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ERGODIC PROPERTIES OF SKEW PRODUCTS WITH FIBRE MAPS OF LASOTA–YORKE TYPE

Abstract. We consider the skew product transformation $T(x,y) = (f(x), T_{e(x)}y)$, where f is an endomorphism of a Lebesgue space $(X,\mathcal{A},p)$, $e : X \to S$ and $\{T_s\}_{s\in S}$ is a family of Lasota–Yorke type maps of the unit interval into itself. We obtain conditions under which the ergodic properties of f imply the same properties for T. Consequently, we get the asymptotical stability of random perturbations of a single Lasota–Yorke type map. We apply this to some probabilistic model of the motion of cogged bits in the rotary drilling of hard rock with high rotational speed.

1. Preliminaries and main results. Let f be a negative nonsingular transformation of a Lebesgue space $(X,\mathcal{A},p)$ into itself. Let I be the unit interval.

Definition. The transformation $\tau : I \to I$ is of the *Lasota–Yorke type* if there exist $0 = a_0 < a_1 < \ldots < a_N = 1$ and a constant λ, $\lambda > 1$, such that for any $j = 0,1,\ldots,N-1$:

(i) $\tau|(a_j,a_{j+1})$ is of class C^1 and the limits $\tau'(a_j^+)$, $\tau'(a_{j+1}^-)$ exist (or are infinite),

(ii) there exists a positive integer n such that $\inf|(\tau^n)'| \ge \lambda$,

(iii) $|1/\tau'|$ is a function of bounded variation.

We denote by R_τ the set $\{a_0,a_1,\ldots,a_N\}$ and by Z_τ the partition of I into closed intervals $I_1 = [a_0,a_1],\ldots,I_N = [a_{N-1},a_N]$.

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Let $\{T_s\}_{s\in S}$ be a family of Lasota–Yorke type maps of I into itself. Consider a function $e : X \to S$ such that the mapping $(x,y) \to T_{e(x)}y$ is measurable. We define the *skew product transformation* by

$$T(x,y) = (f(x), T_{e(x)}y)\,.$$

The transformation T is negative nonsingular with respect to the product measure $p \times m$ (m the Lebesgue measure).

Let P_T denote the *Frobenius–Perron operator* for T, i.e.

$$P_T G = \frac{d}{dp \times m} \int\limits_{T^{-1}(\cdot)} G\, d(p \times m) \quad \text{for } G \in L_1(p \times m)\,.$$

Then (using the Fubini theorem)

$$P_T = P_f P_{e(\cdot)}\,, \tag{1.1}$$

where P_f and $P_{e(x)}$ denote the Frobenius–Perron operators for f and $T_{e(x)}$, respectively. Moreover, fixing the function e we write P_x, T_x instead of $P_{e(x)}$ and $T_{e(x)}$, respectively. For a function F, $F : X \times I \to \mathbb{C}$, let $\mathbf{V}_x F$ denote the total variation of $F(x,\cdot)$, for every $x \in X$. For $G \in L_1(p \times m)$ we introduce the following notation:

$$\begin{aligned} \mathbf{V}\, G &= \inf\Big\{\int \mathop{\mathbf{V}}_x F\, dp : F \text{ is any version of } G\Big\}\,,\\ BV &= \{G \in L_1(p\times m) : \mathbf{V}\, G < \infty\} \text{ and}\\ \mathcal{D} &= \{G \in L_1(p \times m) : G \ge 0,\ \|G\|_1 = 1\}\,. \end{aligned}$$

Our first aim is to estimate the variation of iterations of the Frobenius–Perron operator. By Lemma 2 of [6] we have $\mathbf{V}\, P_f G \le \mathbf{V}\, G$ for $G \in BV$ and consequently by using (1.1) we get

$$\mathbf{V}\, P_T G \le \mathbf{V}\, P_x F\,, \tag{1.2}$$

where F is any version of G.

