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ASYMPTOTIC DISTRIBUTIONS OF LINEAR COMBINATIONS OF ORDER STATISTICS

Abstract. We study the asymptotic distributions of linear combinations of order statistics (L-statistics) which can be expressed as differentiable statistical functionals and we obtain Berry–Esseen type bounds and the Edgeworth series for the distribution functions of L-statistics. We also analyze certain saddlepoint approximations for the distribution functions of L-statistics.

1. Introduction. This paper is concerned with the asymptotic behaviour of linear combinations of order statistics (or L-statistics), i.e. statistics of the form

$$L_n = \frac{1}{n}\sum_{i=1}^{n} c_{in} X_{i:n}, \qquad n \geq 1\,, \tag{1}$$

where c_{in}, $i = 1, \ldots, n$, are fixed real numbers and $X_{i:n}$, $i = 1, \ldots, n$, are the order statistics of a sequence $X_1, \ldots, X_n$ of i.i.d. random variables (rv's) with common distribution function (df) F. L-statistics are widely used in the robust estimation of location and scale parameters.

The first step in the investigation of L-statistics was to find conditions assuring their asymptotic normality. This problem was studied in the sixties and seventies by Chernoff, Gastwirth and Johns (see [7]), Stigler (see [19, 20, 21]) and Shorack (see [17, 18]), and a little later by Boos (see [4, 5]). A short summary of their results is included in the book [16] in Chapter 8.2.4.

The next step in the development of the theory was to obtain Berry–Esseen type bounds for L-statistics and the approximation of their distributions by the first terms of Edgeworth expansions and by the saddlepoint method. Sections 2 and 3 of the present paper give a short summary of the

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already existing results and some new ones achieved under modified assumptions (Theorems 2, 3, 4 and approximations (38) and (48)). Following Boos (see [5]) we treat the L-statistics as differentiable statistical functions and utilize the von Mises representation. For the estimation of the remainder term of the von Mises expansion we apply the result of Inglot and Ledwina from the theory of strong approximations (see [12]). The investigation of the behaviour of the leading term of this expansion requires only some well-known facts concerning the distributions of the mean and U-statistics.

2. Berry–Esseen type bounds. In 1977 Bjerve (see [2]) obtained the Berry–Esseen rate $O(n^{-1/2})$ for generalized L-statistics of the form

$$T_n = \frac{1}{n}\sum_{i=1}^{n} c_{in} h(X_{i:n}),$$

where h is some measurable function, under the assumption that a certain proportion of the observations among the smallest and the largest are discarded. His theorem concerns the situation when the df F of X_1 is considerably smooth. In particular, for L-statistics of the form

$$E_n = \frac{1}{n}\sum_{i=1}^{n} J\left(\frac{i}{n+1}\right) X_{i:n}, \tag{2}$$

where $J(s)$ is a real-valued function which vanishes outside $[a,b]$, $0 < a < b < 1$, the Berry–Esseen bound holds if $J(s)$ and $F^{-1\prime\prime}(s)$ satisfy the Lipschitz condition on the open interval containing $[a,b]$. As usual, we write

$$F^{-1}(s) = \inf\{x : F(x) \ge s\}.$$

In 1979 Boos and Serfling (see [6]) investigated L-statistics of the form

$$I_n = \int_0^1 F_n^{-1}(s) J(s)\, ds, \tag{3}$$

where $F_n(t)$ is the empirical df based on a sample $X_1, \ldots, X_n$. Equivalently, these statistics can be expressed by formula (1) with $c_{in} = n\int_{(i-1)/n}^{i/n} J(s)\, ds$. If $J'(s)$ satisfies the Lipschitz condition of order δ,

$$|J'(s) - J'(t)| \le D|s-t|^{\delta}, \quad D > 0,$$

with $\delta > 1/3$ or if $J(s)$ vanishes outside $[a,b]$, $0 < a < b < 1$, then providing that some additional assumptions on the distributions are made the authors of [6] achieved a Berry–Esseen rate $O(n^{-1/2})$. As in [2], this result is a conclusion from a more general theorem.

A short summary of all the above mentioned results can be found in the book [16] (Ch. 8.2.5).

The following theorem which puts much weaker conditions on the df F of X_1 and a weight function $J(s)$ was obtained by Helmers in his PhD thesis, published in 1978.

Let us consider a statistic L_n of the form (1).

ASSUMPTION 1. Suppose that a sequence of real numbers $0 < s_1, \ldots \ldots, s_k < 1$, $k \in \mathbb{N}$, is such that $F^{-1}(s)$ satisfies the Lipschitz condition of order $\delta \geq 1/2$ in their neighbourhoods. Suppose further that weights $c_{j_l n}$, $1 \leq l \leq k$, $n \geq 1$, where $j_l = [ns_l] + 1$, are uniformly bounded and that there exists some real-valued measurable function $G(s)$ such that

$$\max_{1\leq i\leq n, i\neq j_1,\ldots,j_k} \Big| c_{in} - n \int_{(i-1)/n}^{i/n} G(s)\, ds \Big| = O(n^{-1}) \quad \text{as } n \to \infty\,.$$

Below we use the following notation:

$$\sigma^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} J(F(x))J(F(y))[\min(F(x), F(y)) - F(x)F(y)]\, dx\, dy\,, \tag{4}$$

$$\mu = \int_0^1 J(s)F^{-1}(s)\, ds\,. \tag{5}$$

THEOREM 1. *Let L_n be a statistic which satisfies Assumption* 1. *If $G(s)$ satisfies the Lipschitz condition of order* 1 *on* $[0,1]$ *and at the same time* $E|X_1|^3 < \infty$ *and* $\sigma^2 > 0$, *then*

$$\sup_{x\in\mathbb{R}} \Big| P\Big(\sqrt{n}\, \frac{L_n - \mu}{\sigma} \leq x\Big) - \Phi(x)\Big| = O(n^{-1/2}) \quad \textit{as } n \to \infty\,.$$

As usual, $\Phi(x)$ denotes the df of the standard normal distribution. It is easy to check that Assumption 1 is satisfied by the L-statistics defined by (3), and if $J(s)$ satisfies the Lipschitz condition of order 1, also by the L-statistics given by (2).

None of the above mentioned theorems can be applied when the function $J(s)$ in (2) and (3) is not continuous, although such a function is very useful for obtaining a trimmed mean. The following theorem, dealing with this situation and proved by elementary methods, gives a Berry–Esseen rate only a little weaker than $O(n^{-1/2})$.

THEOREM 2. *Let I_n be a statistic of the form* (3). *Suppose that*

$$J(s) \text{ vanishes outside } [a,b]\,, \quad 0 < a < b < 1\,, \tag{6}$$

and $J(s)$ satisfies the Lipschitz condition of order 1 *on* $[a,b]$, *i.e.*

$$|J(s) - J(t)| \leq D|s-t|\,, \quad s,t \in [a,b]\,. \tag{7}$$

Moreover, assume

$$F^{-1}(a+\eta)-F^{-1}(a-\eta)=O(\eta)$$

and

$$F^{-1}(b+\eta)-F^{-1}(b-\eta)=O(\eta) \quad \text{as } \eta\to 0\,.$$

Suppose also that $\sigma^2>0$. *Then, uniformly in* $x\in\mathbb{R}$,

$$P\biggl(\sqrt{n}\,\frac{I_n-\mu}{\sigma}<x\biggr)=\Phi(x)+O\biggl(\frac{\log n}{\sqrt{n}}\biggr).$$

