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WAVE FRONTS OF SOLUTIONS OF SOME CLASSES OF NON-LINEAR PARTIAL DIFFERENTIAL EQUATIONS

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1. This paper is devoted to the study of wave fronts of solutions of first order symmetric systems of non-linear partial differential equations. A short communication was published in [4]. The microlocal point of view enables us to obtain more precise information concerning the smoothness of solutions of symmetric hyperbolic systems. Our main result is a generalization to the non-linear case of Theorem 1.1 of Ivriĭ [3]. The machinery of paradifferential operators introduced by Bony [1] together with an idea coming from [3], [2] are used.

2. The definition and main properties of paradifferential operators are assumed to be known to the reader [1]. We will use here the same notations as in [1]. We recall the definition of the microlocalized Sobolev space $\mathcal{H}^s_{\mathrm{mcl}}$:

Definition. A distribution $u \in \mathcal{D}'(X)$ belongs to the class $\mathcal{H}^s_{\mathrm{mcl}}(\varrho^0)$, $\varrho^0 \in T^*(X) \setminus 0$, $\varrho^0 = (x^0, \xi^0)$, if there exists a classical properly supported pseudodifferential operator a of order 0 such that $a(\varrho^0) \neq 0$, $au \in \mathcal{H}^s_{\mathrm{loc}}(X)$, where $\mathcal{H}^s_{\mathrm{loc}}$ is the local Sobolev space.

We denote by $W \subset T^*(X) \setminus 0$ an (open) closed set conical with respect to ξ and having a compact base in X. Assume that $F_k(x, u_1, \ldots, u_N, u_{11}, \ldots, u_{ij}, \ldots, u_{Nn})$, $1 \le j \le n$, $1 \le i, k \le N$, are real-valued C^∞-functions of their arguments $x \in X$, $\vec{u} \in \mathbb{R}^N$, $(u_{11}, \ldots, u_{Nn}) \in \mathbb{R}^{Nn}$ and X is a domain in $\mathbb{R}^n$. Define a matrix A_j by

$$A_j = \|\partial F_k / \partial u_{ij}(x, \vec{u}(x), \partial \vec{u}(x))\|_{1 \le i,k \le N}.$$

We now formulate the main result of this paper.

Theorem 1. *Consider the non-linear system of partial differential operators*

$$F_k(x, \vec{u}(x), \partial \vec{u}(x)) = 0\,, \qquad 1 \le k \le N, \tag{1}$$

$\vec{u}(x) = (u_1, \ldots, u_N)$, *and suppose that* (1) *possesses a real-valued solution* $\vec{u} \in \mathcal{H}^s_{\mathrm{loc}}(X)$, $s > 2 + n/2$, *such that*

(i) $\partial F_k/\partial u_{ij}(x, \vec{u}(x), \partial\vec{u}(x)) = \partial F_i/\partial u_{kj}(x, \vec{u}(x), \partial\vec{u}(x))$, $\forall\, x \in X$,

(ii) *the matrix* $A_{j_0}(x) = \|\partial F_k/\partial u_{ij_0}(x, \vec{u}(x), \partial\vec{u}(x))\|_{1\le i,k\le N}$, $x \in X$, *is (positive) negative definite.*

Suppose, moreover, that for each characteristic point $\varrho^0 \in \operatorname{Char} p_1 \cap \partial W \cap \{x_{j_0} \ge \delta\}$ *we have* $u \in \mathcal{H}^t_{\mathrm{mcl}}(\varrho^0)$ *for some* $t < 2s - 2 - n/2$. *Then* $u \in \mathcal{H}^t_{\mathrm{mcl}}(\varrho^0)$, $\forall \varrho^0 \in \operatorname{Char} p_1 \cap W \cap \{x_{j_0} \ge \delta\}$, $\delta = \mathrm{const}$.

