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POISSON SAMPLING FOR SPECTRAL ESTIMATION IN PERIODICALLY CORRELATED PROCESSES

Abstract. We study estimation problems for periodically correlated, non gaussian processes. We estimate the correlation functions and the spectral densities from continuous-time samples. From a random time sample, we construct three types of estimators for the spectral densities and we prove their consistency.

1. Introduction. The processes we encounter in applications are often assumed to be stationary in the wide sense. Many problems pertaining to these processes have been studied. However, in practice, the stationary hypothesis is not always valid. In that case, we say that the processes are nonstationary. In this paper, we consider processes satisfying for all s, t,

$$E(X(t+T)) = E(X(t))$$

and

$$E(X(s+T)X(t+T)) = E(X(s)X(t))$$

where T is a fixed nonnegative number, for example the noise of a periodic oscillator.

Such processes are called periodically correlated up to order two. They are encountered in meteorology, in communications and also in radio-physics. H. L. Hurd [4] has found estimators for the characteristics of periodically correlated processes from continuous-time samples. He showed these estimators are consistent if the process is gaussian. We will show the consistency of these estimators under certain conditions on the fourth cumulant function (the process is not assumed to be gaussian). However, numerical computation of the estimators leads to a discretization of the process. Thus, it is better to estimate the characteristics from discrete samples (periodic or

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random). We will see that these results are valid for almost periodically correlated harmonizable gaussian processes.

2. Generalities

Definition. A second order real process $X(t)$ is called *periodically correlated* up to order 2 with period T if

$$\forall t \in \mathbb{R}, \quad E\{X(t)\} = m(t) = m(t+T)$$

and

$$\forall s,t \in \mathbb{R}, \quad E\{X(s)X(t)\} = R(s,t) = R(s+T, t+T).$$

If $X(t)$ is periodically correlated, then

$$B(t,u) = R(t+u, t),$$

the autocovariance function of the process, is periodic in t with period T. For fixed u, we can suppose that $B(\cdot, u) \in L^1[0,T]$ and so we assume the Fourier series representation

$$B(t,u) = \sum_{k=-\infty}^{\infty} B_k(u) \exp\left(i\frac{2\pi}{T}kt\right)$$

where the coefficient functions are given by

$$B_k(u) = \frac{1}{T}\int_0^T B(t,u)\exp\left(-i\frac{2\pi}{T}kt\right)dt \qquad (k \in \mathbb{Z}).$$

If we assume that $B_k(\cdot) \in L^1(\mathbb{R})$, then

$$g_k(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} B_k(u)\exp(-i\omega u)\,du$$

exists for $k \in \mathbb{Z}$. The functions $g_k(\omega)$ $(k \in \mathbb{Z})$ are called the *spectral density functions* of $X(t)$. For $B_k(\cdot)$ and $g_k(\cdot)$, we have the following properties:

$$\begin{aligned}
&B_k(-u) = \exp\left(-i\frac{2\pi}{T}ku\right)B_k(u),\\
&g_k(\omega) = \frac{1}{2\pi}\int_0^{\infty}\left[\exp(-i\omega u) + \exp\left(i\left(\omega - \frac{2\pi}{T}k\right)\right)\right]B_k(u)\,du,\\
&g_k(\omega) = g_k\left(\frac{2\pi}{T}k - \omega\right),\\
&\overline{g_k(\omega)} = g_{-k}(-\omega).
\end{aligned} \tag{2.1}$$

Definition. A real process $X(t)$ having finite moments of order p is called *periodically correlated* up to order p with period T if for $t_1, \ldots, t_n \in \mathbb{R}$

and for $p_1, \ldots, p_n \in \mathbb{N}$ satisfying $\sum_{i=1}^{n} p_i \le p$,

$$E\{X(t_1+T)^{p_1} \ldots X(t_n+T)^{p_n}\} = E\{X(t_1)^{p_1} \ldots X(t_n)^{p_n}\}.$$

We notice that if $X(t)$ is PC up to order p then $X(t)$ is PC up to order $q \le p$. Throughout the paper, the term *PC processes* will be used for all periodically correlated processes up to order 2.

Assuming $X(t)$ is PC up to order 4, the functions

$$\nu(t, u_1, u_2, u_3) = E(X(t)X(t+u_1)X(t+u_2)X(t+u_3))$$

and

$$\begin{aligned} K(t, u_1, u_2, u_3) = \nu(t, u_1, u_2, u_3) &- B(t, u_1)B(t+u_2, u_3-u_2) \\ &- B(t, u_2)B(t+u_3, u_1-u_2) \\ &- B(t, u_3)B(t+u_1, u_2-u_1) \end{aligned} \tag{2.2}$$

are periodic in t with period T. The last function is called the *fourth cumulant function*. We see that if $X(t)$ is gaussian, then K is identically zero.

3. Estimation from continuous-time samples. We assume throughout that

(1) $X(t)$ is PC up to order 4,
(2) $X(t)$ has uniformly bounded fourth moment, $E\{X(t)^4\} \le M$,
(3) $m(t) \equiv 0$.

From a sample of $X(t)$, $0 \le t \le A$, let us estimate:

- $B_k(u)$ by

$$\widehat{B}_k(A, u) = \begin{cases} \dfrac{1}{A} \displaystyle\int_0^{A-u} X(t)X(t+u) \exp\left(-i\frac{2\pi}{T}kt\right) dt & \text{if } u \ge 0, \\ \dfrac{1}{A} \displaystyle\int_{-u}^{A} X(t)X(t+u) \exp\left(-i\frac{2\pi}{T}kt\right) dt & \text{if } u < 0, \end{cases}$$

- $g_k(\omega)$ by

$$\begin{aligned} \widehat{g}_k(A, \omega) &= \frac{1}{2\pi} \int_{-A}^{A} h(B_A v)\widehat{B}_k(A, v) \exp(-i\lambda v)\, dv \\ &= \frac{1}{B_A} \int_{-\infty}^{\infty} H\left(\frac{u-\lambda}{B_A}\right) g_k(A, u)\, du \end{aligned}$$

where

$$g_k(A, \lambda) = \frac{1}{2\pi A} I_A(\lambda) \overline{I_A\left(\lambda - \frac{2\pi}{T}k\right)}$$

with $I_A(\lambda) = \int_0^A X(t)\exp(-i\lambda t)\,dt$; B_A is a nonnegative function of A for which $\lim_{A\to\infty} B_A = 0$; h is an even, integrable function for which $|h(t)| \le M'$ for t, $h(0) = 1$, and $H(\omega) = \int_{-\infty}^{\infty} h(t)\exp(-i\omega t)\,dt$.

PROPOSITION 3.1. *If* $X(t)$ *is PC up to order* 4 *with* $\int_{-\infty}^{\infty}\int_0^T B(t,u)^2\,dt\,du < \infty$ *and*

$$\forall u \in \mathbb{R}, \qquad \int_{-\infty}^{\infty}\int_0^T |K(t,u,v,v+u)|\,dt\,dv < \infty,$$

then

$$\forall u \in \mathbb{R}, \qquad \lim_{A\to\infty} E\{|\widehat{B}_k(A,u) - B_k(u)|^2\} = 0.$$

Proof. The proof will be given for fixed $u \ge 0$. Let

$$\begin{aligned} J_k(A) &= \frac{1}{A}\int_0^{A-u} (X(t+u)X(t) - B(t,u))\exp\left(-i\frac{2\pi}{T}kt\right)dt \\ &= \widehat{B}_k(A,u) - E\{\widehat{B}_k(A,u)\}. \end{aligned}$$

It suffices to show that $\lim_{A\to\infty} E\{|J_k(A)|^2\} = 0$ since

$$E\{|\widehat{B}_k(A,u) - B_k(u)|^2\}^{1/2} \le E\{|J_k(A)|^2\}^{1/2} + |E(\widehat{B}_k(A,u)) - B_k(u)|.$$

Using (2.2), we have

$$\begin{aligned} E\{|J_k(A)|^2\} \le \frac{1}{A^2}\int_0^{A-u}\int_0^{A-u} &|B(s-t,t+u)B(s-t,t)|\,ds\,dt \\ &+ \frac{1}{A^2}\int_0^{A-u}\int_0^{A-u} |B(t,s-t+u)B(s,t-s+u)|\,ds\,dt \\ &+ \frac{1}{A^2}\int_0^{A-u}\int_0^{A-u} |K(s,u,t-s,t-s+u)|\,ds\,dt. \end{aligned}$$

The first and second terms go to zero as A goes to infinity ([4], Prop. 6). Let us consider the third expression:

$$\begin{aligned} &\frac{1}{A^2}\int_0^{A-u}\int_0^{A-u} |K(s,u,t-s,t-s+u)|\,ds\,dt \\ &\qquad \le \frac{1}{A^2}\int_{-\infty}^{\infty}\int_0^{2A} |K(s,u,t,t+u)|\,ds\,dt \\ &\qquad \le \frac{1}{A^2}\left(1 + \left[\frac{2A}{T}\right]\right)\int_{-\infty}^{\infty} dt \int_0^T |K(s,u,t,t+u)|\,ds = O\left(\frac{1}{A}\right) \end{aligned}$$

where $[x]$ is the integer part of x.