Proof. Set $T(H)=\int_0^1 J(s)H^{-1}(s)\,ds$. Then $I_n=T(F_n)$. Taking one term of the von Mises expansion (see [22]) for $T(F_n)-T(F)=I_n-\mu$ we get

$$I_n-\mu=\frac{1}{n}\sum_{i=1}^n h(X_i)+R_{1n}\,, \tag{8}$$

where

$$h(x)=-\int\limits_{-\infty}^{\infty}[I(y\ge x)-F(y)]J(F(y))\,dy \tag{9}$$

and

$$R_{1n}=-\int\limits_{-\infty}^{\infty}\Biggl\{\int\limits_{F(x)}^{F_n(x)}J(s)\,ds\;-J(F(x))[F_n(x)-F(x)]\Biggr\}\,dx\,,$$

with $I(y\ge x)$ denoting the characteristic function of the set $\{y\ge x\}$ (compare [16], Ch. 8.2.4). So, for every $c>0$ and arbitrary ε_n we get

$$\begin{aligned}
P\biggl(\sqrt{n}\,\frac{I_n-\mu}{\sigma}<x\biggr)&\ge P\biggl(\frac{\sqrt{n}}{n\sigma}\sum_{i=1}^n h(X_i)<x-c\varepsilon_n\biggr)\\
&\qquad -P\biggl(\frac{\sqrt{n}}{\sigma}\,|R_{1n}|>c\varepsilon_n\biggr)\quad\text{and}\\
P\biggl(\sqrt{n}\,\frac{I_n-\mu}{\sigma}<x\biggr)&\le P\biggl(\frac{\sqrt{n}}{n\sigma}\sum_{i=1}^n h(X_i)<x+c\varepsilon_n\biggr)\\
&\qquad +P\biggl(\frac{\sqrt{n}}{\sigma}\,|R_{1n}|>c\varepsilon_n\biggr).
\end{aligned} \tag{10}$$

Next we show that there is a constant $c_0>0$ such that for $\varepsilon_n=n^{-1/2}\log n$,

$$P\biggl(\frac{\sqrt{n}}{\sigma}\,|R_{1n}|>c_0\varepsilon_n\biggr)=O(n^{-1/2})\,. \tag{11}$$

Let $\eta_1 > 0$ be such that for some constant $c_1 > 0$ and for every $0 < \eta < \eta_1$,

$$F^{-1}(a+\eta) - F^{-1}(a-\eta) \le c_1\eta \quad \text{and} \tag{12}$$
$$F^{-1}(b+\eta) - F^{-1}(b-\eta) \le c_1\eta\,.$$

Next, take η_0 such that $0 < \eta_0 < \min\{a, 1-b, \frac{1}{2}(b-a), \eta_1\}$. Put $M_n = \sup_{x\in\mathbb{R}} |F_n(x) - F(x)|$. For every $c > 0$,

$$\text{(13)}\quad P\biggl(\frac{\sqrt{n}}{\sigma}\,|R_{1n}| > c\varepsilon_n\biggr) \le P(M_n > \eta_0) + P\biggl(\{M_n \le \eta_0\} \cap \biggl\{\frac{\sqrt{n}}{\sigma}\,|R_{1n}| > c\varepsilon_n\biggr\}\biggr).$$

Applying (12) and the conditions on J (see (6) and (7)), it is easy to check that under the assumption $M_n \le \eta_0$, we have

$$\biggl|\frac{\sqrt{n}}{\sigma}\,R_{1n}\biggr| \le |\sqrt{n}D_2M_n^2|\,,$$

where $D_2 = (c_2 + D_1)/\sigma$, $c_2 = 4c_1\sup_{0\le s\le 1}|J(s)|$, and $D_1 = D[F^{-1}(b) - F^{-1}(a)]$. Thus using (13) and the Dvoretzky–Kiefer–Wolfowitz (D-K-W) inequality we conclude that for every $c > 0$,

$$P\biggl(\frac{\sqrt{n}}{\sigma}\,|R_{1n}| > c\,\frac{\log n}{\sqrt{n}}\biggr) \le P(M_n > \eta_0) + P\biggl(\sqrt{n}D_2M_n^2 > c\,\frac{\log n}{\sqrt{n}}\biggr) \le D_0\exp(-2n\eta_0^2) + D_0\exp\biggl(-\frac{2c\log n}{D_2}\biggr),$$

where D_0 is the constant from the D-K-W inequality. Therefore (11) holds with the constant $c_0 = D_2/4$.

Next we estimate

$$P\biggl(\frac{\sqrt{n}}{\sigma}\,\frac{1}{n}\sum_{i=1}^n h(X_i) < x \pm c_0\,\frac{\log n}{\sqrt{n}}\biggr).$$

From (9) and (6) it is immediate that for every $x \in \mathbb{R}$,

$$\text{(14)}\quad |h(x)| \le \int_{-\infty}^{\infty} |J(F(y))|\,dy = \int_{F^{-1}(a)}^{F^{-1}(b+\eta_0)} |J(F(y))|\,dy \le [F^{-1}(b+\eta_0) - F^{-1}(a)]\sup_{0\le s\le 1}|J(s)| < \infty\,.$$

Thus $h(X_1)$ is a bounded rv and in particular $E|h(X_1)|^3 < \infty$. Applying Fubini's Theorem to the right-hand side of (9) we have $Eh(X_1) = 0$. Because

$Eh^2(X_1) = \sigma^2 > 0$ (see [16], Ch. 8.2.4) we can apply the classical Berry–Esseen Theorem to get

$$P\left(\frac{\sqrt{n}}{\sigma}\,\frac{1}{n}\sum_{i=1}^{n} h(X_i) < x\right) = \Phi(x) + O(n^{-1/2}),$$

uniformly in $x \in \mathbb{R}$. Because $\Phi(x)$ has a bounded derivative we have

$$P\left(\frac{\sqrt{n}}{\sigma}\,\frac{1}{n}\sum_{i=1}^{n} h(X_i) < x \pm c_0\,\frac{\log n}{\sqrt{n}}\right) = \Phi(x) + O\left(\frac{\log n}{\sqrt{n}}\right) \quad \text{as } n \to \infty.$$

Together with (10) and (11), this completes the proof. ∎

3. Edgeworth expansions. In his 1980 work [11], Helmers gave Edgeworth expansions for distributions of normalized L-statistics expressed by (2) and (3), which produce the error of order $o(n^{-1})$. He achieved his results by analytic methods, under conditions including the existence of $J'''(s)$ on $(0,1)$ and $E|X_1|^4 < \infty$.

In Theorem 3 below, we give the Edgeworth expansion for L-statistics of the form (3) which, in the case when $J'(s)$ satisfies the Lipschitz condition of order 1 on $(0,1)$, produces an error of order $O((\log^2 n)/n)$. In the proof we use probabilistic methods and apply an already known result for U-statistics. In comparison with the work of Helmers [11] we weaken the conditions concerning the smoothness of $J(s)$ but we put some additional requirements on the distribution of X_1.

In the proof of Theorem 3 we need the following lemma.

Lemma 1. *Let $\delta > 0$. Suppose that $E\exp(t|X_1|^\alpha) < \infty$ for some $t > 0$ and $\alpha > 1/(2+\delta)$. If $\{\varepsilon_n\}$ is a sequence of positive numbers satisfying*

$$\varepsilon_n = o(n^{-1/2}) \quad \textit{and} \quad \gamma_n = \varepsilon_n^2 n^{\delta+1} \to \infty \quad \textit{as } n \to \infty,$$

then there exists $c > 0$ such that for sufficiently large n,

$$P\Big(\sqrt{n}\int_{-\infty}^{\infty} |F_n(x) - F(x)|^{\delta+2}\,dx \ge \varepsilon_n\Big) \le \exp(-c\gamma_n^{1/(\delta+2)}),$$

where, as usual, $F(x)$ denotes the df of the rv X_1 and $F_n(x)$ is the empirical df based on $X_1, \ldots, X_n$.

Proof. Let $U_1, \ldots, U_n$ be independent uniform $(0,1)$ rv's. It is well known that the joint distribution of $X_1, \ldots, X_n$ is the same as that of $F^{-1}(U_1), \ldots, F^{-1}(U_n)$. Therefore we identify X_i with $F^{-1}(U_i)$, $i = 1, \ldots, n$.