We point out that conditions (i), (ii) imply that the linearized system $Pv = \sum_{j=1}^n A_j(x) D_j v - iB(x)v$ is symmetric and positive, $B, A_j(x) \in C^{1+\varepsilon}(X)$, $1 > \varepsilon > 0$. As usual,

$$\operatorname{Char} p_1 = \Big\{\varrho = (x,\xi) \in T^*(X) \setminus 0 : \det \sum_{j=1}^n A_j(x)\xi_j = 0\Big\}.$$

It is interesting to note that $u \in \mathcal{H}^{2s-1-\varepsilon-n/2}_{\mathrm{mcl}}(\varrho^0)$, $\varepsilon > 0$, for each $\varrho^0 \notin \operatorname{Char} p_1$ (see Th. 5.4 of [1]).

Standard considerations from the theory of paradifferential operators $P \in \widetilde{O}_p(\Sigma^1_\sigma)$, $\sigma > 1$, σ not an integer, reduce the proof of Theorem 1 to the proof of the following assertion.

Theorem 2. *Consider the first order paradifferential system*

$$P(x, D)u = \sum_{j=1}^n A_j(x) D_j u - iB(x)u = f \qquad (-P(x,D)u = -f) \tag{2}$$

where $P \in \widetilde{O}_p(\Sigma^1_\sigma)$, $\sigma > 1$, σ *not an integer*, $A^*_j(x) = A_j(x)$, $\forall\, x \in X$, *the* $A_j(x)$ *are real-valued* $N \times N$ *matrices and* $(A_{j_0}(x) > 0)$ $A_{j_0}(x) < 0$, $\forall x \in X$. *Assume that* $u \in \mathcal{H}^{t-1/2}_{\mathrm{comp}}(X)$, $Pu \in \mathcal{H}^t_{\mathrm{mcl}}(W \cap \{x_{j_0} \ge \delta\})$, *and* $u \in \mathcal{H}^t_{\mathrm{mcl}}(\varrho^0)$ *for each* $\varrho^0 \in \operatorname{Char} p_1 \cap \partial W \cap \{x_{j_0} \ge \delta\}$. *Then*

$$u \in \mathcal{H}^t_{\mathrm{mcl}}(\varrho^0)\,, \qquad \forall \varrho^0 \in \operatorname{Char} p_1 \cap W \cap \{x_{j_0} \ge \delta\}\,.$$

In the special case when $\varrho^0 \notin \operatorname{Char} p_1$ *the solution* $u \in \mathcal{H}^{t+1}_{\mathrm{mcl}}(\varrho^0)$.

3. Supposing Theorem 2 is proved and $s < t$ we will verify Theorem 1. To do this we apply Theorem 5.3 b) of [1] with the corresponding notations $d = 1$, $\varrho = s - \varepsilon - n/2$, $\varepsilon > 0$, $\sigma = \varrho - 1$ to conclude that there exists a paradifferential operator $P \in \widetilde{O}_p(\Sigma^1_\sigma)$, $\sigma > 1$, satisfying $Pu \in \mathcal{H}^{2s-2-\varepsilon-n/2}_{\mathrm{loc}} \Rightarrow Pu \in \mathcal{H}^t_{\mathrm{loc}}$ for $\varepsilon > 0$ sufficiently small, $u \in \mathcal{H}^s_{\mathrm{loc}}$.

The next remark will be useful later:

Let $u \in \mathcal{H}^s_{\mathrm{loc}}(X)$, $Pu \in \mathcal{H}^t_{\mathrm{mcl}}(W \cap \{x_{j_0} \geq \delta\})$, $u \in \mathcal{H}^{t-1/2}_{\mathrm{mcl}}(W \cap \{x_{j_0} \geq \delta\})$ *and* $u \in \mathcal{H}^t_{\mathrm{mcl}}(\partial W \cap \{x_{j_0} \geq \delta\})$. *Then* $u \in \mathcal{H}^t_{\mathrm{mcl}}(W \cap \{x_{j_0} \geq \delta\})$.