For further considerations we introduce a property (A) of the family $\{T_s\}_{s\in S}$. Let $S^n = \{(s_1,\ldots,s_n) : s_i \in S,\ i = 1,\ldots,n\}$. For $\alpha \in S^n$, $\alpha = (s_1,\ldots,s_n)$, we define $T_\alpha = T_{s_n} \circ \ldots \circ T_{s_1}$. Then

(A) There exists a positive integer n such that

- (A_1) there is a constant $\lambda > 1$ such that $|T'_\alpha| \ge \lambda$ for all $\alpha \in S^n$,
- (A_2) there is a constant $W > 0$ such that $\mathbf{V}\,|1/T'_\alpha| \le W$ for all $\alpha \in S^n$,
- (A_3) there is a constant $\delta > 0$ such that for any $\alpha \in S^{ln}$, there is a finite partition K_α of I into intervals such that for $J \in K_\alpha$, $T_\alpha|J$ is 1-1 and $T_\alpha(J)$ is an interval, and $\min_{J\in K_\alpha} \operatorname{diam}(J) > \delta$. Here l is the minimal integer such that

$$\frac{3}{\lambda^l} + \frac{l}{\lambda^{l-1}} W < 1\,.$$

If the family $\{T_s\}_{s\in S}$ has property (A) then an analysis similar to that in the proof of Theorem 1 of [2] shows that

$$\bigvee_x P_{f^{k-1}(x)} \circ \ldots \circ P_x F \le \alpha(k) \bigvee_x F + c \int |F|\, dm\,, \tag{1.3}$$

where c and $\alpha(k)$ are independent of F and $\lim_{k\to\infty} \alpha(k) = 0$. Therefore by (1.2) and (1.3) we get

$$\mathbf{V}\, P_T^k G \le \alpha(k)\, \mathbf{V}\, G + c\|G\|_1\,.$$

The following result may be proved in the same way as Theorem 6 of [6].

THEOREM 1. *If the family $\{T_s\}_{s\in S}$ has property* (A) *and if for every $G \in L_1(p \times m)$ the limit*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} P_T^k G = Q_T G \quad \textit{exists in } L_1\,,$$

then $\mathbf{V}\, Q_T G \le c\|G\|_1$, where the constant c is independent of G.

The assumption about the existence of $\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} P_T^k G$ implies the existence of a T-invariant absolutely continuous measure (a.c.i.m.) and therefore the existence of an f-a.c.i.m. It turns out that the converse implication is true, i.e. if p is an f-invariant measure and the family $\{T_s\}_{s\in S}$ has property (A) then the limit $\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} P_T^k G$ exists. The description of a T-a.c.i.m. and the ergodic properties of T can be found in Morita [9]. Below we present the Morita theorem with weakened assumptions. Namely, we omit the condition: $\inf_{s\in S} \min_{J\in Z_{T_s}} \mathrm{diam}(J) > 0$ when $\sup_s |T_s'| < \infty$.

MORITA THEOREM. *Suppose f preserves the measure p and the family $\{T_s\}_{s\in S}$ has property* (A).

(1) *The limit $\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} P_T^k G$ exists in L_1 for every $G \in L_1$.*

(2) *If the dynamical system (f, p) is ergodic, then there exists a finite number of a.c.i.m. $\mu_1, \ldots, \mu_r$ such that*

- (i) *for each $i = 1, \ldots, r$, the dynamical system (T, μ_i) is ergodic,*
- (ii) *if μ is an a.c.i.m. for $(T, p \times m)$, then μ is a linear combination of the μ_i.*

(3) *If (f, p) is totally ergodic and μ_i is one of the above mentioned probability measures, then there is an integer N_i and a collection of disjoint sets $L_{i,0}, L_{i,1}, \ldots, L_{i,N_i-1}$ such that*

- (i) $T(L_{i,j}) = L_{i,j+1}$ $(0 \le j < N_i - 1)$, $T(L_{i,N_i-1}) = L_{i,0}$,
- (ii) *for each $j = 0, 1, \ldots, N_i - 1$, the dynamical system $(T^{N_i}, \mu_{i,j})$ is totally ergodic where $\mu_{i,j} = N_i \mu_i | L_{i,j}$.*

(4) *Under the assumptions of* (3), *if moreover the dynamical system (f, p) is exact, so is $(T^{N_i}, \mu_{i,j})$.*

Section 3 contains a simplified version of the proof of the above theorem ([9]). From the Morita Theorem we conclude that if T is totally ergodic with respect to an a.c.i.m. and if f is exact, then T is also exact. Therefore, it seems useful to find some criteria for the ergodic properties of T.