PROPOSITION 3.2. *Let $X(t)$ be PC up to order 4, with $K \in L^1([0,T]\times\mathbb{R}^3)$ and*

$$\int_{-\infty}^{\infty}\Big(\int_0^T B(t,u)^2\,dt\Big)^{1/2}du<\infty\,.$$

Then, for all $\omega_1,\omega_2\ge 0$ and for all $j,k\in\mathbb{Z}$,

$$\lim_{A\to\infty} AB_A|\operatorname{cov}\{\widehat{g}_j(A,\omega_1),\widehat{g}_k(A,\omega_2)\}| \le \frac{2}{2\pi^2}\int_0^{\infty}|h(z)|^2dz\int_{-\infty}^{\infty}du_1\int_{-\infty}^{\infty}S(u+u_1)S(u)\,du<\infty$$

where $S(u)^2=T^{-1}\int_0^T B(t,u)^2\,dt=\sum_{k=-\infty}^{\infty}|B_k(u)|^2$.

Proof. We have

$$\begin{aligned}AB_A\operatorname{cov}\{\widehat{g}_j(A,\omega_1),g_k(A,\omega_2)\} &= \frac{AB_A}{2\pi^2}\int_{-A}^{A}\int_{-A}^{A}h(B_Av_1)h(B_Av_2)\\ &\quad\times\operatorname{cov}\{\widehat{B}_j(A,v_1),\widehat{B}_k(A,v_2)\}\exp(-i\omega_1v_1+i\omega_2v_2)\,dv_1\,dv_2\\ &= I_1+I_2+I_3\end{aligned}$$

where

$$I_1=\frac{1}{A^2}\int_0^{A-v_1}\int_0^{A-v_2}B(t+v_2,s+v_1-t-v_2)B(t,s-t)\times\exp\Big(-i\frac{2\pi}{T}(js-kt)\Big)\,ds\,dt\,,$$

$$I_2=\frac{1}{A^2}\int_0^{A-v_1}\int_0^{A-v_2}B(t,s-t+v_2)B(t+v_2,s-t-v_2)\times\exp\Big(-i\frac{2\pi}{T}(js-kt)\Big)\,ds\,dt\,,$$

$$I_3=\frac{1}{A^2}\int_0^{A-v_1}\int_0^{A-v_2}K(t,v_2,s-t,s-t+v_1)\exp\Big(-i\frac{2\pi}{T}(js-kt)\Big)\,ds\,dt\,,$$

and

$$\lim_{A\to\infty}(|I_1|+|I_2|)\le\frac{C}{2\pi^2}\int_0^{\infty}|h(z)|^2dz\int_{-\infty}^{\infty}du_1\int_{-\infty}^{\infty}S(u+u_1)S(u)\,du$$

([4], Th. 3 and Prop. 7).

On the other hand,

$$|I_3| \le \frac{4B_A}{2\pi^2 A} \int_{-A}^{A} \int_{-A}^{A} |h(B_A v_1) h(B_A v_2)|$$
$$\times \Big(\int_0^{2A} \int_0^{2A} |K(t, v_2, s-t, s-t+v_1)| \, ds \, dt \Big) \, dv_1 \, dv_2$$
$$\le \frac{2M^2 B_A}{\pi^2 A} \left(1 + \left[\frac{2A}{T} \right] \right) \int_0^T \int_{\mathbb{R}^3} |K(t, u_1, u_2, u_3)| \, dt \, du_1 \, du_2 \, du_3$$
$$= O(B_A) \, .$$

If we choose B_A so that $\lim_{A\to\infty} AB_A = \infty$, we obtain the consistency of the estimator $\widehat{g}_k(A, \omega)$.

4. Estimation from periodic sampling. Let $X(t)$ be a PC process with period T and with autocovariance function

$$B(t, u) = \sum_{k\in\mathbb{Z}} B_k(u) \exp\left(i\frac{2\pi}{T} kt \right) .$$

Let $h > 0$; consider the process $Y_n = X(nh)$ with $n \in \mathbb{Z}$. By analogy with the stationary case, we can look if it is possible to estimate the characteristics of $X(t)$ with a sample from Y_n. Let $B^h(m, n) = E(Y_{m+n} Y_m)$, which is the autocovariance function of Y_n. Assume first that $T/h = N$ is a nonnegative integer; then for all integers m and n, $B^h(m + N, n) = B^h(m, n)$, therefore Y_n is PC in discrete time. In this case, we can write

$$B^h(m, n) = \sum_{l=0}^{N-1} B_l^h(n) \exp\left(i\frac{2\pi}{N} ml \right)$$

with

$$B_l^h(n) = \frac{1}{N} \sum_{m=0}^{N-1} B^h(m, n) \exp\left(-i\frac{2\pi}{N} ml \right) .$$

On the other hand,

$$B^h(m, n) = B(mh, nh) = \sum_{k=-\infty}^{\infty} B_k(nh) \exp\left(i\frac{2\pi}{T} kmh \right)$$
$$= \sum_{l=0}^{N-1} \sum_{k=-\infty}^{\infty} B_{jN+l}(nh) \exp\left(i\frac{2\pi}{N} (jN + l)m \right)$$
$$= \sum_{l=0}^{N-1} \Big(\sum_{j=-\infty}^{\infty} B_{jN+l}(nh) \Big) \exp\left(i\frac{2\pi}{N} lm \right) ,$$

so $B_l^h(n) = \sum_{j=-\infty}^{\infty} B_{jN+l}(nh)$.

We will show that periodic sampling involves aliasing, that is, we can find two different autocovariance functions $B_1(t,u), B_2(t,u)$ such that $B_1^h(m,n) = B_2^h(m,n)$ for all m and n. In this case, we say that $B_1(t,u)$ and $B_2(t,u)$ are *aliases*. To show that, for fixed u, assume $B(t,u) \in L^2[0,T]$ and let for $k \in \mathbb{Z}$,

$$B_{p,k}(u) = \cos\left(\frac{2\pi}{T} pu\right) B_k(u)$$

where p is a nonzero integer. For fixed p, $B_{p,k}$ and B_k are two different functions of u, so, in $L^2[0,T]$, the function

$$B_p(t,u) = \sum_{k\in\mathbb{Z}} B_{p,k}(u) \exp\left(i\frac{2\pi}{T}kt\right)$$

is different from $B(t,u)$.

$B_p(t,u)$ is the autocovariance function of a PC process with period T. In fact ([3], Theorem 1), it suffices to show that for integers $k_1,\ldots,k_n$, real numbers $u_1,\ldots,u_n$, and complex numbers $x_1,\ldots,x_n$,

$$A = \sum_{p,q=1}^{n} x_p \overline{x}_q B_{p,k_p k_q}(u_p - u_q) \geq 0$$

with

$$\begin{aligned} B_{p,jk}(u) &= B_{p,k-j}(u) \exp\left(i\frac{2\pi}{T} ju\right) \\ &= \frac{1}{2}\left\{ \exp\left(i\frac{2\pi}{h} pu\right) + \exp\left(-i\frac{2\pi}{h}pu\right)\right\} B_{jk}(u) \end{aligned}$$

where $B_{jk}(u) = B_{k-j}\exp(i\frac{2\pi}{T}ju)$. Now

$$\begin{aligned} 2A = & \sum_{i,j=1}^{n} x_i\overline{x}_j \exp\left\{i\frac{2\pi}{h}(u_i-u_j)\right\} B_{k_ik_j}(u_i-u_j) \\ & + \sum_{i,j=1}^{n} x_i\overline{x}_j \exp\left\{-i\frac{2\pi}{h}(u_i-u_j)\right\} B_{k_ik_j}(u_i-u_j) \\ = & \sum_{i,j=1}^{n} x_i \exp\left(i\frac{2\pi}{h}u_i\right) \overline{x_j \exp\left(i\frac{2\pi}{h}u_j\right)} B_{k_ik_j}(u_i-u_j) \\ & + \sum_{i,j=1}^{n} x_i \exp\left(-i\frac{2\pi}{h}u_i\right) \overline{x_j \exp\left(-i\frac{2\pi}{h}u_j\right)} B_{k_ik_j}(u_i-u_j) \geq 0\,, \end{aligned}$$

since $B(t,u)$ is the autocovariance function of a PC process with period T.