In fact, consider a classical pseudodifferential operator $T \in S^0_{1,0}$, $T \equiv 1$ in a small conic neighbourhood (ngbhd) of $W \cap \{x_{j_0} \geq \delta\}$, $T \equiv 0$ outside a larger conic ngbhd of $W \cap \{x_{j_0} \geq \delta\}$. Then $Tu \in \mathcal{H}^{t-1/2}_{\mathrm{comp}}(X)$, $Tu \in \mathcal{H}^t_{\mathrm{mcl}}(\partial W \cap \{x_{j_0} \geq \delta\})$ and $P(Tu) \in \mathcal{H}^t_{\mathrm{mcl}}(W \cap \{x_{j_0} \geq \delta\})$ as $P(Tu) = Pu + P((T-I)u)$ and $P((I-T)u) \in \mathcal{H}^{s-1+\sigma}_{\mathrm{mcl}} \subset \mathcal{H}^t_{\mathrm{mcl}}(W \cap \{x_{j_0} \geq \delta\})$ according to Corollary 3.5 of [1]. Thus $Tu \in \mathcal{H}^t_{\mathrm{mcl}}(W \cap \{x_{j_0} \geq \delta\}) \;\Rightarrow\; u \in \mathcal{H}^t_{\mathrm{mcl}}(W \cap \{x_{j_0} \geq \delta\})$. To complete the proof of Theorem 1 we observe that there exists a uniquely determined integer $k \geq 1$ for which $(k-1)/2 \leq t - s < k/2$ and therefore

$$t - k/2 \leq s \leq t - (k-1)/2 < t - (k-2)/2 < \ldots < t - 1/2 < t.$$

Setting $t' = t - (k-1)/2$ we get $u \in \mathcal{H}^s_{\mathrm{loc}} \subset \mathcal{H}^{t-k/2}_{\mathrm{loc}} = \mathcal{H}^{t'-1/2}_{\mathrm{loc}}$, $Pu \in \mathcal{H}^t_{\mathrm{loc}} \subset \mathcal{H}^{t-(k-1)/2}_{\mathrm{loc}} = \mathcal{H}^{t'}_{\mathrm{loc}}$, $u \in \mathcal{H}^{t'}_{\mathrm{mcl}}(\partial W \cap \{x_{j_0} \geq \delta\})$. So $u \in \mathcal{H}^{t'}_{\mathrm{mcl}}(W \cap \{x_{j_0} \geq \delta\})$ as $s \leq t'$. Put now $t'' = t - (k-2)/2 = t' + 1/2$. Obviously $u \in \mathcal{H}^{t''-1/2}_{\mathrm{mcl}}(W \cap \{x_{j_0} \geq \delta\})$, $Pu \in \mathcal{H}^{t''}_{\mathrm{loc}}$, $u \in \mathcal{H}^{t''}_{\mathrm{mcl}}(\partial W \cap \{x_{j_0} \geq \delta\})$.

The remark above and $s \leq t' \leq t''$ give us $u \in \mathcal{H}^{t''}_{\mathrm{mcl}}(W \cap \{x_{j_0} \geq \delta\})$. Thus we conclude that $u \in \mathcal{H}^t_{\mathrm{mcl}}(W \cap \{x_{j_0} \geq \delta\})$.