Let $\Gamma_n(x)$ denote the empirical df based on a sample $U_1, \ldots, U_n$ and let α_n denote the classical empirical process, i.e.

$$\alpha_n(u) = \sqrt{n}\,[\Gamma_n(u) - u], \quad u \in (0,1).$$

It is easy to check that $\Gamma_n(F(x)) = F_n(x)$ for every $x \in \mathbb{R}$. Thus we obtain

$$\int_{-\infty}^{\infty} |F_n(x) - F(x)|^{\delta+2}\,dx = \int_{-\infty}^{\infty} |\Gamma_n(F(x)) - F(x)|^{\delta+2}\,dx = \int_0^1 |\Gamma_n(s) - s|^{\delta+2}\,dF^{-1}(s)\,.$$

Therefore

$$(15)\qquad P\Big(\sqrt{n}\int_{-\infty}^{\infty} |F_n(x) - F(x)|^{\delta+2}\,dx \ge \varepsilon_n\Big) = P\Big(\sqrt{n}\int_0^1 |\Gamma_n(s) - s|^{\delta+2}\,dF^{-1}(s) \ge \varepsilon_n\Big) = P\bigg(\Lambda(\alpha_n) \ge \sqrt{n}\left(\frac{\varepsilon_n}{\sqrt{n}}\right)^{1/(\delta+2)}\bigg),$$

where

$$\Lambda(x) = \Big\{\int_0^1 |x(s)|^{\delta+2}\,dF^{-1}(s)\Big\}^{1/(\delta+2)}.$$

For any two measurable functions $x(s)$ and $y(s)$,

$$|\Lambda(x) - \Lambda(y)| \le \Lambda(x-y) \le A \sup_{0\le s\le 1} \frac{|x(s) - y(s)|}{|\omega(s)|}\,,$$

where

$$\omega(s) = \left(\log\frac{1}{s(1-s)}\right)^{-1}, \qquad s \in (0,1)\,,$$

and

$$A = \Big(\int_0^1 |\omega(s)|^{\delta+2}\,dF^{-1}(s)\Big)^{1/(\delta+2)}.$$

Applying the Markov inequality and the condition $\alpha(\delta+2) > 1$, it is easy to check that $A < \infty$.

Hence the functional Λ satisfies the assumptions of Proposition 3.2 of [12]. So there exists a number $a > 0$ such that for every sequence of positive numbers x_n satisfying

$$x_n \to 0 \quad\text{and}\quad nx_n^2 \to \infty \qquad \text{as } n \to \infty\,,$$

we get

$$(16)\qquad P(\Lambda(\alpha_n) \ge x_n\sqrt{n}) = \exp\Big\{-\frac{a}{2}nx_n^2 + o(nx_n^2)\Big\}.$$

Therefore, combining (15) and (16), we obtain

$$P\Big(\sqrt{n}\int_{-\infty}^{\infty}|F_n(x)-F(x)|^{\delta+2}\,dx\ge\varepsilon_n\Big)=P\bigg(\Lambda(\alpha_n)\ge\sqrt{n}\,\bigg(\frac{\varepsilon_n}{\sqrt{n}}\bigg)^{1/(\delta+2)}\bigg)$$
$$=\exp\bigg\{-\frac{a}{2}n\left(\frac{\varepsilon_n^2}{n}\right)^{1/(\delta+2)}+o\bigg(n\left(\frac{\varepsilon_n^2}{n}\right)^{1/(\delta+2)}\bigg)\bigg\}$$
$$=\exp\bigg\{-\frac{a}{2}\gamma_n^{1/(\delta+2)}+o(\gamma_n^{1/(\delta+2)})\bigg\}.$$

Thus, for sufficiently large n and $c=a/4$ we have

$$P\Big(\sqrt{n}\int_{-\infty}^{\infty}|F_n(x)-F(x)|^{\delta+2}\,dx\ge\varepsilon_n\Big)\le\exp(-c\gamma_n^{1/(\delta+2)})\,.\ \blacksquare$$

Before we give Edgeworth expansions for L-statistics we introduce the following notation. Let $h(x)$, σ and μ be as in Section 1 (see (9), (4) and (5)). Let

$$\begin{aligned}
&\beta(x,y)=-\int_{-\infty}^{\infty}[I(x\le z)-F(z)][I(y\le z)-F(z)]J'(F(z))\,dz\,,\\
&\alpha(x,y)=\tfrac{1}{2}[h(x)+h(y)+\beta(x,y)]\,,\\
&\kappa_3=\frac{Eh^3(X_1)+3E\{h(X_1)h(X_2)\beta(X_1,X_2)\}}{\sigma^3}\,,\\
&\mu_n=\mu+\frac{E\alpha(X_1,X_1)}{n}\,,\\
&\vartheta_n(x)=\Phi(x)-\phi(x)\frac{\kappa_3}{6}(x^2-1)n^{-1/2}\,.
\end{aligned}\tag{17}$$

Φ and ϕ denote, as usual, the df and the density of the standard normal distribution.

We call λ an *eigenvalue* of the function $\beta(x,y)$ with respect to the df F if there exists a function $\Psi(x)$ (an eigenfunction) such that

$$\int_{-\infty}^{\infty}\beta(x,y)\Psi(x)\,dF(x)\equiv\lambda\Psi(y)\,.$$

THEOREM 3. *Let I_n be a statistic given by* (3). *Suppose that $J'(x)$ satisfies the Lipschitz condition of order $\delta>0$ with a constant $D<\infty$, and*

$$E\exp(t|X_1|^{\gamma})<\infty \text{ for some } t>0 \text{ and } \gamma>\frac{1}{\delta+2}\,,\tag{18}$$

$$\limsup_{|t|\to\infty}|Ee^{ith(X_1)}|<1\,,\tag{19}$$

(20) $$\limsup_{|t|\to\infty} |Ee^{it\alpha(X_1,X_1)}| < 1\,,$$

(21) *$\beta(x,y)$ has at least 5 nonzero eigenvalues with respect to F.*

Then, uniformly in $x \in \mathbb{R}$,

(22) $$P\bigg(\frac{\sqrt{n}(I_n-\mu_n)}{\sigma} \le x\bigg) = \vartheta_n(x) + O\bigg(\frac{\log^{\delta/2+1} n}{n^{(\delta+1)/2}}\bigg) \quad \text{as } n \to \infty\,.$$

P r o o f. From the definition (3) of I_n we have $I_n = T(F_n)$, where $T(H) = \int_0^1 J(s)H^{-1}(s)\,ds$, for any df H. To the expression $I_n - \mu = T(F_n) - T(F)$ we apply the following von Mises expansion obtained by Serfling (see [16], Ch. 8.2.5):

$$I_n - \mu = \frac{1}{n^2}\sum_{1\le i,j\le n} \alpha(X_i,X_j) + R_{2n}\,,$$

where

(23) $$R_{2n} = -\int\limits_{-\infty}^{\infty}\bigg\{\int\limits_{F(x)}^{F_n(x)} J(s)\,ds - J(F(x))[F_n(x)-F(x)] \\ -\tfrac{1}{2}J'(F(x))[F_n(x)-F(x)]^2\bigg\}\,dx\,.$$

Notice that

(24) $$\frac{1}{n^2}\sum_{1\le i,j\le n}\alpha(X_i,X_j) = \frac{n-1}{n}U_n + \frac{1}{n}W_n\,,$$

where $U_n = \binom{n}{2}^{-1}\sum_{1\le i<j\le n}\alpha(X_i,X_j)$ is a U-statistic with kernel $\alpha(x,y)$, and

(25) $$W_n = \frac{1}{n}\sum_{1\le i\le n}\alpha(X_i,X_i)\,.$$

Thus we have

$$I_n - \mu_n = \frac{n-1}{n}U_n + Z_n + R_{2n}\,,$$

where

$$Z_n = \frac{1}{n}(W_n - E\alpha(X_1,X_1))\,.$$

From (19) we conclude that $\sigma^2 = \operatorname{Var} h(X_1) > 0$, so for every $\varepsilon_n > 0$ we obtain