On the other hand,

$$\sum_{j=-\infty}^{\infty} B_{p,jN+l}(nh) = \sum_{j=-\infty}^{\infty} B_{jN+l}(nh)\cos\left(\frac{2\pi}{h}pnh\right) = \sum_{j=-\infty}^{\infty} B_{jN+l}(nh)$$

and therefore $B_p^h(m,n) = B^h(m,n)$.

When T/h is not an integer, let $r(u)$ denote any real even function in $L^1 \cap L^2$, having a continuous second derivative and vanishing at the points $t_n = nh$ for which $r'(0) = r''(0) = 0$. For example, we can take r so that on $]t_n, t_{n+1}[$,

$$r(u) = \exp\left[\frac{1}{(u + t_n + h/2)^2 - h^2/4}\right].$$

Define $a_k(u)$ by

$$a_0(u) = a_0(-u) = r(u) \quad \text{for } u \ge 0,$$

$$a_k(u) = \frac{1}{k^2} r(u) \quad \text{and} \quad a_k(-u) = \exp\left(-i\frac{2\pi}{T}ku\right) a_k(u) \qquad \text{for } k \ge 1 \text{ and } u \ge 0\,,$$

$$a_{-k}(u) = \overline{a_k(u)} \quad \text{for } k \ge 1 \text{ and } u \in \mathbb{R}\,.$$

The functions $a_k(u)$ have continuous second derivatives since $r'(0) = r''(0) = 0$, vanishing at the points $t_n = nh$. We also have $a_k \in L^1 \cap L^2$.

For j, $k \in \mathbb{Z}$, let

$$a_{jk}(u) = a_{k-j}(u)\exp\left(i\frac{2\pi}{T}ju\right),$$

$$A_{jk}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} a_{jk}(\omega)\exp(-i\omega u)\,du\,.$$

We have

$$a_{jk}(-u) = \overline{a_{jk}(u)}\,, \qquad a_{p-j,q-j}(u)\exp\left(i\frac{2\pi}{T}ju\right) = a_{pq}(u)\,,$$

$$A_{jk}(\omega) = \overline{A_{kj}(\omega)}\,, \qquad A_{p-j,q-j}\left(\omega - \frac{2\pi}{T}j\right) = A_{pq}(\omega) \quad \text{and} \quad A_{jk} \in L^1 \cap L^2\,.$$

For $\omega \in [0, 2\pi/T[$, there exists $A_{jk}^{(i)}$, $i = 1,2$, so that $[A_{jk}^{(i)}(\omega)]_{j,k}$, $i = 1,2$, are nonnegative hermitian matrices, $\sup_{i=1,2}\{|A_{jk}^{(i)}(\omega)|\} \le A_{jk}(\omega)$ and $A_{jk}(\omega) = A_{jk}^{(1)}(\omega) - A_{jk}^{(2)}(\omega)$. Thus for $\omega \in [\frac{2\pi}{T}j, \frac{2\pi}{T}(j+1)[$, let for $i = 1,2$,

$$A_{jk}^{(i)}(\omega) = A_{p-j,q-j}^{(i)}\left(\omega - \frac{2\pi}{T}j\right), \qquad p, q \in \mathbb{Z}\,,$$

$$B_{jk}^{(i)}(u) = \int_{-\infty}^{\infty} A_{jk}^{(i)}(u) \exp(i\omega u)\, d\omega = B_{0,k-j}^{(i)}(u) \exp\left(i\frac{2\pi}{T} ju\right).$$

Therefore putting $B_k^{(i)}(u) = B_{0,k}^{(i)}(u)$, we have

$$B_{jk}^{(i)}(u) = B_{k-j}^{(i)}(u) \exp\left(i\frac{2\pi}{T} ju\right).$$

We can easily verify that $|B_k^{(i)}(u)| \le 1/k^2$ for $k \neq 0$, therefore

$$B^{(i)}(t,u) = \sum_{k\in\mathbb{Z}} B_k^{(i)}(u) \exp\left(i\frac{2\pi}{T} ju\right)$$

is well defined. We have

$$\begin{aligned} B_{jk}^{(1)}(t_n) - B_{jk}^{(2)}(t_n) &= \int_{-\infty}^{\infty} (A_{jk}^{(1)}(\omega) - A_{jk}^{(2)}(\omega)) \exp(i\omega t_n)\, d\omega \\ &= a_{jk}(t_n) = 0\,, \end{aligned}$$

therefore $B_k^{(1)}(t_n) = B_k^{(2)}(t_n)$ and for all $m, n \in \mathbb{Z}$, $B^{(1)}(t,t_n) = B^{(2)}(t,t_n)$, $B^{(1)}(t_m,t_n) = B^{(2)}(t_m,t_n)$. We can easily show that for $i = 1,2$, $B^{(i)}(t,u)$ is the autocovariance function of a PC process with period T. We conclude therefore that $B^{(1)}(t,u)$ and $B^{(2)}(t,u)$ are aliases. More generally, the family $B_\alpha(t,u) = \alpha B^{(1)}(t,u) + (1-\alpha)B^{(2)}(t,u)$, with $0 \le \alpha \le 1$, is a family of alias functions.

5. Estimation from random samples

(a) *General case*. Let $X(t)$ be a PC process with period T for which the autocovariance function

$$B(t,u) = \sum_{k\in\mathbb{Z}} B_k(u) \exp\left(-i\frac{2\pi}{T} kt\right)$$

and $B(t,u) \in L^1([0,T]\times\mathbb{R})$. We assume that the spectral density functions g_k belong to $L^1 \cap L^2$. Let $(t_n)_{n\ge0}$ be a nondecreasing sequence of random variables such that $t_0 = 0$ and $t_n = \alpha_n + t_{n-1}$, where $(\alpha_n)_{n\ge1}$ is a sequence of identically distributed, independent random variables, with $E\alpha_n < \infty$ and common probability density $f(u)$, independent of $X(t)$. We assume that $f \in L^2(\mathbb{R})$ and that $f(u) = 0$ for $u \le 0$, which implies that $t_n \ge t_{n-1}$. Let $f_k(u)$ be the probability density function of the sum of k random variables α_i. Then if $\phi(\omega)$ is the characteristic function of α_i,

$$\phi(\omega)^k = \int_{-\infty}^{\infty} f_k(u) \exp(i\omega u)\, du$$

is the characteristic function of the sum of the α_i. Using Parseval's theorem, we get

$$\int_{-\infty}^{\infty} f_k(u)^2\,du = \int_{-\infty}^{\infty} |\phi(\omega)|^{2k}\,d\omega \le \frac{1}{2\pi}\int_{-\infty}^{\infty} |\phi(\omega)|^2\,d\omega$$
$$\le \int_{-\infty}^{\infty} f(u)^2\,du < \infty\,,$$

therefore $f_k \in L^2$ and $\phi^k \in L^2$. Finally, we assume that for $u \ge 0$,

$$\sum_{k\ge 1} f_k(u) = K < \infty\,.$$

Let us now consider the process $Y_n = X(t_n)$, $n \ge 0$; for $n \ge 0$, let

$$C_k^{\alpha}(n) = \frac{1}{KT}\sum_{l\ge 1} E\Big[Y_{l+n}Y_l \exp\Big(-i\frac{2\pi}{T}kt_l\Big)1_{\{t_l<T\}}\Big]\,. \tag{5.1}$$

For $n \ge 1$, we have

$$\begin{aligned}
E\Big[&Y_{l+n}Y_l \exp\Big(-i\frac{2\pi}{T}kt_l\Big)1_{\{t_l<T\}}\Big]\\
&= E\Big[E\Big\{X(t_{l+n})X(t_l)\exp\Big(-i\frac{2\pi}{T}kt_l\Big)1_{\{t_l<T\}}\,\Big|\,(t_n)\Big\}\Big]\\
&= E\Big[B(t_l, t_{l+n}-t_l)\exp\Big(-i\frac{2\pi}{T}kt_l\Big)1_{\{t_l<T\}}\Big]\\
&= \int_{-\infty}^{\infty}\int_0^T B(t,u)\exp\Big(-i\frac{2\pi}{T}kt\Big)f_l(t)f_n(u)\,dt\,du\,,
\end{aligned}$$

since $t_{l+n} - t_l$ with density f_n and t_l with density f_l are independent. On the other hand,

$$\begin{aligned}
\sum_{l\ge 1}\int_{-\infty}^{\infty}\int_0^T \Big|B(t,u)\exp\Big(-i\frac{2\pi}{T}kt\Big)f_l(t)f_n(u)\Big|\,dt\,du&\\
\le K\int_{-\infty}^{\infty}\int_0^T |B(t,u)|\,dt\,du < \infty\,,&
\end{aligned}$$

so, for $n \ge 1$, we have

$$\begin{aligned}
C_k^{\alpha}(n) &= \frac{1}{KT}\sum_{l\ge 1} E\Big[Y_{l+n}Y_l \exp\Big(-i\frac{2\pi}{T}kt_l\Big)1_{\{t_l<T\}}\Big]\\
&= \int_{-\infty}^{\infty}\Big\{\frac{1}{T}\int_0^T B(t,u)\exp\Big(-i\frac{2\pi}{T}kt\Big)\,dt\Big\}f_n(u)\,du
\end{aligned}$$

$$= \int_{-\infty}^{\infty} B_k(u) f_n(u)\, du = \int_{0}^{\infty} B_k(u) f_n(u)\, du\,.$$