Suppose the family $\{T_x\}_{x\in X}$ has property (A). Let $D_G = \{(x,y) : G(x,y) > 0\}$, where $P_T G = G$ and $G \in \mathcal{D} \cap BV$. Then $T(D_G) = D_G$ up to $(p\times m)$-null sets. Fixing the density G we write $\mu = \mu_G$ and $D = D_G$. Here μ_G is a T-a.c.i.m. such that $d\mu_G/d(m \times p) = G$. Theorem 1 and arguments similar to those in [6] imply:

Lemma 1. *Let A be a T-invariant set such that $\mu(A) > 0$. Then there exists a set $B \in \mathcal{A}$, $p(B) > 0$, such that $\bigcup_{x\in B} x \times I_x \subset A \cap D$ for some nonempty open intervals I_x.*

Lemma 2. *If T is not weakly mixing, then there exists a $T\times T$-invariant set A with $0 < (\mu\times\mu)(A) < 1$ such that*

$$\bigcup_{(x,v)\in B} (x,v) \times I_x \times I_v \subset A \cap D \times D$$

for some set $B \in \mathcal{A} \times \mathcal{A}$ with $(p \times p)(B) > 0$, and for some nonempty intervals I_x, I_v.

Next, we introduce a new property (B) of the family $\{T_x\}_{x\in X}$:

(B) For a.e. x and for every nonempty open interval J there exists $k(J)$ such that

(B_1) $k(J) = 1$ when $J = I$,
(B_2) $T_{f^{k(J)-1}(x)} \circ \ldots \circ T_x(J) = I$.

Remarks. 1) In the case $|T'_x| \ge \lambda > 2$ for a.e. x it suffices to take under consideration only maximal intervals of continuity and monotonicity for T_x.

2) If τ is a Lasota–Yorke type map with invariant measure equivalent to m then the condition: for every nonempty interval J there exists $k(J)$ such that $\tau^{k(J)}(J) = I$, is equivalent to the total ergodicity of τ ([4]).

Lemma 3. *Suppose the family $\{T_x\}_{x\in X}$ has properties* (A) *and* (B).

(i) *If A is a T-invariant set such that $\mu(A) > 0$ then there exists a set $B \in \mathcal{A}$ such that $A \cap D = B \times I$.*

(ii) *If $T \times T$ is not weakly mixing, then there exists a $T \times T$-invariant set A such that $0 < (\mu \times \mu)(A) < 1$ and $A \cap D \times D = B \times I \times I$ for some $B \in \mathcal{A} \times \mathcal{A}$.*

Proof. (i) By Lemma 1, $A \cap D \supset \bigcup_{x\in B_1} x \times I_x$. From (B) we conclude that there exist a positive integer k and a set $B_2 \subset B$ with $p(B_2) > 0$ such that $T_{f^{k-1}(x)} \circ \ldots \circ T_x(I_x) = I$ for every $x \in B_2$. Hence

$$A \cap D \supset T^k\Big(\bigcup_{x\in B_1} x \times I_x\Big) \supset f^k(B_2) \times I\,.$$

For $B = \{x \in X : \{y : (x,y) \in A \cap D\} = I\}$ we have $p(B) > 0$ and $f(B) = B$ and so the set $A \cap D - B \times I$ is T-invariant. The assumption $\mu(A \cap D - B \times I) > 0$ leads to a contradiction with the definition of B (by repeating the above considerations).

(ii) can be proved in a similar manner. ■

THEOREM 2. *Suppose the family* $\{T_x\}_{x\in X}$ *has properties* (A) *and* (B). *If f is ergodic* (*totally ergodic*, *weakly mixing*, *exact*), *then T is ergodic* (*respectively totally ergodic*, *weakly mixing*, *exact*).

We assume, for the rest of this paper, that if (f,p) is a Bernoulli endomorphism then the random variables $\xi_n(x) = e(f^n(x))$, $n = 0,1,\ldots,$ are mutually independent and $\mathcal{A} = \mathcal{F}(\xi_0, \xi_1, \ldots)$.

In the case when (f,p) is a Bernoulli endomorphism and property (A) holds we can use Theorem 3.1 of [10] to get the following result:

If E is a T-invariant set, then $E \cap D = X \times B$ for some set $B \in \mathcal{B}$ and $\mu = p \times m_1$.