4. Proof of Theorem 2. To simplify the proof we will assume that $W = \Delta \times \Gamma_\xi$, $\Delta = [a_1, b_1] \times \ldots \times [a_n, b_n]$, Γ_ξ is a closed cone in $T^*(\mathbb{R}^n)$ and $A_1(x) < 0$. Choose $\kappa_j \in C_0^\infty(\mathbb{R})$ so that $\kappa_j \equiv 1$ on $[a_j, b_j]$, $\kappa_j'(x_j) = \kappa_j^-(x_j) - \kappa_j^+(x_j)$, $0 \leq \kappa_j^+$, $0 \leq \kappa_j^-$, $x_j \leq a_j$ in $\operatorname{supp} \kappa_j^-$, $x_j \geq b_j$ in $\operatorname{supp} \kappa_j^+$ and $\delta = a_1$ but no information on the $\mathcal{H}^t_{\mathrm{mcl}}$-smoothness of u at $\{x_1 = a_1\} \times \Gamma_\xi$ is given. For $\lambda, \delta_1 > 0$ put

$$Q = Q_{\lambda,\delta_1} = e^{\lambda x_1} \kappa(x)(1 + |\delta_1 \xi|^2)^{-1} h(\xi),$$

$\operatorname{ord}_\xi h = t$ and $\operatorname{cone\,supp} Q_{\lambda,\delta_1}$ is concentrated in a small conic ngbhd of W. Obviously, $Q_{\lambda,\delta_1} \in S^{t-2}_{1,0}$ and the factor $\kappa(x)(1 + (\delta_1|\xi|)^2)^{-1}$ is bounded in Σ^0_ϱ, $S^0_{1,0}$, $\forall \varrho > 0$, ϱ not an integer, uniformly with respect to $\delta_1 \in (0,1]$ and $\kappa(x) = \kappa_1(x) \ldots \kappa_n(x)$. Thus for each fixed $\lambda > 0$ and arbitrary $\delta_1 \in (0,1]$, $Q_{\lambda,\delta_1} \in S^t_{1,0}$.

Consider now the identity

$$(QPu, Qu)_{L_2} = (PQu, Qu)_{L_2} + ([Q,P]u, Qu)_{L_2}.$$

It is legitimate as $Pu \in \mathcal{H}^t_{\mathrm{mcl}}(W) \Rightarrow QPu \in \mathcal{H}^2_{\mathrm{comp}}(X)$, $Qu \in \mathcal{H}^{3/2}_{\mathrm{comp}}(X)$ (in our notations $W = W \cap \{x_1 \geq \delta\}$). So

$$\operatorname{Im}(QPu, Qu)_{L_2} = \operatorname{Im}(PQu, Qu)_{L_2} + \operatorname{Im}([Q,P]u, Qu)_{L_2}. \tag{3}$$

We first estimate

$$I = \operatorname{Im}(PQu, Qu)_{L_2}, \tag{4}$$

i.e. we have to consider the terms $(A_j(x)D_jQu, Qu)$, $(B(x)Qu, Qu)$ $((\cdot\,,\cdot)_{L_2} = (\cdot\,,\cdot))$. It can easily be seen that

$$|(B(x)Qu, Qu)| \le C_1\|Qu\|_0^2 + C_{1\lambda}\|u\|_{t-\sigma/2}^2$$

where C_1 is an absolute constant and $C_{1\lambda}$ depends on $\lambda > 0$ but does not depend on $\delta_1 \in (0, 1]$. Now,

$$\begin{aligned}(A_j(x)D_jQu, Qu) &= (D_jA_jQu, Qu) + ([A_j, D_j]Qu, Qu)\\ &= (Qu, A_jD_jQu) + ([A_j, D_j]Qu, Qu),\end{aligned}$$

i.e. $2|\operatorname{Im}(A_jD_jQu, Qu)| \le |([A_j, D_j]Qu, Qu)|$. The principal symbol of the commutator $[A_j, D_j]$ is $-i\{A_j, \xi_j\} = i\partial A_j(x)/\partial\xi_j \in \Sigma^0_{\sigma-1}$, $\sigma - 1 > 0$, i.e. $\operatorname{Im}|(A_jD_jQu, Qu)| \le C_2\|Qu\|_0^2$. In other words,

$$|I| \le C_3\|Qu\|_0^2 + C_{3\lambda}\|u\|_{t-\sigma/2}^2. \tag{5}$$

To estimate $II = \operatorname{Im}([Q, P]u, Qu)$ we use Theorem 3.2 of [1]. Since the principal symbol of $[Q, P]$ is $(1/i)\{Q, p_1\}$ we have

$$II = -\operatorname{Re}(\{Q, p_1\}u, Qu) + C'_{3\lambda}\|u\|_{t+(1-\sigma)/2}^2.$$