Since

$$B_k(u) = \int_{-\infty}^{\infty} g_k(\omega)\exp(i\omega u)\, d\omega \quad \text{and} \quad f_n(u) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \phi(\omega)^n \exp(-i\omega u)\, d\omega,$$

we have, for $n \geq 1$,

$$C_k^{\alpha}(n) = \int_{-\infty}^{\infty} g_k(\omega)\phi(\omega)^n\, d\omega\,.$$

On the other hand,

$$\begin{aligned} C_k^{\alpha}(0) &= \frac{1}{KT}\sum_{l\geq 1} E\Big[X(t_l)^2 \exp\Big(-i\frac{2\pi}{T}kt_l\Big)1_{\{t_l<T\}}\Big] \\ &= \sum_{l\geq 1}\int_{0}^{T} B(t,0)\exp\Big(-i\frac{2\pi}{T}kt\Big) f_l(t)\, dt \\ &= B_k(0) = \int_{-\infty}^{\infty} g_k(\omega)\, d\omega\,. \end{aligned}$$

Finally, for $n \geq 0$ and $k \in \mathbb{Z}$, we have

$$C_k^{\alpha}(n) = \int_{-\infty}^{\infty} g_k(\omega)\phi(\omega)^n\, d\omega\,. \tag{5.2}$$

PROPOSITION 5.1. *Random sampling is alias-free if the characteristic function $\phi(\omega)$ takes no value more than once on the real axis.*

Proof. For $k \in \mathbb{Z}$, $g_k \in L^1 \cap L^2$, and it suffices to apply Theorem 1 of [8] and (5.2).

PROPOSITION 5.2. *If the characteristic function $\phi(s)$ (where $s \in \mathbb{C}$, $\Im(s) \leq 0$) takes the same value at two different points of the open upper half-plane, then aliasing occurs with random sampling.*

Proof. We will show that for $k \geq 0$, there exist $A_k \in L^1 \cap L^2$ not identically zero so that $A_k(\omega) = A_k(\frac{2\pi}{T}k - \omega)$ and $\int_{-\infty}^{\infty} A_k(\omega)\phi(\omega)^n\, d\omega = 0$ for $n \geq 0$. As Fourier transformation is a unitary transformation of L^2 onto itself, if $a_k(u) = \int_{-\infty}^{\infty} A_k(\omega)\exp(i\omega u)\, d\omega$ for $k \geq 0$ and $\mathcal{A}$ is the class of Fourier transforms of L^1 functions, we have to show that there exists a function $a_k \in \mathcal{A} \cap L^2$ not identically zero so that

$$a_k(-u) = \exp\Big(-i\frac{2\pi}{T}ku\Big)a_k(u)\,, \tag{a}$$

(b) $$\int_0^\infty a_k(u) f_n(u)\, du = 0 \qquad (n \ge 1)\,,$$

(c) $$a_k(0) = \int_{-\infty}^\infty A_k(\omega)\, d\omega = 0\,.$$

The existence of a_k may be proved as in the proof of Theorem 2 of [8], p. 239. Assume that A_k is defined for $k \ge 0$. For $k < 0$, let $A_k(-\omega) = \overline{A_{-k}(\omega)}$ and for $k, j \in \mathbb{Z}$, $A_{jk}(\omega) = A_{k-j}(\omega - \frac{2\pi}{T} j)$. We have

$$\begin{aligned}\overline{A_{jk}(\omega)} &= \overline{A_{k-j}\left[\frac{2\pi}{T}(k-j) - \left(\omega - \frac{2\pi}{T} j\right)\right]}\\ &= A_{j-k}\left(\omega - \frac{2\pi}{T} k\right) = A_{kj}(\omega)\,.\end{aligned}$$

We find $A_{jk}(\omega)$ which have the same properties as the properties used to construct alias autocovariance functions.

(b) *Poisson sampling case*. We assume throughout the paper that α_n are r.v. with common exponential density probability, $f(u) = \beta \exp(-\beta u) 1_{\mathbb{R}^+}(u)$; thus

$$f_n(u) = \beta \exp(-\beta u) \frac{(\beta u)^{n-1}}{(n-1)!} 1_{\mathbb{R}^+}(u)\,, \qquad \phi(\omega) = \frac{\beta}{\beta - i\omega}$$

and $\sum_{n\ge 1} f_n(u) = \beta < \infty$. It is easily seen from Proposition 5.1 that Poisson sampling is alias-free. The functions $\{f_n(u) : n \ge 1\}$ form a complete system in $L^2[0,\infty[$, therefore we have a complete orthogonal sequence $\{q_n(u) : n \ge 1\}$ in $L^2[0,\infty[$ such that $q_n(u) = b_{n,1} f_1(u) + \ldots + b_{n,n} f_n(u)$ where

$$b_{n,l} = \left(\frac{2}{\beta}\right)^{1/2} (-2)^{l-1} \mathcal{C}_{n-1}^{l-1}\,.$$

For $n \ge 1$, let

$$\gamma_k^\alpha(n) = \int_0^\infty B_k(u) q_n(u)\, du = b_{n,1} C_k^\alpha(1) + \ldots + b_{n,n} C_k^\alpha(n); \tag{5.3}$$

then in $L^2[0,\infty[$,

$$B_k(u) = \sum_{n\ge 1} \gamma_k^\alpha(n) q_n(u) \qquad (k \in \mathbb{Z})\,. \tag{5.4}$$

Using (2.4), we have in $L^2(\mathbb{R})$,

$$g_k(\omega) = \sum_{n\ge 1} \gamma_k^\alpha(n) \left\{\psi_n(\omega) + \psi_n\left(\frac{2\pi}{T} k - \omega\right)\right\} \qquad (k \in \mathbb{Z}) \tag{5.5}$$

where

$$\psi_n(\omega) = \frac{1}{2\pi} \int_0^\infty \exp(-i\omega u) q_n(u)\, du\,.$$

Further, we will use (5.4) and (5.5) to find estimators of $g_k(\omega)$ and $B_k(u)$. First, we study kernel estimators of $g_k(\omega)$.

Let $H(\omega)$ be an even, absolutely integrable function, with $\int_{-\infty}^{\infty} H(\omega)\, d\omega = 1$. Let M_N be a sequence of nonnegative numbers so that $\lim_{N\to\infty} M_N = \infty$. Consider $W_N(\omega) = M_N H(M_N\omega)$, $h(t) = \int_{-\infty}^{\infty} H(\omega)\exp(i\omega t)\, dt$ and $w_N(t) = h(t/M_N)$. Assuming that we get a sample $X(t_1), \ldots, X(t_N)$, we estimate $g_k(\omega)$ by the following estimators:

$$g_k^{(1)}(N,\omega) = \frac{1}{2\pi\beta N} \sum_{n=1}^{N-1} \sum_{l=1}^{N-n} w_N(t_{l+n}-t_l) X(t_{l+n}) X(t_l) \exp\Big(-i\frac{2\pi}{T}kt_l\Big) \times \Big\{\exp\Big(i\Big(\omega - \frac{2\pi}{T}k\Big)(t_{l+n}-t_l)\Big) + \exp(-i\omega(t_{l+n}-t_l))\Big\}$$

and

$$g_k^{(2)}(N,\omega) = \frac{1}{2\pi\beta N} \sum_{n=1}^{M_N} \sum_{l=1}^{N-n} X(t_{l+n}) X(t_l) \exp\Big(-i\frac{2\pi}{T}kt_l\Big) \times \Big\{\exp\Big(i\Big(\omega - \frac{2\pi}{T}k\Big)(t_{l+n}-t_l)\Big) + \exp(-i\omega(t_{l+n}-t_l))\Big\}.$$