(26) $$P\bigg(\frac{\sqrt{n}}{\sigma}\,(I_n-\mu_n) \le x\bigg) \le P\bigg(\frac{\sqrt{n}}{\sigma}\,\frac{n-1}{n}U_n \le x + 2\varepsilon_n\bigg) \\ + P\bigg(\frac{\sqrt{n}}{\sigma}\,|Z_n + R_{2n}| > 2\varepsilon_n\bigg)$$

and

$$P\bigg(\frac{\sqrt{n}}{\sigma}(I_n-\mu_n)\le x\bigg)\ge P\bigg(\frac{\sqrt{n}}{\sigma}\frac{n-1}{n}U_n\le x-2\varepsilon_n\bigg) - P\bigg(\frac{\sqrt{n}}{\sigma}|Z_n+R_{2n}|>2\varepsilon_n\bigg). \tag{27}$$

It is evident that

$$P\bigg(\frac{\sqrt{n}}{\sigma}|Z_n+R_{2n}|>2\varepsilon_n\bigg)\le P\bigg(\frac{\sqrt{n}}{\sigma}|Z_n|>\varepsilon_n\bigg)+P\bigg(\frac{\sqrt{n}}{\sigma}|R_{2n}|>\varepsilon_n\bigg).$$

We examine the expression $P\big(\frac{\sqrt{n}}{\sigma}|R_{2n}|>\varepsilon_n\big)$. Using the Lipschitz condition for J' we obtain

$$|R_{2n}|\le\frac{D}{2}\int_{-\infty}^{\infty}|F_n(x)-F(x)|^{\delta+2}\,dx.$$

Applying Lemma 1 with

$$\varepsilon_n=c_1\frac{\log^{\delta/2+1}n}{n^{(\delta+1)/2}}, \tag{28}$$

where $c_1=\frac{D}{2\sigma}c^{(\delta+2)/2}$, we have

$$P\bigg(\frac{\sqrt{n}}{\sigma}|R_{2n}|>\varepsilon_n\bigg)=O(n^{-1}). \tag{29}$$

Next we consider the expression $P\big(\frac{\sqrt{n}}{\sigma}|Z_n|>\varepsilon_n\big)$. We have

$$P\bigg(\frac{\sqrt{n}}{\sigma}|Z_n|>\varepsilon_n\bigg)\le P\bigg(\frac{\sqrt{n}}{\sigma}(W_n-E\alpha(X_1,X_1))>n\varepsilon_n\bigg) + P\bigg(\frac{\sqrt{n}}{\sigma}(W_n-E\alpha(X_1,X_1))<-n\varepsilon_n\bigg).$$

It is easily seen that

$$\int_{-\infty}^{\infty}|I(X_1\le x)-F(x)|\,dx\le|X_1|+E|X_1|.$$

Thus, using (18), we see that all the moments of the rv's $\alpha(X_1,X_1)$ and $\alpha(X_1,X_2)$ are finite. Because $\alpha(X_1,X_1)$ also satisfies the standard condition (20) we can apply the Edgeworth expansion of order $O(n^{-1})$ for the df of the mean W_n. Thus, it is easy to check that for ε_n given by (28) we have

$$\begin{aligned}P\bigg(\frac{\sqrt{n}}{\sigma}(W_n-E\alpha(X_1,X_1))>n\varepsilon_n\bigg)&=O(n^{-1}),\\ P\bigg(\frac{\sqrt{n}}{\sigma}(W_n-E\alpha(X_1,X_1))<-n\varepsilon_n\bigg)&=O(n^{-1}).\end{aligned} \tag{30}$$

Hence

$$P\left(\frac{\sqrt{n}}{\sigma}\,|Z_n| > \varepsilon_n\right) = O(n^{-1})\,. \tag{31}$$

Next we consider the expression

$$P\left(\frac{\sqrt{n}}{\sigma}\,\frac{n-1}{n}U_n \le x \pm 2\varepsilon_n\right).$$

From Fubini's Theorem we have $E\alpha(X_1, X_2) = 0$ and $E\{\alpha(X_1, X_2) \mid X_1 = x\} = \frac{1}{2}h(x)$. So the assumptions (19) and (21) allow us to apply Corollary 1.1 of [1] (taking $g(x) = h(x)/2$, $\Psi(x, y) = \beta(x, y)/2$, $k = 5$ and $r = 11$). As a result, after some simple calculations exploiting the uniform boundedness of $\vartheta_n(x)$, we obtain

$$P\left(\frac{\sqrt{n}}{\sigma}\frac{n-1}{n}\,U_n \le x \pm 2\varepsilon_n\right) = \vartheta(x) + O(n^{-1}) + O(\varepsilon_n) \quad \text{as } n \to \infty\,. \tag{32}$$

This result combined with (26)–(29) and (31) gives (22). ■

R e m a r k 1. The assumption of the existence of k eigenfunctions (in our case $k = 5$) for the function $\beta(x, y)$ can be replaced by a condition easier to verify: there exist points $x_1, \ldots, x_k$ in the support of the df of X_1 such that the functions $\beta(\cdot, x_1), \ldots, \beta(\cdot, x_k)$ are linearly independent (see [1], p. 1478).

Let us consider the case when $J'(s)$ vanishes outside $[a, b]$, $0 < a < b < 1$. Then we show that (22) holds even if the assumption (18) is not satisfied. To this end we prove the following lemma.

LEMMA 2. *Suppose* $J(s)$ *vanishes outside* $[a, b]$, $0 < a < b < 1$. *Let* $J'(s)$ *satisfy the Lipschitz condition of order* $\delta \in (0, 1]$ *on* $[0, 1]$. *Then there exists* $c > 0$ *such that for* $d_n = cn^{-(\delta+1)/2}\log^{\delta/2+1} n$,

$$P(\sqrt{n}|R_{2n}| > d_n) = O(n^{-1})\,.$$

P r o o f. In this proof we repeat some parts of the proof of Theorem 2. Define $M_n = \sup_{x\in\mathbb{R}} |F_n(x) - F(x)|$. Fix a number η such that $0 < \eta < \min\{a, 1 - b\}$. Then

$$P(\sqrt{n}|R_{2n}| > d_n) \le P(M_n > \eta) + P(\{M_n \le \eta\} \cap \{\sqrt{n}|R_{2n}| > d_n\})\,.$$

It is easy to check that under the assumption $M_n \le \eta$ we have $|R_{2n}| \le D_1 M_n^{\delta+2}$, where

$$D_1 = \frac{D}{2}[F^{-1}(b + \eta) - F^{-1}(a - \eta)] < \infty$$

and D is the constant from the Lipschitz condition for $J'(s)$.

Finally, using the D-K-W inequality, we get for $c = (2^{\delta/2+1}D_1)^{-1}$,

$$\begin{aligned} P(\sqrt{n}|R_{2n}| > d_n) &\le P(M_n > \eta) + P(\sqrt{n}D_1 M_n^{\delta+2} > d_n) \\ &\le D_0 e^{-2n\eta^2} + D_0 \exp\bigg(-2n\bigg(\frac{d_n}{\sqrt{n}D_1}\bigg)^{2/(\delta+2)}\bigg) \\ &= O(n^{-1}), \end{aligned}$$

where D_0 is the constant from the D-K-W inequality. ■

THEOREM 4. *Let $J(s)$ and $J'(s)$ satisfy the assumptions of Lemma* 2. *If the assumptions* (19)–(21) *of Theorem* 3 *are satisfied then* (22) *holds.*

Proof. Since $J(s)$ is continuous and vanishes outside $[a,b]$,

$$|h(x)| \le \sup_{0\le s\le 1} |J(s)|[F^{-1}(b) - F^{-1}(a)] < \infty$$

(compare (14)). Similarly we get

$$|\beta(x,x)| \le \sup_{0\le s\le 1} |J'(s)|[F^{-1}(b) - F^{-1}(a)] < \infty$$

and

$$|\beta(x,y)| \le \sup_{0\le s\le 1} |J'(s)|[F^{-1}(b) - F^{-1}(a)] < \infty\,.$$

So $E|\alpha(X_1,X_1)|^4 < \infty$ and $E|\beta(X_1,X_2)|^{11} < \infty$. Thus we can prove (31) and (32) in the same way as in the proof of Theorem 3, without using the condition (18). We also get (29) as a result of Lemma 2. So, repeating the proof of (26) and (27), we get (22). ■

4. The saddlepoint approximation. In a fundamental 1954 paper, Daniels derived a very accurate approximation to the density of the mean of a sample of independent, identically distributed observations using the saddlepoint technique of asymptotic analysis (see [8]). The resulting approximation is in most cases more accurate (especially in the tails) than the two-term Edgeworth series approximation.