THEOREM 3. *If the family* $\{T_x\}$ *has property* (A) *and f is a Bernoulli endomorphism*, *then T is exact provided* $\{B : T_x^{-1}(B) = T_y^{-1}(B)\ p\times p\text{-a.e.}\} = \{\emptyset, I\}$ *up to m_1-null sets.*

Proof. In order to show this we replace m by m_1 and the unit interval I by $\operatorname{supp} m_1$. Now, we prove the property of weak mixing of T as in the proof of Theorem 1 of [5]. By the Morita Theorem we conclude the proof. ■

2. Applications. We investigate two kinds of random perturbations of a Lasota–Yorke type transformation.

I. Let τ be a Lasota–Yorke type transformation which satisfies the following assumptions:

(a) $\tau|(a_i, a_{i+1})$ can be extended to a C^2-function $\overline{\tau}$ on $[a_i, a_{i+1}]$ for $a_i \in R_\tau$,

(b) if $k(\tau)$ is the first integer such that $\inf|(\tau^{k(\tau)})'| > 2$, then

$$\bigcup_{i=1}^{k(\tau)-1} \overline{\tau}^i(R_\tau) \cap (R_\tau - \{0,1\}) = \emptyset\,.$$

THEOREM 4. *If τ satisfies conditions* (a) *and* (b), *then there exists a number δ*, $0 < \delta < 1$, *such that for every Bernoulli dynamical system* (f,p)

and for every measurable function $e : X \to [1-\delta, 1]$ which is not constant, the dynamical system $T(x,y) = (f(x), e(x)\tau(y))$ is exact.

Proof. We obtain the assertion by applying Theorem 3. Here we prove inequality (1.3) instead of property (A). By conditions (a), (b) and by the estimation of variation (as in [8]) we get the existence of a δ, $0 < \delta < 1$, such that for every function $e : X \to [1-\delta, 1]$ the inequality (1.3) holds.

Now, if a set B belongs to the family

$$\left\{B : \tau^{-1}\left(\frac{1}{e(x)}B\right) = \tau^{-1}\left(\frac{1}{e(y)}B\right)\ p \times p\text{-a.e.}\right\}$$

then

$$B \cap (0, e(x)) = \frac{e(x)}{e(y)}(B \cap (0, e(y))) \quad \text{for } p \times p\text{-a.e. } (x,y)\,.$$

It is not difficult to see (by Lemma 2) that if $m_1(B) > 0$ then there exists an interval I_1 such that $I_1 \subset B$ and $m(I_1) > d_\delta$ ($\lim_{\delta\to 0} d_\delta > 0$). If we take a maximal interval I_0 in B then for small δ we obtain $B \supset I_0 \supset (0, e(x))$ for p-a.e. x and hence $m_1(B) = 1$. ∎

The exactness means that $\lim_{n\to\infty} \|P_T^n G - Q_T G\|_1 = 0$ for every $G \in L_1(p \times m)$. Therefore the operator P_T is asymptotically stable.

Let $\tau = \tau_\lambda$, $\lambda > 2$, where τ_λ is the Lasota–Yorke type transformation which appears in the mathematical model (see [7]) describing the motion of cogged bits in the rotary drilling of hard rock with high rotational speed. The transformation τ_λ satisfies conditions (a) and (b), except possibly a finite number of values of λ. Theorem 4 is a generalization of the result of K. Horbacz [3], which concerns the asymptotic stability of P_T for $T(x,y) = (f(x), e(x)\tau_\lambda(y))$.

II. Let τ be a totally ergodic Lasota–Yorke type transformation such that $\inf|\tau'| = \lambda > 1$, $\mu_\tau \approx m$, where μ is an a.c.i.m. We will denote by R the set R_τ and by Z the set Z_τ. Let $\{\tau_m\}_{m\ge 1}$ be a family of Markovian transformations associated with τ (defined in [2]). Let $R^n = \bigcup_{j=0}^n \overline{\tau}^{-j}(R)$, $n = 0, 1, \ldots$, and $Z^n = \bigvee_{j=0}^n \tau^{-j}(Z)$. The transformation τ_n has the following properties:

$$\overline{\tau}_n(R^n) \subseteq R^n \text{ where } \overline{\tau}_n(b) = \overline{\tau}(b) \text{ for } b \in R^n - R \text{ and } Z_{\tau_n} = Z^n\,, \tag{2.1}$$

$$\inf|\tau_n'| \ge \inf|\tau'|\,, \tag{2.2}$$

$$\mathop{\mathbf{V}}_J |1/\tau_n'| \le \mathop{\mathbf{V}}_J |1/\tau'| \quad \text{for } J \in Z^n\,, \tag{2.3}$$

$$\tau_n(J) \supset \tau(J) \quad \text{for } J \in Z^n\,. \tag{2.4}$$

The family $\{\tau_s\}_{s\ge l}$ has property (A). To see this, we take $n = 1$. Conditions (A_1) and (A_2) follow from (2.2) and (2.3). We take Z^l for K_α in (A_3), where l is defined in (A_3).

Let k be the least integer such that $d = \lambda^k/2 > 1$. Moreover, set

$$k_0 = ([-\ln((\lambda/2)^k b)/\ln d] + 1)k \quad \text{and} \quad n_0 = \max\{k(J) : J \in Z^k\},$$

where $k(J)$ is such that $\tau^{k(J)}(J) = I$, $b = \inf_{J\in Z^k} m(J)$ and $[x]$ denotes the integer part of x. Let the dynamical system (f, p) be ergodic and let $e_n : X \to \{n, n+1, \ldots\}$ for $n \ge \max\{n_0, l\}$. We define $T_n(x, y) = (f(x), \tau_{e_n(x)} y)$.

Theorem 5. *If (f, p) is ergodic and there exists a sequence $n \le n_1 < n_2 < \ldots < n_{n_0+k_0}$ such that*

$$p(f^{-n_0-k_0+1}(e_n^{-1}(n_{n_0+k_0})) \cap \ldots \cap e_n^{-1}(n_1)) > 0,$$

then $\{\tau_{e_n(x)}\}$ has properties (A) *and* (B).

Proof. Since $\{\tau_{e_n(x)}\}$ has property (A) for $n \ge l$, it remains to prove property (B). Let J be a fixed nonempty interval. For some integer r with $r \le -\ln m(J)/\ln d$ and for any positive integers $i_{rk}, \ldots, i_1$ there exists $J_1 \in Z^k$ such that $\tau_{i_{rk}} \circ \ldots \circ \tau_{i_1}(J_1) \supset J_1$. Therefore, for any $j \ge 0$ there exists an interval $J_2 \subset J_2' \in Z^k$ such that $\tau_{i_{rk+j}} \circ \ldots \circ \tau_{i_1}(J) \supset J_2$ and $m(J_2) \ge (\lambda/2)^k b$. By the assumption, for a.e. x there exists $r \ge (-\ln m(J)/\ln d)k$ such that

$$n_1 = e_n(f^r(x)) < n_2 = e_n(f^{r+1}(x)) < \ldots < n_{n_0+k_0} = e_n(f^{r+n_0+k_0-1}(x)).$$

Hence

$$\tau_{e_n(f^{r+n_0+k_0-1}(x))} \circ \ldots \circ \tau_{e_n(x)}(J) \supset \tau_{n_{n_0+k_0}} \circ \ldots \circ \tau_{n_{r_2}}(J_3),$$

for some $r_2 \le k_0$ and $J_3 \in Z^k$.

By definition of n_0, $\tau^{n_0}(J_3) = I$. Let $J_4 \in Z^{n_0}$ and $J_4 \subset J_3$. Then $\tau_{n_{r_2}}(J_4) \supset \tau(J_4)$. This is a consequence of (2.4) and of the inequality $Z^{n_{r_2}} \ge Z^{n_0}$, for $n_{r_2} \ge n \ge n_0$. The set $\tau_{n_{r_2}}(J_4)$ is a union of intervals from $Z^{n_{r_2}}$ (by (2.1)) and, consequently, a union of intervals from $Z^{n_{r_2+1}}$, because $n_{r_2+1} \ge n_{r_2}$. Therefore,

$$\tau_{n_{r_2+1}} \tau_{n_{r_2}}(J_4) \supset \tau(\tau_{n_{r_2}}(J_4)) \supset \tau^2(J_4).$$