Asymptotic properties of $g_k^{(1)}(N,\omega)$. We have

$$\begin{aligned} g_k^{(1)}(N,\omega) &= \frac{1}{4\pi\beta N} \int_{-\infty}^{\infty} W_N(u) \sum_{n=1}^{N-1} \sum_{l=1}^{N-n} X(t_{l+n}) X(t_l) \exp\Big(-i\frac{2\pi}{T}kt_l\Big) \\ &\quad \times \Big\{\exp\Big(i\Big(\omega + u - \frac{2\pi}{T}k\Big)(t_{l+n}-t_l)\Big) \\ &\qquad\qquad + \exp(-i(\omega+u)(t_{l+n}-t_l))\Big\}\, du \\ &\quad + \frac{1}{4\pi\beta N} \int_{-\infty}^{\infty} W_N(u) \sum_{n=1}^{N-1} \sum_{l=1}^{N-n} X(t_{l+n}) X(t_l) \exp\Big(-i\frac{2\pi}{T}kt_l\Big) \\ &\quad \times \Big\{\exp\Big(i\Big(\omega - u - \frac{2\pi}{T}k\Big)(t_{l+n}-t_l)\Big) \\ &\qquad\qquad + \exp(-i(\omega-u)(t_{l+n}-t_l))\Big\}\, du \\ &= I_1 + I_2\,. \end{aligned}$$

Setting $\lambda = \omega + u$ (resp. $\lambda = \omega - u$) in I_1 (resp. I_2) and using the evenness of W_N, we notice that $I_1 = I_2$, therefore

$$g_k^{(1)}(N,\omega) = \frac{1}{2\pi\beta N}\int_{-\infty}^{\infty} W_N(\omega-\lambda)\sum_{n=1}^{N-1}\sum_{l=1}^{N-n} X(t_{l+n})X(t_l)\exp\Big(-i\frac{2\pi}{T}kt_l\Big)$$
$$\times\Big\{\exp\Big(i\Big(\lambda-\frac{2\pi}{T}k\Big)(t_{l+n}-t_l)\Big)+\exp(-i\lambda(t_{l+n}-t_l))\Big\}d\lambda.$$

We notice that if $\omega \neq 0$ then

$$\Big|\sum_{n=1}^{N}\phi(\omega)^n\Big| = \Big|\phi(\omega)\frac{\phi(\omega)^N-1}{\phi(\omega)-1}\Big| \le 2\Big|\frac{\phi(\omega)}{\phi(\omega)-1}\Big| \le \frac{2\beta}{|\omega|}\,. \tag{5.6}$$

Theorem 5.1. *If $X(t)$ is a PC process with*

$$\int_{-\infty}^{\infty}\Big(\sum_{j\neq k}\frac{|B_j(u)|}{|j-k|}+|B_k(u)|\Big)\,du<\infty\,,$$

then

$$E\{g_k^{(1)}(N,\omega)\} = \int_{-\infty}^{\infty} W_N(\omega-\lambda)g_k(\lambda)\,d\lambda + o(1)\,.$$

Proof. We have $E\{g_k^{(1)}(N,\omega)\} = \int_{-\infty}^{\infty} W_N(\omega-\lambda)I(\lambda)\,d\lambda$ where

$$I(\lambda) = \frac{1}{2\pi\beta N}E\Big(\sum_{n=1}^{N-1}\sum_{l=1}^{N-n}\Big\{\exp\Big(i\Big(\lambda-\frac{2\pi}{T}k\Big)(t_{l+n}-t_l)\Big)$$
$$+\exp(-i\lambda(t_{l+n}-t_l))\Big\}X(t_{l+n})X(t_l)\exp\Big(-i\frac{2\pi}{T}kt_l\Big)\Big)$$
$$= \frac{1}{2\pi\beta N}\sum_{n=1}^{N-1}\sum_{l=1}^{N-n}\int_0^{\infty}\int_0^{\infty}\sum_{j\in\mathbb{Z}}B_j(u)\exp\Big(i\frac{2\pi}{T}jt\Big)\exp\Big(-i\frac{2\pi}{T}kt\Big)$$
$$\times\Big\{\exp\Big(i\Big(\lambda-\frac{2\pi}{T}k\Big)u\Big)+\exp(-i\lambda u)\Big\}f_n(u)f_l(t)\,du\,dt\,.$$

Since

$$\int_{-\infty}^{\infty}\Big(\sum_{j\neq k}\frac{|B_j(u)|}{|j-k|}+|B_k(u)|\Big)\,du<\infty$$

(using Fubini's theorem), we have

$$I(\lambda) = \frac{1}{2\pi\beta N}\sum_{n=1}^{N-1}\int_0^{\infty}\sum_{j\in\mathbb{Z}}B_j(u)\Big\{\exp\Big(i\Big(\lambda-\frac{2\pi}{T}k\Big)u\Big)$$

$$+\exp(-i\lambda u)\Big\}f_n(u)\bigg(\sum_{l=1}^{N-n}\int_0^\infty \exp\Big(i\frac{2\pi}{T}(j-k)t\Big)f_l(t)\,dt\bigg)\,du$$
$$= I_1(\lambda)+I_2(\lambda)\,,$$

where

$$I_1(\lambda)=\frac{1}{2\pi\beta N}\sum_{n=1}^{N-1}(N-n)\int_0^\infty B_k(u)\Big\{\exp\Big(i\Big(\lambda-\frac{2\pi}{T}k\Big)u\Big)$$
$$+\exp(-i\lambda u)\Big\}f_n(u)\,du$$

and

$$I_2(\lambda)=\frac{1}{2\pi\beta N}\sum_{n=1}^{N-1}\int_0^\infty\bigg(\sum_{j\neq k}B_j(u)\Big\{\exp\Big(i\Big(\lambda-\frac{2\pi}{T}k\Big)u\Big)$$
$$+\exp(-i\lambda u)\Big\}\sum_{l=1}^{N-n}\phi\Big(\frac{2\pi}{T}(j-k)\Big)^l\bigg)f_n(u)\,du\,.$$

Using (5.6) and Lemma 1 of [6], we have

$$|I_2(\lambda)|\le\frac{4}{2\pi\beta N}\sum_{n=1}^{N-1}\int_0^\infty\bigg(\sum_{j\neq k}|B_j(u)|\frac{\beta}{\frac{2\pi}{T}|j-k|}\bigg)f_n(u)\,du=O\Big(\frac{1}{N}\Big)$$

where the $O(1/N)$ term is uniform in λ. Put

$$e_n^k(\lambda)=\int_0^\infty B_k(u)\Big\{\exp\Big(i\Big(\lambda-\frac{2\pi}{T}k\Big)u\Big)+\exp(-i\lambda u)\Big\}f_n(u)\,du\,.$$

Then

$$\frac{1}{2\pi\beta}\sum_{n=1}^\infty e_n^k(\lambda)=\frac{1}{2\pi}\int_0^\infty B_k(u)\Big\{\exp\Big(i\Big(\lambda-\frac{2\pi}{T}\Big)u\Big)$$
$$+\exp(-i\lambda u)\Big\}\,du=g_k(\lambda)$$

and therefore

$$I_1(\lambda)=\frac{1}{2\pi\beta}\sum_{n=1}^{N-1}\Big(1-\frac{n}{N}\Big)e_n^k(\lambda)$$
$$=g_k(\lambda)-\frac{1}{2\pi\beta}\sum_{n=N}^{\infty}e_n^k(\lambda)-\frac{1}{2\pi\beta N}\sum_{n=1}^{N-1}ne_n^k(\lambda)\,.$$

Since $|e_n^k(\lambda)|\le 2\int_0^\infty|B_k(u)|f_n(u)\,du$, the series $\sum_{n\ge1}e_n^k(\lambda)$ converges uniformly (Lemma 1 of [6]); so $\sum_{n\ge N}e_n^k(\lambda)$ and $\frac{1}{N}\sum_{n=1}^{N-1}ne_n^k(\lambda)$ (Kronecker's lemma) converge uniformly to 0. Finally,

$$\begin{aligned}E\{g_k^{(1)}(N,\omega)\} &= \int_{-\infty}^{\infty} W_N(\omega-\lambda)[g_k(\lambda)+o(1)+O(1/N)]\,d\lambda\\ &= \int_{-\infty}^{\infty} W_N(\omega-\lambda)g_k(\lambda)\,d\lambda+o(1)\,.\end{aligned}$$

Corollary 5.1. *If $X(t)$ is a PC process with*

$$\int_{-\infty}^{\infty}\Big(\sum_{j\neq k}\frac{|B_j(u)|}{|j-k|}+|B_k(u)|\Big)\,du<\infty$$

and $\int_0^\infty |uB_k(u)|\,du\le\infty$, then

$$E\{g_k^{(1)}(N,\omega)\}=\int_{-\infty}^{\infty} W_N(\omega-\lambda)g_k(\lambda)\,d\lambda+O\Big(\frac{1}{N}\Big)\,.$$

Proof. We have

$$I_1=g_k(\lambda)-\frac{1}{2\pi\beta}\sum_{n\ge N}e_n^k(\lambda)-\frac{1}{2\pi\beta N}\sum_{n=1}^{N-1}ne_n^k(\lambda)$$

and using Lemma 1(ii) of [6], we have

$$\Big|\sum_{n\ge N}e_n^k(\lambda)\Big|\le\frac{2}{N}\sum_{n=N}^{\infty}n\int_0^\infty|B_k(u)|f_n(u)\,du=O\Big(\frac{1}{N}\Big)$$

and

$$\Big|\frac{1}{2\pi\beta N}\sum_{n=1}^{N-1}ne_n^k(\lambda)\Big|\le\frac{1}{N}\sum_{n=1}^{N-1}n\int_0^\infty|B_k(u)|f_n(u)\,du=O\Big(\frac{1}{N}\Big)\,.$$

We notice that the O term is independent of λ, which gives the result.