The saddlepoint approximations have been found very useful in a variety of problems in statistics. Reid in [15] gives the general review of their applications and suggests using them for approximations of distributions of L-statistics (p. 222). In this paper we investigate such approximations. At first we present the saddlepoint approximations to the density and df of the mean of a sample of independent rv's.

Let $X_1, \ldots, X_n$ be i.i.d. rv's. Denote the moment generating function of the rv X_1 by $M(t) = E\exp(tX_1)$ and its cumulant generating function by $K(t) = \log M(t)$. Assume that $M(t)$ and $K(t)$ exist in an open neighbourhood of the origin. Then the density of $\overline{X}_n = \frac{1}{n}\sum_{i=1}^n X_i$ is expressed as the inversion integral of its moment generating function:

$$f_{\overline{X}_n}(x) = \frac{n}{2\pi i} \int_{r-i\infty}^{r+i\infty} \exp\{n[K(t) - tx]\}\, dt\,, \tag{33}$$

where $r \in \mathbb{R}$ is such that $M(r) < \infty$. The leading contribution to the value of the integral (33) comes from a small region near the real saddlepoint $\widehat{t} = \widehat{t}(x)$ of the function $K(t) - tx$, i.e. the real number defined by

$$K'(\widehat{t}\,) = x, \qquad \widehat{t} \in \mathbb{R}\,, \tag{34}$$

and the saddlepoint approximation to the density of the mean $\overline{X}_n$ is

$$f_{\overline{X}_n}(x) = \sqrt{\frac{n}{2\pi K''(\widehat{t}\,)}} \exp\{n[K(\widehat{t}\,) - \widehat{t}x]\}[1 + O(n^{-1})] \tag{35}$$

(see [8]). In the same paper Daniels showed that a unique real root of the saddlepoint equation (34) exists under very broad conditions.

In 1980, Lugannani and Rice, applying the idea of Bleinstein (see [3]), derived the approximation for the tail probability of $\overline{X}_n$ which proved to be very accurate over the whole range of arguments for which the saddlepoints exist (see [14]). Their result is

$$P(\overline{X}_n > x) = 1 - \Phi(\xi) + \phi(\xi)\left\{\frac{1}{z} - \frac{1}{\xi} + O(n^{-3/2})\right\}, \tag{36}$$

where

$$z = \widehat{t}\sqrt{nK''(\widehat{t}\,)}\,, \qquad \xi = \sqrt{2n[\widehat{t}x - K(\widehat{t}\,)]}\,\mathrm{sgn}(\widehat{t}\,)\,.$$

At $x = EX_1$, (36) reduces to

$$P(\overline{X}_n > EX_1) = \frac{1}{2} - \frac{1}{6}\frac{\lambda}{\sqrt{2\pi n}} + O(n^{-3/2})\,,$$

where $\lambda = K^{(3)}(0)/[K''(0)]^{3/2}$.

The approximation (36) has been discussed by Daniels in [9]. Some remarks on the uniformity of the error in (35) and (36) can be found in [13]. For many standard distributions of X_i's the error in (36) can be bounded uniformly in some neighbourhood of $x = 0$ if the saddlepoints exist for all x from some larger neighbourhood of 0. In that case from (36) we have

$$\begin{aligned} P\left(\sqrt{\frac{n}{\mathrm{Var}\, X_1}}\overline{X}_n > x\right) &= P(\overline{X}_n > x\sqrt{n^{-1}\,\mathrm{Var}\, X_1}) \\ &= 1 - \Phi(\xi_n) + \phi(\xi_n)\left\{\frac{1}{z_n} - \frac{1}{\xi_n} + O(n^{-3/2})\right\}, \end{aligned} \tag{37}$$

where

$$z_n = \widehat{t}_n \sqrt{nK''(\widehat{t}_n)}\,,$$
$$\xi_n = \sqrt{2n[\widehat{t}_n x \sqrt{n^{-1}\operatorname{Var} X_1} - K(\widehat{t}_n)]}\,\operatorname{sgn}(\widehat{t}_n)\,,$$

and the saddlepoint $\widehat{t}_n$ is given by

$$K'(\widehat{t}_n) = x\sqrt{n^{-1}\operatorname{Var} X_1}\,, \qquad \widehat{t}_n \in \mathbb{R}\,.$$

For L-statistics of the form (3) we have

$$I_n - \mu = \frac{1}{n}\sum_{i=1}^{n} h(X_i) + R_{1n}$$

(see (8)). In most cases we can apply (37) for the mean $\overline{h}_n = \frac{1}{n}\sum_{i=1}^{n} h(X_i)$. Therefore we would like to know how accurate is the approximation of the df of a normalized L-statistic by

$$LR(x) = \Phi(\xi_n) - \phi(\xi_n)\left\{\frac{1}{z_n} - \frac{1}{\xi_n}\right\}, \tag{38}$$

where ξ_n and z_n are given by (37) with $K(t) = \log E[\exp(th(X_1))]$. To answer this question we compare the Edgeworth series for the mean $\overline{h}_n$ and for the L-statistic I_n.

The Edgeworth series for the statistics I_n and E_n (see (3) and (2)) can be found in [11]. They were obtained under assumptions on the smoothness of $J(s)$ (the existence of $J'''(s)$) and on the existence of $E|X_1|^4$. Denote by S_n any of the statistics I_n and E_n. We have

$$P\left(\sqrt{n}\,\frac{S_n - \mu}{\sigma} \le x\right) = \Phi(x) - \phi(x)\left[\frac{\kappa_3}{6\sigma^3}(x^2-1) - a_3\right] n^{-1/2} + O(n^{-1})\,, \tag{39}$$

with σ, μ, κ_3 as in (4), (5) and (17). The parameter a_3 for E_n is given by

$$a_3 = \frac{1}{\sigma}\left[\frac{1}{2}\int_0^1 s(1-s)J'(s)\,dF^{-1}(s) - \int_0^1 F^{-1}(s)\left(\frac{1}{2} - s\right)J'(s)\,ds\right]$$

and for I_n by

$$a_3 = \frac{1}{2\sigma}\int_0^1 s(1-s)J'(s)\,dF^{-1}(s)$$

(see [11], p. 1363).

Since $Eh(X_1) = 0$ and $\operatorname{Var} h(X_1) = \sigma^2$, the Edgeworth series for the mean $\overline{h}_n$ is

$$P\left(\frac{\sqrt{n}}{\sigma}\,\overline{h}_n \le x\right) = \Phi(x) - \phi(x)(x^2-1)\frac{Eh^3(X_1)}{6\sigma^3} n^{-1/2} + O(n^{-1})\,. \tag{40}$$

Comparing (39) and (40) we obtain

$$P\left(\sqrt{n}\,\frac{S_n-\mu}{\sigma}\le x\right)=P\left(\frac{\sqrt{n}}{\sigma}\,\overline{h}_n\le x\right)+D(x)n^{-1/2}+O(n^{-1})\,, \tag{41}$$

where

$$D(x)=\phi(x)\left[\frac{Eh^3(X_1)-\kappa_3}{6\sigma^3}\,(x^2-1)+a_3\right].$$

Applying (37) for $\overline{h}_n$ we get

$$P\left(\frac{\sqrt{n}}{\sigma}\,\overline{h}_n\le x\right)=LR(x)+O(n^{-3/2})\,, \tag{42}$$

where $LR(x)$ is given by (38). Thus from (41) and (42) we have

$$P\left(\sqrt{n}\,\frac{S_n-\mu}{\sigma}\le x\right)=LR(x)+D(x)n^{-1/2}+O(n^{-1})\,. \tag{43}$$

In most cases $D(x)\ne 0$, so we conclude that the approximation of $P(\sqrt{n}(S_n-\mu)\le\sigma x)$ by $LR(x)$ gives an error of order $O(n^{-1/2})$. However, if the density function of X_1 is symmetric about EX_1 and $J(s)$ is symmetric about $1/2$, then $Eh^3(X_1)=0$, $\kappa_3=0$ and $a_3=0$. Therefore in that case $D(x)\equiv 0$ and (43) reduces to

$$P\left(\sqrt{n}\,\frac{S_n-\mu}{\sigma}\le x\right)=LR(x)+O(n^{-1})\,. \tag{44}$$

On the other hand, from (39) we get

$$P\left(\sqrt{n}\,\frac{S_n-\mu}{\sigma}\le x\right)=\Phi(x)+O(n^{-1})\,.$$

Thus we have shown that in such a symmetric case the approximations of $P(\sqrt{n}(S_n-\mu)\le\sigma x)$ by the Edgeworth series and $LR(x)$ are asymptotically equivalent. We compare the behaviour of these approximations by calculating some examples (see Section 5).