Obviously

$$-\{Q, p_1\} = -\sum_{j,k=1}^{n}(\partial Q/\partial\xi_k)(\partial A_j(x)/\partial x_k)\xi_j + \sum_{j=1}^{n}(\partial Q/\partial x_j)A_j(x)$$

and therefore

$$\partial Q/\partial x_1 = \lambda Q + e^{\lambda x_1}(\partial\kappa/\partial x_1)(1 + |\delta_1\xi|^2)^{-1}h(\xi).$$

The inequality $\partial\kappa/\partial x_1 \ge -\kappa_1^+(x_1)\kappa_2\ldots\kappa_n = -\kappa^+(x)$ will enable us to apply the sharp Gårding estimate. In fact,

$$\operatorname{Re}((\partial Q/\partial x_1)A_1u, Qu) = \lambda\operatorname{Re}(QA_1u, Qu) + \operatorname{Re}(\widetilde{Q}^+A_1u, Qu),$$

where $\widetilde{Q}^+ = e^{\lambda x_1}(\partial\kappa/\partial x_1)(1 + |\delta_1\xi|^2)^{-1}h(\xi)$. It is clear that $(QA_1u, Qu) = (A_1Qu, Qu) + ([Q, A_1]u, Qu)$, thus

$$\operatorname{Re}(QA_1u, Qu) \le -C_4\|Qu\|_0^2 + C_{4\lambda}\|u\|_{t-1/2}^2\,, \qquad C_4 > 0. \tag{6}$$

On the other hand,

$$\operatorname{Re}(\widetilde{Q}^+A_1u, Qu) \le \operatorname{Re}(A_1(x)\kappa(\partial\kappa/\partial x_1)v, v) + C_{5\lambda}\|u\|_{t-1/2}^2 \tag{7}$$

where $v = e^{\lambda x_1}h(D)(1 + |\delta_1D|^2)^{-1}u$. The commutator

$$[A_1,\ \kappa e^{\lambda x_1}(\partial\kappa/\partial x_1)h(D)(1 + |\delta_1D|^2)^{-1}]$$

is bounded in $\Sigma^{t-1}_{\sigma-1}$ uniformly with respect to $\delta_1 > 0$. We apply the sharp Gårding inequality to the symmetric non-positive matrix $A_1\kappa(\partial\kappa/\partial x_1) + \kappa\kappa^+A_1$ and we get

$$\operatorname{Re}(\kappa(\partial\kappa/\partial x_1)A_1v, v) \le -\operatorname{Re}(\kappa\kappa^+A_1v, v) + C_{6\lambda}\|u\|_{t-\mu/2}^2, \tag{8}$$

with $\mu < \sigma/2$ if $1 < \sigma < 2$ and $\mu = 1$ if $\sigma > 2$ (see [1]). Then

$$(9)\qquad \begin{aligned}|(\kappa\kappa^+ A_1 v, v)| &\le |(A_1\kappa v, \kappa^+ v)| + |([A_1, \kappa]v, \kappa^+ v)|\\ &\le C_7(\|\kappa v\|\cdot\|\kappa^+ v\| + \|v\|_{-1}\cdot\|\kappa^+ v\|)\\ &\le C_7\|Qu\|\cdot\|Q^+ u\| + C_{7\lambda}\|u\|_{t-1}\cdot\|Q^+ u\|\end{aligned}$$

and $Q^+ = Q^+_{\lambda,\delta_1}$ is defined as Q_{λ,δ_1} with κ replaced by κ^+. Note that $\|Q^+ u\| < \infty$ as $Q^+(x,\xi)$ concentrates in a ngbhd of $\{x_1 = b_1\} \times \Gamma_\xi$ and $u \in \mathcal{H}^t_{\mathrm{mcl}}(\partial W \cap \{x_1 \ge \delta\})$, $\delta = a_1$.