Corollary 5.2. *If $X(t)$ is a PC process with*

$$\int_{-\infty}^{\infty}\Big(\sum_{j\neq k}\frac{|B_j(u)|}{|j-k|}+|B_k(u)|\Big)\,du<\infty\,,$$

then $g_k^{(1)}$ is an asymptotically unbiased estimator.

Proof. By (2.1) we see that $g_k(\omega)$ is a continuous and bounded function. On the other hand,

$$E\{g_k^{(1)}(N,\omega)\}=\int_{-\infty}^{\infty} W_N(\omega-\lambda)g_k(\lambda)\,d\lambda+o(1)$$

$$= \int_{-\infty}^{\infty} M_N H(M_N(\omega-\lambda)) g_k(\lambda)\,d\lambda + o(1)$$

$$= \int_{-\infty}^{\infty} H(u) g_k\left(\omega - \frac{u}{M_N}\right) du + o(1)$$

and we use Lebesgue's theorem to conclude.

COROLLARY 5.3. *In addition to the hypothesis of Theorem* 5.1, *let* q *be a positive integer such that* $u^q B_k(u) \in L^1[0,\infty[$ *and assume* $h(t)$ *is* q *times differentiable with bounded derivatives. Then*

$$E\{g_k^{(1)}(N,\omega)\} = g_k(\omega) + \sum_{l=1}^{q-1} \frac{i^l h^{(l)}(0)}{M_N^l} g_k^{(l)}(\omega) + O\left(\frac{1}{M_N^q}\right) + O\left(\frac{1}{N}\right)$$

where $f^{(l)}$ *is the derivative of* f *of order* l.

The proof is identical to the proof of Corollary 1.2 of [5], p. 175.

To prove the consistency of the estimator $g_k^{(1)}(N,\lambda)$, we make the following assumptions:

$(\mathcal{H}_1)$ (i) $\displaystyle\int_{-\infty}^{\infty} \left(\sum_{j\neq k} \frac{|B_j(u)|}{|j-k|} + |B_k(u)|\right) du < \infty\,, \qquad \int_0^{\infty} |uB_k(u)|\,du < \infty\,,$

(ii) $\displaystyle\sum_{j\in\mathbb{Z}} |B_j(0)| \int_0^{\infty} |B_j(u)|\,du < \infty\,, \qquad \sum_{j\in\mathbb{Z}} |B_j(0)|^2 < \infty\,.$

$(\mathcal{H}_2)$ For $j \ge 0$, there exists $h_j(u)$ such that

(i) $h_j(u)$ is a continuous, even, nonnegative, integrable and nondecreasing function on $[0,\infty[$,

(ii) $|B_j(u)| \le h_j(u)$,

(iii) $$\int_0^{\infty}\int_0^{\infty} \left(\sum_{j_1+j_2\neq 0} \frac{1}{|j_1+j_2|} h_{j_1}(u_1) h_{j_2}(u_2) + \sum_{j_1+j_2=0} h_{j_1}(u_1)h_{j_2}(u_2)\right) du_1\,du_2 < \infty\,,$$

(iv) $$\int_0^{\infty}\int_0^{\infty} \left(\sum_{j_1+j_2\neq 0} \frac{1}{|j_1+j_2|} h_{j_1}(u_1) h_{j_2}(u_2) + \sum_{j_1+j_2=0} h_{j_1}(u_1)h_{j_2}(u_2)\right)(u_1+u_2)\,du_1\,du_2 < \infty\,.$$

$(\mathcal{H}_3)$ $X(t)$ is a PC process up to order 4 and for $j\in\mathbb{Z}$, there exist nonnegative functions q_j^1, q_j^2, q_j^3 so that

$$|K_j(u_1,u_2,u_3)| \le \prod_{i=1}^{3} q_j^i(u_i)\,,$$

$$\int_0^\infty\int_0^\infty\int_0^\infty \Big(\sum_{j\ne k}\frac{q_j^1(u_1)q_j^2(u_2)q_j^3(u_3)}{|j-k|} + q_k^1(u_1)q_k^2(u_2)q_k^3(u_3)\Big)\,du_1\,du_2\,du_3 < \infty\,.$$

We also assume that the sequence M_N satisfies $\lim_{N\to\infty} M_N/N = 0$.

Let

$$p(l,n,k,\lambda) = \bigg(\exp\Big(i\Big(\lambda - \frac{2\pi}{T}k\Big)(t_{l+n}-t_l)\Big) + \exp(-i\lambda(t_{l+n}-t_l))\bigg)\exp\Big(-i\frac{2\pi}{T}kt_l\Big)\,.$$

Then

$$\operatorname{var}\{g_k^{(1)}(N,\omega)\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W_N(\omega-\lambda)W_N(\omega-\mu)\operatorname{cov}\{J_k(N,\lambda);J_k(N,\mu)\}\,d\lambda\,d\mu \tag{5.7}$$

where

$$J_k(N,\lambda) = \frac{1}{2\pi\beta N}\sum_{n=1}^{N-1}\sum_{l=1}^{N-n} X(t_{l+n})X(t_l)p(l,n,k,\lambda)\,.$$

We have

$$E\{J_k(N,\lambda)\overline{J_k(N,\mu)}\} = \frac{1}{(2\pi\beta N)^2}\sum_{n_1,n_2=1}^{N-1}\sum_{l_1=1}^{N-n_1}\sum_{l_2=1}^{N-n_2} E(p(l_1,n_1,k,\lambda)\overline{p(l_2,n_2,k,\mu)} \times E\{X(t_{l_1+n_1})X(t_{l_1})X(t_{l_2+n_2})X(t_{l_2}) \mid (t_n)\})\,,$$

so $\operatorname{cov}\{J_k(N,\lambda);J_k(N,\mu)\} = \sum_{i=1}^4 U_{N,i}(k,\lambda,\mu) - E\{J_k(N,\lambda)\}E\{\overline{J_k(N,\mu)}\}$ where

$$U_{N,1}(k,\lambda,\mu) = \frac{1}{(2\pi\beta N)^2}\sum_{n_1,n_2=1}^{N-1}\sum_{l_1,l_2\in R} E(p(l_1,n_1,k,\lambda)\overline{p(l_2,n_2,k,\mu)} \times B(t_{l_1},t_{l_1+n_1}-t_{l_1})B(t_{l_2},t_{l_2+n_2}-t_{l_2}))\,, \tag{5.8a}$$

$$U_{N,2}(k,\lambda,\mu) = \frac{1}{(2\pi\beta N)^2}\sum_{n_1,n_2=1}^{N-1}\sum_{l_1,l_2\in R} E(p(l_1,n_1,k,\lambda)\overline{p(l_2,n_2,k,\mu)} \times B(t_{l_1},t_{l_2}-t_{l_1})B(t_{l_1+n_1},t_{l_2+n_2}-t_{l_1+n_1}))\,, \tag{5.8b}$$

(5.8c) $U_{N,3}(k,\lambda,\mu)$

$$= \frac{1}{(2\pi\beta N)^2} \sum_{n_1,n_2=1}^{N-1} \sum_{l_1,l_2\in R} E(p(l_1,n_1,k,\lambda)\overline{p(l_2,n_2,k,\mu)} \times B(t_{l_1}, t_{l_2+n_2}-t_{l_1}) B(t_{l_2}, t_{l_1+n_1}-t_{l_2}))\,,$$

(5.8d) $U_{N,4}(k,\lambda,\mu)$

$$= \frac{1}{(2\pi\beta N)^2} \sum_{n_1,n_2=1}^{N-1} \sum_{l_1,l_2\in R} E(p(l_1,n_1,k,\lambda)\overline{p(l_2,n_2,k,\mu)} \times K(t_{l_1}, t_{l_1+n_1}-t_{l_1}, t_{l_2+n_2}-t_{l_1}, t_{l_2}-t_{l_1}))\,,$$

with $R = \{(l_1,l_2)\in\mathbb{N}^2 : 1\le l_1\le N-n_1,\ 1\le l_2\le N-n_2\}$.

We give here the main results, the proofs are in the appendix.