Easton and Ronchetti [10] have proposed another application of the saddlepoint method for approximating the density functions of L-statistics. We briefly recall their approach and also suggest an alternative way of using the Lugannani–Rice formula to approximate the df of S_n. The Easton–Ronchetti approach can be applied when the Edgeworth expansion up to and including the term of order $o(n^{-1})$ for the density $f_n(x)$ of the considered statistic S_n is available, i.e.

$$f_n(x)=\widetilde{f}_n(x)+o(n^{-1})\,, \tag{45}$$

where

$$\widetilde{f}_n(x) = \phi(x)\Big[1 + \frac{\kappa_{3n}}{6}(x^3 - 3x) + \frac{\kappa_{4n}}{24}(x^4 - 6x^2 + 3) + \frac{\kappa_{3n}^2}{72}(x^6 - 15x^4 + 45x^2 - 15)\Big]$$

and κ_{3n} and κ_{4n} are known numbers. Their approach is as follows: let

$$\widetilde{K}_n(t) = \log \int_{-\infty}^{\infty} e^{tx} \widetilde{f}_n(x)\, dx \quad \text{and} \quad \widetilde{R}_n(t) = \widetilde{K}_n(nt)/n\,.$$

By Fourier inversion,

$$\widetilde{f}_n(x) = \frac{n}{2\pi i} \int_{\tau - i\infty}^{\tau + i\infty} \exp[n(\widetilde{R}_n(t) - tx)]\, dt\,.$$

Applying the saddlepoint technique to this integral Easton and Ronchetti have obtained

$$\widetilde{f}_n(x) = \widehat{f}_n(x) + O(n^{-1})\,,$$

where

$$\widehat{f}_n(x) = \sqrt{\frac{n}{2\pi \widetilde{R}_n''(\widehat{t})}} \exp[n(\widetilde{R}_n(\widehat{t}) - \widehat{t}x)]\,, \tag{46}$$

and $\widehat{t}$ is the saddlepoint of the function $\widetilde{R}_n(t) - tx$, i.e. $\widetilde{R}_n'(\widehat{t}) = x$, $\widehat{t} \in \mathbb{R}$. They have also noticed that

$$\widetilde{R}_n(t) = m_n t + n\frac{\sigma_n^2 t^2}{2} + \frac{\kappa_{3n}\sigma_n^3 n^2 t^3}{6} + \frac{\kappa_{4n}\sigma_n^4 n^3 t^4}{24}\,, \tag{47}$$

where m_n is the mean and σ_n^2 the variance of S_n, and that the replacement in (47) of m_n and σ_n by

$$m_n = m_1 + \frac{a_1}{n} + o(n^{-1})\,,$$
$$\sigma_n = \frac{b_0}{n^{1/2}} + \frac{b_1}{n^{3/2}} + o(n^{-3/2})\,,$$

does not change the order of the approximation of $f_n(x)$ by $\widehat{f}_n(x)$. Finally, Easton and Ronchetti obtained the df of S_n by numerical integration of the approximated density $\widehat{f}_n(x)$.

In this paper, by analogy with the above presented method of approximating a density function, we propose approximating the df of S_n by utilizing the Lugannani–Rice formula (36) with $K(t) = \widetilde{R}_n(t)$. Thus to estimate $P(\sqrt{n}(S_n - \mu) \le \sigma x)$ we use the expression

$$Q_n(x) = \Phi(\xi_n) - \phi(\xi_n)\left(\frac{1}{z_n} - \frac{1}{\xi_n}\right), \tag{48}$$

where

$$z_n = \widehat{t}_n \sqrt{n\widetilde{R}''_n(\widehat{t}_n)}\,, \qquad \xi_n = \sqrt{2n\left[\widehat{t}_n x \frac{\sigma}{\sqrt{n}} - \widetilde{R}_n(t) + \mu t\right]}\,\mathrm{sgn}(\widehat{t}_n)\,,$$

and $\widehat{t}_n$ is given by

$$\widetilde{R}'_n(\widehat{t}_n) - \mu = x\frac{\sigma}{\sqrt{n}}\,.$$

We verify the above mentioned approximations in the examples below.

Notice that while approximating the density of S_n and $P\big(\frac{\sqrt{n}}{\sigma}(S_n - \mu) \le x\big)$ by $\widehat{f}_n(x)$ and $Q_n(x)$ respectively, we only use the information given by the Edgeworth series (45), so we should not expect our results to be much better than the Edgeworth expansion.

5. Examples

Example 1. Consider the asymptotically first-order efficient L-estimator Δ_n for the centre θ of the logistic distribution

$$F(x) = \frac{1}{1 + \exp(\theta - x)}\,,$$

which is given by (2), with $J(s) = 6s(1-s)$, i.e.

$$\Delta_n = \frac{1}{n}\sum_{i=1}^{n} 6\frac{i}{n+1}\left(1 - \frac{i}{n+1}\right)X_{i:n}\,.$$

Some approximations of the df of this estimator were investigated by Helmers [11] and Easton and Ronchetti [10].

The results of the approximations of $P(\sqrt{n}(\Delta_n - \mu) \le \sigma x)$, obtained by several different methods, for sample sizes $n = 3, 4, 10, 25$, can be found in Tables 1–4.

In column 2 of Tables 1–4 we denote by $P_n(x)$ the exact values of $P(\sqrt{n}(\Delta_n - \mu) \le \sigma x)$, taken from the work of Helmers [11]. They were calculated by numerical integration for $n = 3$ and 4 and by Monte Carlo simulation for $n = 10$ and 25.

Helmers [11] has given the Edgeworth expansion of order $o(n^{-1})$ for the df of the normalized Δ_n (see p. 1364)

$$P\left(\sqrt{n}\,\frac{\Delta_n - \mu}{\sigma} \le x\right) = H_n(x) + o(n^{-1})\,,$$

where $\mu = \theta$, $\sigma^2 = 3$ and

$$H_n(x) = \Phi(x) - \phi(x)\left[\frac{1}{20n}(x^3 - 3x) + \frac{11 - \pi^2}{n}x\right]. \tag{49}$$

The values of $H_n(x)$ calculated by Helmers can be found in column 5 of Tables 1–4. The values of $\Phi(x)$ are given in column 3 of these tables. It follows from (49) that $\Phi(x)$ approximates the df of Δ_n with an error of order $O(n^{-1})$.

In column 4 the values of $LR(x)$ given by (38) can be found.