By the identity $\partial Q/\partial x_j = e^{\lambda x_1}(\partial\kappa/\partial x_j)(1 + |\delta_1\xi|^2)^{-1}h(\xi)$, $j \ge 2$, $\partial Q/\partial x_j$ concentrates in a ngbhd of $\{x_j = a_j\}\times\Gamma_\xi$, $\{x_j = b_j\}\times\Gamma_\xi$ and simple computations show that

$$(10)\qquad \begin{aligned}&|(\partial Q/\partial x_j(x,D)A_j(x)u, Qu)|\\ &\quad\le |(A_j(\partial Q/\partial x_j)u, Qu)| + |([A_j, \partial Q/\partial x_j]u, Qu)|\\ &\quad\le C_{8\lambda}\|u\|_{\mathcal{H}^t_{\mathrm{mcl}}(\partial W\cap\{x_1\ge\delta\})}\|Qu\|_0 + C_{9\lambda}\|u\|_{t-1}\|Qu\|_0\\ &\quad\le \|Qu\|_0^2 + C_{10\lambda}(\|u\|^2_{t-1} + \|u\|_{\mathcal{H}^t_{\mathrm{mcl}}(\partial W\cap\{x_1\ge\delta\})}).\end{aligned}$$

Now we will estimate $((\partial A_j/\partial x_k)D_j(\partial Q/\partial\xi_k)(x,D)u, Qu)$. To do this two terms will be considered, namely

$$\begin{aligned}III_1 &= ((\partial A_j/\partial x_k)e^{\lambda x_1}\kappa(x)D_j(\partial h/\partial\xi_k)(D)(1 + |\delta_1 D|^2)^{-1}u, Qu),\\ III_2 &= ((\partial A_j/\partial x_k)e^{\lambda x_1}\kappa(x)h(D)D_j\delta_1^2 D_k(1 + |\delta_1 D|^2)^{-2}u, Qu).\end{aligned}$$

Obviously, $\delta_1^2\xi_k(1 + \delta_1^2|\xi|^2)^{-2}$ is uniformly bounded in $S^{-1,0}_{1,0}$, Σ^{-1}_ϱ, $\varrho > 0$, ϱ not an integer, $\forall\delta_1 \in (0,1]$. The observations that $\delta_1^2\xi_j\xi_k(1 + |\delta_1\xi|^2)^{-1}$ is uniformly bounded in $S^0_{1,0}$ with respect to $\delta_1 > 0$ and

$$e^{\lambda x_1}\kappa(x)h(D)\delta_1^2 D_j D_k(1 + |\delta_1 D|^2)^{-2}u = Q(\delta_1^2 D_j D_k(1 + |\delta_1 D|^2)^{-1}u)$$

enable us to conclude that

$$(11)\qquad |III_2| \le C_{11}\|Qu\|_0^2 + C_{11\lambda}\|u\|^2_{t-1/2}.$$

The cut-off symbol $h(\xi)$ can be written as $h(\xi) = |\xi|^t c(\xi)$, $\mathrm{ord}_\xi\, c = 0$, $0 \le c \le 1$, $c \equiv 1$ in a conic ngbhd of Γ_ξ and $c \equiv 0$ outside a larger conic ngbhd of Γ_ξ. The inequality

$$(12)\qquad |\partial h/\partial\xi_k|^2 \le 2t^2(h^2/|\xi|^2) + 2|\xi|^{2t}|\partial c/\partial\xi_k|^2$$

will be useful later. Thus

$$\begin{aligned}&\|e^{\lambda x_1}\kappa D_j(\partial h/\partial\xi_k)(D)(1 + |\delta_1 D|^2)^{-1}u\|_0\\ &\qquad\le \|D_j(\partial h/\partial\xi_k)(D)(1 + |\delta_1 D|^2)^{-1}(e^{\lambda x_1}\kappa(x)u)\|_0 + C_{12\lambda}\|u\|_{t-1}.\end{aligned}$$