PROPOSITION 5.3. (a) *Under assumptions* $(\mathcal{H}_1)$ *and* $(\mathcal{H}_2)$,

$$U_{N,1}(k,\lambda,\mu) - E\{J_k(N,\lambda)\}E\{\overline{J_k(N,\mu)}\} = O(1/N)$$

and

$$U_{N,3}(k,\lambda,\mu) = O(1/N)\,.$$

(b) *Under assumption* $(\mathcal{H}_3)$,

$$U_{N,4}(k,\lambda,\mu) = O(1/N)\,.$$

The O terms are uniform in λ and μ.

PROPOSITION 5.4. *Under assumptions* $(\mathcal{H}_1)$ *and* $(\mathcal{H}_2)$, *if H is a bounded function with a bounded derivative*, *then*

$$U_{N,2}(k,\lambda,\mu) = O(M_N/N)$$

uniformly in λ and μ.

THEOREM 5.2. *If H is a bounded function with a bounded derivative*, *then under assumptions* $(\mathcal{H}_1)$, $(\mathcal{H}_2)$ *and* $(\mathcal{H}_3)$, *we have for* $\lambda,\mu\in\mathbb{R}$ *and* $j,k\in\mathbb{Z}$,

$$\mathrm{var}\{g_k^{(1)}(N,\mu)\} = O(M_N/N)\,.$$

Proof. We just have to apply Propositions 5.3 and 5.4 and use (5.7) and (5.8a)–(5.8d).

Asymptotic properties of $g_k^{(2)}(N,\omega)$

THEOREM 5.3. *If $X(t)$ is a PC process with*

$$\int_{-\infty}^{\infty} \Big(\sum_{j\ne k} \frac{|B_j(u)|}{|j-k|} + |B_k(u)|\Big)\,du < \infty\,,$$

then

$$E\{g_k^{(2)}(N,\omega)\} = g_k(\omega) + o(1)$$

where the $o(1)$ *term is uniform in* ω.

Proof. Using previous expressions, we get

$$E\{g_k^{(2)}(N,\omega)\} = \frac{N-n}{2\pi\beta N}\sum_{n=1}^{M_N} e_n^k(\omega) + O\left(\frac{1}{N}\right).$$

On the other hand, $\sum_{n=1}^{\infty} e_n^k(\omega) = 2\pi\beta g_k(\omega)$, therefore

$$E\{g_k^{(2)}(N,\omega)\} = g_k(\omega) - \frac{1}{2\pi\beta}\sum_{n=M_N+1}^{\infty} e_n^k(\omega) - \frac{1}{2\pi\beta N}\sum_{n=1}^{M_N} n e_n^k(\omega) + O\left(\frac{1}{N}\right).$$

Since $|\sum_{n=M_N+1}^{\infty} e_n^k(\omega)| = o(1)$ uniformly in ω as $N\to\infty$, therefore

$$\left|\frac{1}{N}\sum_{n=1}^{M_N} n e_n^k(\omega)\right| \le \frac{M_N}{N}\frac{1}{M_N}\left|\sum_{n=1}^{M_N} n e_n^k(\omega)\right| = \frac{M_N}{N}o(1),$$

which gives the result.

Corollary 5.4. *If* $X(t)$ *is a PC process with*

$$\int_{-\infty}^{\infty}\left(\sum_{j\neq k}\frac{|B_j(u)|}{|j-k|} + |B_k(u)|\right)du < \infty$$

and $\int_0^{\infty} u|B_k(u)|\,du < \infty$, *then*

$$E\{g_k^{(2)}(N,\omega)\} = g_k(\omega) + O(1/M_N).$$

Proof. In this case the series $ne_n^k(\omega)$ converges uniformly, therefore

$$\Big|\sum_{n=M_N+1}^{\infty} e_n^k(\omega)\Big| \le \frac{1}{M_N}\sum_{n=M_N+1}^{\infty} n|e_n^k(\omega)| = \frac{1}{M_N}o(1)$$

and $|\frac{1}{N}\sum_{n=1}^{M_N} ne_n^k(\omega)| = O(1/N)$, which gives the result.

To prove the consistency of the estimator $g_k^{(2)}(N,\omega)$, we make the following assumptions:

$(\mathcal{H}_1')$ (i) $$\int_0^{\infty}\left(\sum_{j_1\neq -j_2}\frac{|B_{j_1}(u_1)B_{j_2}(u_2)|}{|j_1+j_2|}\right)du_1\,du_2 + \int_0^{\infty}\left(\sum_{j\in\mathbb{Z}}|B_j(u)|^2\right)du < \infty,$$

(ii) $$\int_0^{\infty}\left(\sum_{j_1\neq -j_2}\frac{|B_{j_1}(u_1)B_{j_2}(u_2)|}{|j_1+j_2|}\right)(u_1+u_2)\,du_1\,du_2 + \int_0^{\infty}\left(\sum_{j\in\mathbb{Z}}|B_j(u)|^2\right)u\,du < \infty.$$

$(\mathcal{H}_2')$ $X(t)$ is a PC process up to order 4 and for all $j \in \mathbb{Z}$, we have

$$|K_j(u_1, u_2, u_3)| \le h_j(u_1, u_2, u_3)$$

where h_j is a nondecreasing function on $[0, \infty[$, even in each variable and such that

$$\sum_{j\in\mathbb{Z}} \int_0^\infty \frac{h_j(0,u,0)}{|j-k|}\, du < \infty\,, \qquad \sum_{j\in\mathbb{Z}} h_j(0,0,0) < \infty\,.$$

Let

$$\Gamma_k(n,N,\omega) = \frac{1}{N}\sum_{l=1}^{N-n}\left\{\exp\left(i\left(\omega - \frac{2\pi}{T}k\right)(t_{l+n}-t_l)\right) + \exp(-i\omega(t_{l+n}-t_l))\right\} \times X(t_{l+n})X(t_l)\exp\left(-i\frac{2\pi}{T}kt_l\right).$$

Then

$$E\{\Gamma_k(n,N,\omega)\} = \frac{N-n}{N}e_n^k(\omega) + \frac{1}{N}o(1)\,,$$

therefore

$$|E\{\Gamma_k(n,N,\omega)\}|^2 = \left(\frac{N-n}{N}\right)^2 |e_n^k(\omega)|^2 + \frac{1}{N}o(1)\,.$$

PROPOSITION 5.5. *Under assumptions* $(\mathcal{H}_1')$ *and* $(\mathcal{H}_2')$, *if* $M_N^2/N \to 0$, *then*

$$\operatorname{var}\{\Gamma_k(n,N,\omega)\} = O(1/N)\,.$$

THEOREM 5.4. *Under assumptions* $(\mathcal{H}_1')$ *and* $(\mathcal{H}_2')$, *if* $M_N^2/N \to 0$, *then*

$$\operatorname{var}\{g_k^{(2)}(N,\omega)\} = O(M_N^2/N)\,.$$

Proof. Using the above proposition, we have

$$\begin{aligned}[\operatorname{var}\{g_k^{(2)}(N,\omega)\}]^{1/2} &\le \frac{1}{2\pi\beta}\sum_{n=1}^{M_N}[\operatorname{var}\{\Gamma_k(n,N,\omega)\}]^{1/2} \\ &\le \frac{M_N}{2\pi\beta}O\left(\frac{1}{N^{1/2}}\right),\end{aligned}$$

which gives the result.

Estimation by means of an orthogonal series with truncation. Earlier, we showed that there exists a complete orthonormal system $\{q_n(u) : n \ge 1\}$ on $L^2[0,\infty[$ such that $q_n(u) = \sum_{l=1}^n b_{n,l} f_l(u)$ where $b_{n,l} = (2/\beta)^{1/2}(-2)^{l-1}\mathcal{C}_{n-1}^{l-1}$. We have $g_k(\omega) = \sum_{n\ge 1}\gamma_k(n)Q_n^k(\omega)$ where

$$Q_n^k(\omega) = \frac{1}{2\pi}\int_0^\infty\left\{\exp\left(i\left(\omega - \frac{2\pi}{T}k\right)u\right) + \exp(-i\omega u)\right\}q_n(u)\, du\,,$$

and

$$\gamma_k(n) = \int_0^\infty B_k(u) q_n(u)\, du = \sum_{l=1}^{n} b_{n,l} C_k^\alpha(l)\,.$$

As in F. Messaci's thesis [7], for all nonnegative integers n and for all integers k,

$$|Q_n^k(\omega)| \le \frac{1}{\pi}\left(\frac{2}{\beta}\right)^{1/2}.$$

Assuming that we get a sample $\{X(t_k) : k = 1, \ldots, N\}$, using (5.1), we estimate $C_k^\alpha(n)$ by

$$\widehat{C}_k^\alpha(n, N) = \frac{1}{N}\sum_{l=1}^{N-n} X(t_{l+n})X(t_l)\exp\left(-i\frac{2\pi}{T}kt_l\right).$$

It is natural to estimate $\gamma_k(n)$ using (5.3), by

$$\widehat{\gamma}_k(n, N) = \sum_{l=1}^{n} b_{n,l}\widehat{C}_k^\alpha(l, N)\,.$$

Then we estimate $g_k(\omega)$ by

$$g_k^{(3)}(N, \omega) = \sum_{n=1}^{M_N} Y_n(N)\widehat{\gamma}_k(n, N) Q_n^k(\omega)$$

where M_N is the integer part of $\frac{b}{2\alpha}\ln(N) + 1$ and

$$Y_n(N) = h\left(\frac{\exp(n\alpha)}{N^b}\right), \qquad \alpha > 0\,,\ 0 < b < \frac{\alpha}{\ln 3}\,,$$

where h is a function satisfying

(i) $|h(u)| \le 1\ \forall u \in \mathbb{R}$,
(ii) $|1 - h(u)| \le |u|\ \forall u \in \mathbb{R}$.