Finally, $\tau_{n_{n_0+r_2}} \circ \ldots \circ \tau_{n_{r_2}}(J_4) \supset \tau^n(J_4)$, which implies $\tau_{n_{n_0+r_2}} \circ \ldots \circ \tau_{n_{r_2}}(J_3) = I$. ∎

Corollary 1. *If (f, p) is a Bernoulli endomorphism then the endomorphism T_n is exact for $n \ge \max\{n_0, l\}$.*

Let T_n be as in Corollary 1. Then the a.c.i.m. has the form $p \times m_n$. Let $g_n = dm_n/dm$.

Theorem 6. *If $\tau_n \to \tau$ uniformly on $I - \bigcup_{i=0}^{\infty} R^i$, then $\lim_{n\to\infty} g_n = g$ in L_1, where g is an invariant density of τ.*

Proof. By Theorem 1, the set $\{g_n\}$ is relatively compact in L_1. It suffices to show that any limit point of $\{g_n\}$ is an invariant density of τ. Without loss of generality we can assume that $\lim_{n\to\infty} g_n = g^*$. By Lemma 4 of [2], $\lim_{n\to\infty} \|P_{n_x}h - P_\tau h\|_1 = 0$, for every x. Here $P_{n_x} = P_{\tau_{e_n(x)}}$. Hence

$$\int \|P_{n_x}h - P_\tau h\|_1 \, dp \underset{n}{\to} 0\,,$$

because $\|P_{n_x}h - P_\tau h\|_1 \le 2\|h\|_1$, and next we have

$$\begin{aligned}\|P_\tau g^* - g^*\|_1 &= \|P_f P_\tau g^* - g^*\|_1 \\ &\le \|P_f P_\tau g^* - P_f P_{n_x} g^*\|_1 \\ &\quad + \|P_f P_{n_x} g^* - P_f P_{n_x} g_n\|_1 + \|g_n - g^*\|_1 \\ &\le \int \|P_\tau g^* - P_{n_x} g^*\|_1 \, dp + 2\|g^* - g_n\|_1\,. \ \blacksquare\end{aligned}$$

The piecewise linear Markov approximations of τ satisfy the assumptions of Theorem 6.

In case I, i.e. $T_\varepsilon(x,y) = (f(x), e_\varepsilon(x)\tau(y))$, where $e_\varepsilon : X \to [1-\varepsilon, 1]$ and $0 < \varepsilon < \delta$, we can show in the same manner that the set $\{g_\varepsilon\}_{\varepsilon<\delta}$, where $g_\varepsilon = dm_\varepsilon/dm$ and $\mu_\varepsilon = p \times m_\varepsilon$ is a T-a.c.i.m., is relatively compact in L_1 and any limit point of $\{g_\varepsilon\}_{\varepsilon<\delta}$ is an invariant density of τ.

3. Proof of the Morita Theorem. (1) Let $G \in BV \cap \mathcal{D}$. Then

$$\|P_T^n G\|_\infty = \|P_f^n P_{f^{n-1}(x)} \circ \ldots \circ P_x G\|_\infty \le \|P_{f^{n-1}(x)} \circ \ldots \circ P_x G\|_\infty\,.$$

By inequality (1.3),

$$\begin{aligned}|P_{f^{n-1}(x)} \circ \ldots \circ P_x G| &\le \int |G|\, dm + \bigvee_x P_{f^{n-1}(x)} \circ \ldots \circ P_x G \\ &\le \alpha(n) \bigvee_x G + (c+1) \int |G|\, dm \le M\|G\|_\infty\end{aligned}$$

for some constant $M > 0$. Therefore, the sequence $\frac{1}{n}\sum_{k=0}^{n-1} P_T^k G$ is relatively weakly compact in L_1. By the Kakutani–Yosida Theorem [1], $\frac{1}{n}\sum_{k=0}^{n-1} P_T^k G$ converges strongly in L_1.

(2)–(4). We obtain these by proving (1) and (2) of Lemma 4.1 of [9] and next by applying without change the reasoning from [9], p. 661. The proof of Lemma 4.1 of [9] turns out to be simple by using the equality (1.1) and the inequality (1.3). $\blacksquare$

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