On the other hand, according to (12),

$$\begin{aligned}
&\|D_j(\partial h/\partial\xi_k)(D)(1+|\delta_1 D|^2)^{-1}(e^{\lambda x_1}\kappa(x)u)\|_0^2\\
&\qquad = \int \xi_j^2(\partial h/\partial\xi_k)^2(1+|\delta_1\xi|^2)^{-2}|(e^{\lambda x_1}\kappa(x)u)^\wedge|^2(\xi)\,d\xi\\
&\qquad \le 2t^2\|h(D)(1+|\delta_1 D|^2)^{-1}(e^{\lambda x_1}\kappa u)\|_0^2\\
&\qquad\quad + 2\int |\xi|^{2t+2}(\partial c/\partial\xi_k)^2|(e^{\lambda x_1}\kappa(x)u)^\wedge|^2(\xi)\,d\xi\\
&\qquad \le 2t^2\|Qu\|_0^2 + C_{13\lambda}\|u\|_{t-1}^2 + 2\|\,|D|^{t+1}(\partial c/\partial\xi_k)(e^{\lambda x_1}\kappa u)\|_0^2\\
&\qquad \le 2t^2\|Qu\|_0^2 + C_{13\lambda}\|u\|_{t-1}^2 + C_{14\lambda}\|u\|^2_{\mathcal{H}^t_{\mathrm{mcl}}(\partial W\cap\{x_1\ge\delta\})}.
\end{aligned}$$

We remind the reader that $\mathrm{ord}_\xi\,|\xi|^{t+1}(\partial c/\partial\xi_k)=t$ and $\kappa(x)(\partial c/\partial\xi_k)$ concentrates in a conic ngbhd of $\Delta\times\partial\Gamma_\xi$. In other words,

$$|III_1| \le C_{15}\|Qu\|_0^2 + C_{16\lambda}(\|u\|_{t-1}^2 + \|u\|^2_{\mathcal{H}^t_{\mathrm{mcl}}(\partial W\cap\{x_1\ge\delta\})}). \tag{13}$$

Combining the identity (3) and the corresponding estimates (5) for I, (6)–(11), (13) for II and

$$\mathrm{Im}(QPu,Qu) \ge -2\|QPu\|_0^2 - 2\|Qu\|_0^2$$

we come to the conclusion that

$$\begin{aligned}
(\lambda - C)\|Qu\|_0^2 &\le 2\|QPu\|_0^2 + C\|Q^+u\|_0^2\\
&+K_\lambda(\|u\|_{t-1/2}^2 + \|u\|_{t-\mu/2}^2 + \|u\|_{t+(1-\sigma)/2}^2 + \|u\|^2_{\mathcal{H}^t_{\mathrm{mcl}}(\partial W\cap\{x_1\ge\delta\})}).
\end{aligned} \tag{14}$$

The constant C does not depend on $\lambda>0$ and $\delta_1>0$, and K_λ depends on $\lambda>0$ only. Taking λ sufficiently large and letting $\delta_1\to 0$ we prove Theorem 2 for $\sigma>2$.

To consider the case $1<\sigma<2$ we have to modify the proof of our Theorem 2 assuming $Pu\in\mathcal{H}^t_{\mathrm{mcl}}$, $u\in\mathcal{H}^t_{\mathrm{mcl}}(\partial W)$ and $u\in\mathcal{H}^{t-\gamma}_{\mathrm{comp}}(X)$, $0<\gamma<1/2$, instead of $\gamma=1/2$ etc.

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