For the function $J(s) = 6s(1-s)$,

$$h(X_1) = -\int\limits_{-\infty}^{\infty} [I(y \geq X_1) - F(y)]J(F(y))\,dy = 6F(X_1) - 3$$

is a uniform rv on $[-3,3]$. For the uniform distribution the Lugannani–Rice formula (36) gives a uniformly bounded error in some neighbourhood of its mean. Therefore using (37) we have

$$P\left(\frac{\sqrt{n}}{\sigma}\,\overline{h}_n \leq x\right) = LR(x) + O(n^{-3/2}) \quad \text{as } n \to \infty\,.$$

Because J is symmetric about $1/2$ and the density of X_1 is symmetric about θ, we conclude that (44) holds for Δ_n, so the approximation $LR(x)$ for the df of the normalized Δ_n gives an error not larger than $O(n^{-1})$ (it is easy to check that this error is not $o(n^{-1})$). Taking the Edgeworth series (49), Easton and Ronchetti in [10] have approximated $P(\sqrt{n}(\Delta_n - \mu) \leq \sigma x)$ by numerical integration of $\widehat{f}_n(x)$ given by (46). In the considered case

$$\widetilde{R}_n(t) = m_n t + \frac{1}{2}n\sigma_n^2 t^2 + \frac{1}{20}\sigma_n^4 n^3 t^4\,, \tag{50}$$

where $m_n = \theta + O(n^{-2})$ and

$$\sigma_n = \sqrt{\frac{3}{n}} + \frac{11-\pi^2}{n}\sqrt{\frac{3}{n}} + O(n^{-2})$$

(see [10], equations (2.5), (4.2), (4.3) and the remark below equation (4.1)).

Their results, denoted by $ER1(x)$, can be found in column 6 of Tables 1–4. Since usually $\widehat{f}_n(x)$ does not integrate to 1, Easton and Ronchetti have also calculated the values of $ER1(x)$ rescaled in such a way that the approximation obtained has got the features of df. These modified results, denoted by $ER2(x)$, are given in column 7 of Tables 1–4. In column 8 the values of Q_n given by (48) (with $\widetilde{R}_n(t)$ as in (50)) can be found.

TABLE 1

Exact and approximate df of Δ_n; $n = 3$

1	2	3	4	5	6	7	8
x	$P_n(x)$	$\Phi(x)$	$LR(x)$	$H_n(x)$	$ER1(x)$	$ER2(x)$	$Q_n(x)$
0.2	.5640	.5793	.5754	.5536	.5735	.5617	.5606
0.4	.6262	.6554	.6484	.6069	.6320	.6217	.6196
0.6	.6850	.7257	.7167	.6592	.6874	.6787	.6755
0.8	.7391	.7881	.7786	.7099	.7387	.7314	.7273
1.0	.7875	.8413	.8327	.7582	.7850	.7790	.7741
1.2	.8248	.8849	.8783	.8032	.8259	.8210	.8154
1.4	.8658	.9192	.9152	.8439	.8610	.8572	.8512
1.6	.8958	.9452	.9439	.8796	.8908	.8877	.8816
1.8	.9202	.9641	.9651	.9100	.9154	.9130	.9070
2.0	.9397	.9772	.9800	.9348	.9353	.9335	.9278
2.2	.9550	.9861	.9897	.9543	.9513	.9499	.9446
2.4	.9669	.9918	.9956	.9691	.9638	.9628	.9580
2.6	.9758	.9953	.9987	.9798	.9734	.9727	.9685
2.8	.9825	.9974	–	.9873	.9807	.9802	.9766
3.0	.9875	.9987	–	.9923	.9862	.9858	.9828

TABLE 2

Exact and approximate df of Δ_n; $n = 4$

1	2	3	4	5	6	7	8
x	$P_n(x)$	$\Phi(x)$	$LR(x)$	$H_n(x)$	$ER1(x)$	$ER2(x)$	$Q_n(x)$
0.2	.5663	.5793	.5763	.5601	.5750	.5650	.5642
0.4	.6307	.6554	.6501	.6190	.6366	.6281	.6266
0.6	.6919	.7257	.7190	.6758	.6949	.6877	.6856
0.8	.7469	.7881	.7811	.7295	.7484	.7424	.7397
1.0	.7963	.8413	.8350	.7790	.7962	.7914	.7882
1.2	.8391	.8849	.8801	.8236	.8379	.8341	.8305
1.4	.8752	.9192	.9163	.8627	.8732	.8703	.8665
1.6	.9049	.9452	.9442	.8960	.9026	.9003	.8966
1.8	.9287	.9641	.9647	.9235	.9264	.9247	.9211
2.0	.9474	.9772	.9790	.9454	.9453	.9440	.9407
2.2	.9618	.9861	.9885	.9622	.9600	.9591	.9561
2.4	.9726	.9918	.9942	.9748	.9712	.9705	.9679
2.6	.9807	.9953	.9975	.9837	.9796	.9791	.9769
2.8	.9865	.9974	.9991	.9898	.9857	.9854	.9836
3.0	.9907	.9987	.9998	.9939	.9902	.9899	.9885

TABLE 3

Exact and approximate df of Δ_n; $n = 10$

1	2	3	4	5	6	7	8
x	$P_n(x)$	$\Phi(x)$	$LR(x)$	$H_n(x)$	$ER1(x)$	$ER2(x)$	$Q_n(x)$
0.2	.5734	.5793	.5781	.5716	.5776	.5725	.5723
0.4	.6445	.6554	.6533	.6409	.6468	.6426	.6423
0.6	.7089	.7257	.7231	.7058	.7115	.7080	.7075
0.8	.7680	.7881	.7854	.7647	.7698	.7670	.7665
1.0	.8196	.8413	.8389	.8164	.8208	.8186	.8180
1.2	.8629	.8849	.8831	.8604	.8638	.8622	.8615
1.4	.8985	.9192	.9181	.8966	.8990	.8978	.8971
1.6	.9275	.9452	.9448	.9255	.9269	.9260	.9254
1.8	.9486	.9641	.9643	.9478	.9483	.9477	.9472
2.0	.9646	.9772	.9790	.9645	.9644	.9639	.9635
2.2	.9764	.9861	.9869	.9766	.9760	.9757	.9753
2.4	.9845	.9918	.9926	.9850	.9842	.9840	.9837
2.6	.9905	.9953	.9961	.9907	.9898	.9897	.9895
2.8	.9937	.9974	.9980	.9944	.9936	.9935	.9934
3.0	.9959	.9987	.9991	.9967	.9961	.9960	.9959

TABLE 4

Exact and approximate df of Δ_n; $n = 25$

1	2	3	4	5	6	7	8
x	$P_n(x)$	$\Phi(x)$	$LR(x)$	$H_n(x)$	$ER1(x)$	$ER2(x)$	$Q_n(x)$
0.2	.5785	.5793	.5788	.5762	.5787	.5763	.5763
0.4	.6492	.6554	.6546	.6496	.6518	.6499	.6498
0.6	.7152	.7257	.7247	.7178	.7196	.7181	.7181
0.8	.7728	.7881	.7870	.7787	.7803	.7791	.7791
1.0	.8295	.8413	.8404	.8314	.8326	.8317	.8316
1.2	.8756	.8849	.8842	.8751	.8761	.8754	.8753
1.4	.9100	.9192	.9188	.9102	.9108	.9103	.9103
1.6	.9376	.9452	.9450	.9373	.9377	.9373	.9373
1.8	.9580	.9641	.9641	.9576	.9577	.9575	.9574
2.0	.9732	.9772	.9775	.9721	.9722	.9720	.9720
2.2	.9830	.9861	.9864	.9823	.9822	.9821	.9821
2.4	.9895	.9918	.9921	.9891	.9890	.9889	.9889
2.6	.9942	.9953	.9956	.9935	.9933	.9933	.9933
2.8	.9963	.9974	.9977	.9962	.9961	.9961	.9961
3.0	.9982	.9987	.9988	.9979	.9978	.9978	.9978

Comparing the asymptotically equivalent approximations $\Phi(x)$ and $LR(x)$, which are shown in columns 3 and 4, we notice that for larger sample sizes ($n = 10$, 20) they give very similar results over the whole range of x. For very small sample sizes ($n = 3$, 4) and $x \leq 1.6$, the approximation $LR(x)$ is a little better but for $x \geq 2.2$ it becomes much worse than $\Phi(x)$. We could expect this to happen because near the ends of the support ($[-3, 3]$) of the density of the mean $\overline{h}_n$ its df cannot be an accurate approximation for the df of the normalized Δ_n, whose support is the whole real line. For larger sample sizes this phenomenon is not so significant.

