : i = 1, \ldots, p\}$ with $s_1 = \ldots = s_p = s$.

COROLLARY 4. *The vectors $\mathbf{w} \in W^*$ determine an association scheme with $p$ associate classes. The association matrices of this scheme are of the form*

$$\mathbf{A}_0 = \mathbf{I}, \quad \mathbf{A}_i = \sum_i{}' \mathbf{A}_{a_1 \ldots a_p}, \tag{13}$$

*where $\sum_i'$ is the sum over all $a_1, \ldots, a_p$ satisfying $a_1 + \ldots + a_p = i$.*

Proof. Substituting $\mathbf{w} \in W_i$ in place of $\mathbf{p}_{ij}$ on the left hand side of (6) we get

$$\mathbf{A}_i^{\#} = \sum_{\mathbf{w} \in W_i} \mathbf{w}\mathbf{w}' = \sum_i{}'' \mathbf{A}^{\#}_{c_1 \ldots c_p},$$

where $\sum_i''$ is the sum over all $c_1, \ldots, c_p$ with $c_1 + \ldots + c_p = i$. Hence, and from (12), we get $\mathbf{A}_i^{\#} = \sum_{j=1}^{m} z^{ji} \mathbf{A}_j$, where $\mathbf{A}_j$ is defined by (13), $z^{ji} = v^{-1} \sum_i \{(s-1)^q (-1)^{i-q}\}$ for every $a_1, \ldots, a_p$ with $a_1 + \ldots + a_p = j$, $q$ is the number of zeros in $\{a_i : c_i = 1, i = 1, \ldots, p\}$ and $v = s^p$. So, condition (b) of Theorem 1 is satisfied by the matrices in (13). Since (a) is clear, this ends the proof.

In the literature (see [4], p. 203, [2], p. 572, and [3]) a $C_p$ type association scheme or hypercubic association scheme is known as a special case of an $F_p$ type association scheme. If we assume that there are $v = s^p$ treatments $\phi(\beta_1, \ldots, \beta_p)$, $\beta_i = 0, 1, \ldots, s-1$; $i = 1, \ldots, p$, then a $C_p$ association scheme among these treatments with $m = p$ associate classes is defined as follows:

DEFINITION 3. Two treatments $\phi(\beta_1, \ldots, \beta_p)$ and $\phi(\beta'_1, \ldots, \beta'_p)$ are *i-th associates* if $\sum_{k=1}^{p} \varepsilon(\beta_k - \beta'_k) = i$, where $\varepsilon(x) = 0$ if $x = 0$ and $\varepsilon(x) = 1$ otherwise.

After numbering the $s^p$ treatments in lexicographical order, we find that the association matrices occurring in the above definition are the same as in (13). Therefore Corollary 4 is another definition of a $C_p$ association scheme.

## References

[1] R. C. Bose and D. M. Mesner, *On linear associative algebras corresponding to association schemes of partially balanced designs*, Ann. Math. Statist. 30 (1959), 21–38.

[2] S. Kageyama, *Reduction of associate classes for block designs and related combinatorial arrangements*, Hiroshima Math. J. 4 (1974), 527–618.

[3] D. Raghavarao and K. Chandrasekharao, *Cubic designs*, Ann. Math. Statist. 35 (1964), 389–397.

[4] S. Yamamoto, Y. Fujii and N. Hamada, *Composition of some series of association algebras*, J. Sci. Hiroshima Univ. Ser. A-I 29 (1965), 181–215.

HENRYK BRZESKWINIEWICZ
DEPARTMENT OF MATHEMATICAL & STATISTICAL METHODS
ACADEMY OF AGRICULTURE
UL. WOJSKA POLSKIEGO 28
60-637 POZNAŃ, POLAND

*Received on 19.6.1989;*
*revised version on 8.3.1992*