The approximations from columns 5–8 ($H_n(x)$, $ER1(x)$, $ER2(x)$ and $Q_n(x)$) are much more accurate than $LR(x)$ and $\Phi(x)$. For larger n their results are very similar to each other, and the differences are within the bounds of the error of the Monte Carlo method. Also for very small n and $x > 1.8$, the values in columns 5–8 are comparable.

For very small n and $x \leq 1.8$, the results of the approximation by the Edgeworth series $H_n(x)$ are worse than those of $Q_n(x)$, $ER1(x)$ and $ER2(x)$, which are still similar to each other. We should remark that to compute $Q_n(x)$ we do not need to integrate numerically, unlike in the cases of $ER1(x)$ and $ER2(x)$, so $Q_n(x)$ is easier to calculate.

Example 2. We consider the estimator χ_n for the centre θ of the logistic distribution, given by (3) with $J(s) = 6s(1-s)$.

In this case the Edgeworth series obtained by Helmers [11] is of the form

$$P\left(\sqrt{n}\,\frac{\chi_n - \mu}{\sigma} \leq x\right) = H_n(x) + o(n^{-1}),$$

where $\mu = \theta$, $\sigma^2 = 3$ and

$$H_n(x) = \Phi(x) - \phi(x)\left[\frac{1}{20n}(x^3 - 3x) + \frac{10-\pi^2}{n}x\right]. \tag{51}$$

So, similarly to Example 1, we have

$$P\left(\sqrt{n}\,\frac{\chi_n - \mu}{\sigma} \leq x\right) = \Phi(x) + O(n^{-1}).$$

Furthermore,

$$P\left(\sqrt{n}\,\frac{\chi_n - \mu}{\sigma} \leq x\right) = LR(x) + O(n^{-1}),$$

where $LR(x)$ is as in (38) with $h(x) = 6F(x) - 3$.

Tables 5–8 are similar to Tables 1–4. In column 2 the exact values of $P\big(\sqrt{n}\frac{\chi_n-\mu}{\sigma} \leq x\big)$ are given. For $n = 3$ and 4 they were calculated by numerical integration of the joint density of the random vector $X_{1:n}, \ldots, X_{n:n}$. For $n = 10$ and 25 we have applied the Monte Carlo method.

In columns 3 and 4 of Tables 5–8 the values of $\Phi(x)$ and $LR(x)$, respectively, are given, which are the same as in Tables 1–4. In column 5, the values of the Edgeworth series $H_n(x)$ given by (51) can be found. In column 6 the values of $Q_n(x)$ calculated from (48) are shown, with

$$\widetilde{R}_n(t) = m_n t + \frac{1}{2} n\sigma_n^2 t^2 + \frac{1}{20} n^3 \sigma_n^4 t^4 \,,$$

where $m_n = \theta + O(n^{-1})$ and

$$\sigma_n = \sqrt{\frac{3}{n}} + \frac{10-\pi^2}{n}\sqrt{\frac{3}{n}} + O(n^{-2})\,.$$

Comparing different approximations of $P\big(\sqrt{n}\,\frac{\chi_n-\mu}{\sigma} \le x\big)$ we notice that they are all very accurate, even for small n. This happens because the statistics given by (3) are more regular than those given by (2).

Summary of the examples. The analysis of our examples shows that approximations based on the saddlepoint method ($LR(x)$, $ER1(x)$, $ER2(x)$, $Q_n(x)$) can be applied for small x and n. For larger n ($n \ge 10$), $\Phi(x)$ gives an approximation comparable with $LR(x)$, and the Edgeworth series $H_n(x)$ comparable with $ER1(x)$, $ER2(x)$ and $Q_n(x)$. In that case serious numerical difficulties resulting from the saddlepoint method disqualify it.

TABLE 5

Exact and approximate df of χ_n; $n = 3$

1	2	3	4	5	6
x	$P_n(x)$	$\Phi(x)$	$LR(x)$	$H_n(x)$	$Q_n(x)$
0.2	.5801	.5793	.5754	.5797	.5797
0.4	.6568	.6554	.6484	.6560	.6557
0.6	.7270	.7257	.7167	.7259	.7252
0.8	.7885	.7881	.7786	.7872	.7861
1.0	.8404	.8413	.8327	.8389	.8374
1.2	.8825	.8849	.8783	.8809	.8793
1.4	.9154	.9192	.9152	.9138	.9127
1.6	.9404	.9452	.9439	.9388	.9376
1.8	.9588	.9641	.9651	.9573	.9567
2.0	.9721	.9772	.9800	.9706	.9702
2.2	.9813	.9861	.9897	.9803	.9800
2.4	.9877	.9918	.9956	.9870	.9868
2.6	.9920	.9953	.9987	.9916	.9914
2.8	.9948	.9974	–	.9947	.9945
3.0	.9967	.9987	–	.9967	.9966

TABLE 6

Exact and approximate df of χ_n; $n = 4$

1	2	3	4	5	6
x	$P_n(x)$	$\Phi(x)$	$LR(x)$	$H_n(x)$	$Q_n(x)$
0.2	.5798	.5793	.5763	.5796	.5796
0.4	.6563	.6554	.6501	.6558	.6557
0.6	.7264	.7257	.7190	.7258	.7255
0.8	.7882	.7881	.7811	.7874	.7868
1.0	.8403	.8413	.8350	.8395	.8386
1.2	.8828	.8849	.8801	.8819	.8809
1.4	.9161	.9192	.9163	.9151	.9142
1.6	.9413	.9452	.9442	.9404	.9396
1.8	.9599	.9641	.9647	.9590	.9584
2.0	.9731	.9772	.9790	.9724	.9720
2.2	.9824	.9861	.9885	.9818	.9815
2.4	.9886	.9918	.9942	.9882	.9881
2.6	.9928	.9953	.9975	.9925	.9924
2.8	.9955	.9974	.9991	.9954	.9953
3.0	.9972	.9987	.9998	.9972	.9971

TABLE 7

Exact and approximate df of χ_n; $n = 10$

1	2	3	4	5	6
x	$P_n(x)$	$\Phi(x)$	$LR(x)$	$H_n(x)$	$Q_n(x)$
0.2	.5818	.5793	.5781	.5794	.5794
0.4	.6569	.6554	.6533	.6556	.6556
0.6	.7256	.7257	.7231	.7258	.7257
0.8	.7893	.7881	.7854	.7879	.7877
1.0	.8412	.8413	.8389	.8406	.8404
1.2	.8841	.8849	.8831	.8837	.8835
1.4	.9172	.9192	.9181	.9176	.9174
1.6	.9427	.9452	.9448	.9433	.9431
1.8	.9622	.9641	.9643	.9620	.9619
2.0	.9763	.9772	.9790	.9753	.9752
2.2	.9856	.9861	.9869	.9844	.9843
2.4	.9904	.9918	.9926	.9904	.9903
2.6	.9941	.9953	.9961	.9942	.9942
2.8	.9961	.9974	.9980	.9966	.9966
3.0	.9976	.9987	.9991	.9981	.9981

TABLE 8

Exact and approximate df of χ_n; $n = 25$

1	2	3	4	5	6
x	$P_n(x)$	$\Phi(x)$	$LR(x)$	$H_n(x)$	$Q_n(x)$
0.2	.5782	.5793	.5788	.5793	.5793
0.4	.6552	.6554	.6546	.6555	.6555
0.6	.7240	.7257	.7247	.7258	.7257
0.8	.7875	.7881	.7870	.7880	.7880
1.0	.8412	.8413	.8404	.8411	.8410
1.2	.8827	.8849	.8842	.8844	.8844
1.4	.9164	.9192	.9188	.9186	.9185
1.6	.9433	.9452	.9450	.9444	.9444
1.8	.9614	.9641	.9641	.9633	.9632
2.0	.9758	.9772	.9775	.9765	.9764
2.2	.9850	.9861	.9864	.9854	.9854
2.4	.9910	.9918	.9921	.9912	.9912
2.6	.9946	.9953	.9956	.9949	.9949
2.8	.9974	.9974	.9977	.9971	.9971
3.0	.9982	.9987	.9988	.9984	.9984

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