Let $AC^r[0, \infty[$ be the class of functions which are r times differentiable with absolutely continuous derivatives. To study the asymptotic behavior of $g_k^{(3)}(N, \omega)$, we make the following assumption:

($\mathcal{H}_1''$) There exists $r > 2$ so that for $u \ge 0$ and an integer $k,$ we have
(i) $B_k(u) \in AC^{r-1}[0, \infty[\,,$
(ii) $u^{r/2}B_k^{(l)}(u) \in L^2[0, \infty[, \quad l = 0, 1, \ldots, r.$

We follow F. Messaci's method [7].

Lemma 5.3. *Under assumption* ($\mathcal{H}_1''$), *for all integers* $n \ge 0$ *and* k,

(i) $|\gamma_k(n)| \le A_n^k(r) n^{-r/2}$,
(ii) $g_k(\omega) = \sum_{n=1}^\infty \gamma_k(n) Q_n^k(\omega)$ *uniformly on* $\mathbb{R}$.

PROPOSITION 5.6. *Under assumptions* $(\mathcal{H}_1')$ *and* $(\mathcal{H}_2')$,

$$\text{(i)} \qquad E\{\widehat{C}_k^{\alpha}(n,N)\} = \frac{N-n}{N} C_k^{\alpha}(n) + O\left(\frac{1}{N}\right),$$

$$\text{(ii)} \qquad \operatorname{var}\{\widehat{C}_k^{\alpha}(n,N)\} \le \frac{A_2}{N}\,.$$

The proof of this proposition is analogous to the proofs of Proposition 5.5 and Theorem 5.4.

PROPOSITION 5.7. *Under assumptions* $(\mathcal{H}_1')$ *and* $(\mathcal{H}_2')$,

$$\text{(i)} \qquad E\{\widehat{\gamma}_k(n,N)\} = \gamma_k(n) - \frac{1}{N}\{n\gamma_k(n) - (n-1)\gamma_k(n-1)\} + O\left(\frac{1}{N}\right)\sum_{l=1}^{n} b_{n,l}\,,$$

$$\text{(ii)} \qquad \operatorname{var}\{\widehat{\gamma}_k(n,N)\} \le A_3 \frac{3^n}{N}\,.$$

The proof of this proposition is analogous to the proof of Theorem 2 of [7], p. 25.

THEOREM 5.5. *If* $X(t)$ *satisfies assumptions* $(\mathcal{H}_1')$, $(\mathcal{H}_2')$ *and* $(\mathcal{H}_1'')$, *then* $g_k^{(3)}(N,\omega)$ *is uniformly consistent*, *with*

$$E\{|g_k^{(3)}(N,\omega) - g_k(\omega)|^2\} = O\left(\frac{1}{\ln N}\right)^{r-2}.$$

Proof. From the proof of Theorem 3 of [7], p. 27, we have

$$E\{|g_k^{(3)}(N,\omega) - g_k(\omega)|^2\} \le V_k(N,\omega) + \left|O\left(\frac{1}{N}\right)\right| \sum_{n=1}^{M_N} |\gamma_k(n) Q_n^k(\omega)| \sum_{l=1}^{n} |b_{n,l}|$$

with $V_k(N,\omega) = O(1/\ln N)^{r-2}$. On the other hand, we have

$$\sum_{n=1}^{M_N} |\gamma_k(n) Q_n^k(\omega)| \sum_{l=1}^{n} |b_{n,l}| \le \frac{2}{\pi\beta} A_1(r) \sum_{n=1}^{M_N} n^{-r/2} \sum_{l=1}^{n} \mathcal{C}_{n-1}^{l-1} 2^{l-1}$$

and

$$\begin{aligned}\frac{1}{N}\sum_{n=1}^{M_N} |\gamma_k(n) Q_n^k(\omega)| \sum_{l=1}^{n} |b_{n,l}| &\le \frac{2}{\pi\beta} A_1(r) \sum_{n=1}^{M_N} 3^n \\ &\le \frac{2}{\pi\beta} A_1(r) \frac{3}{2N} 3^{M_N-1} \\ &\le \frac{2}{\pi\beta} A_1(r) \frac{3}{2} N^{-(1-\frac{b}{2\alpha}\ln 3)}\,.\end{aligned}$$

Therefore
$$\frac{1}{N}\sum_{n=1}^{M_N}|\gamma_k(n)Q_n^k(\omega)|\sum_{l=1}^{n}|b_{n,l}| \le O(N^{-q})$$
with $q = 1 - \frac{b}{2\alpha}\ln 3 > 0$. Thus
$$\frac{1}{N}\sum_{n=1}^{M_N}|\gamma_k(n)Q_n^k(\omega)|\sum_{l=1}^{n}|b_{n,l}| = O\bigg(\frac{1}{\ln N}\bigg)^{r-2}.$$

Generalization to almost periodically correlated harmonizable and gaussian processes. If $X(t)$ is almost periodically correlated, then
$$B(t,u) = E\{X(t+u)X(t)\} = \sum_{\alpha\in S(X)} B_\alpha(u)\exp(i\alpha t)$$
where the set $S(X)$ is at most countable [1]. If $B_\alpha \in L^1(\mathbb{R})$ for some α, we define the spectral density function g_α by
$$g_\alpha(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} B_\alpha(u)\exp(-iu\omega)\,du\,.$$
If we assume that $X(t)$ is harmonizable, gaussian and
$$\int_0^\infty \Big(\sum_{\lambda\in S(X)}|B_\lambda(u)|^2\Big)\,du < \infty\,,$$
then the above results are valid.

6. Examples of periodically correlated processes. We give here some examples of PC processes which will be used for simulations in a next paper.

Let $Y(t)$ be a stationary process with continuous autocovariance function $r_Y(u)$. Let $P(t)$ be a bounded periodic function with period T. Put
$$X(t,\omega) = P(t)Y(t,\omega)\,.$$
Then
$$B(t,u) = E\{X(t+u)X(t)\} = P(t+u)P(t)r_Y(u)\,.$$
We have
$$\begin{aligned}B_k(u) &= \frac{1}{T}r_Y(u)\int_0^T P(t+u)P(t)\exp\bigg(-i\frac{2\pi}{T}kt\bigg)\,dt\\ &= r_Y(u)\sum_{n\in\mathbb{Z}}\exp\bigg(i\frac{2\pi}{T}nu\bigg)p_n\overline{p}_{n-k}\end{aligned}$$
where
$$p_n = \frac{1}{T}\int_0^T P(t)\exp\bigg(-\frac{2\pi}{T}nt\bigg)\,dt\,.$$

1) If we choose $P(t)$ so that $p_n = 0$ for $|n| > A$ and $Y(t)$ is a Markov gaussian process with $EY(t) = 0$ and autocovariance function $r_Y(u) = e^{-|u|}$, then $X(t)$ is a PC process which satisfies the assumptions of the above theorems.

2) If we choose $P(t)$ so that $\sum_{n\in\mathbb{Z}} |p_n| < \infty$ and $Y(t)$ as before, then we have the same conclusion as in 1).

Other examples can be found in [2].

7. Appendix. We follow and generalize Masry and Lui's method [6]. Let

$$\begin{aligned}
R_1 &:= \{(l_1, l_2) \in R : l_1 < l_1 + n_1 \le l_2 < l_2 + n_2\}\,,\\
R_2 &:= \{(l_1, l_2) \in R : l_1 < l_2 < l_1 + n_1 < l_2 + n_2\}\,,\\
R_3 &:= \{(l_1, l_2) \in R : l_1 = l_2 < l_1 + n_1 < l_2 + n_2\}\,,\\
R_4 &:= \{(l_1, l_2) \in R : l_1 < l_2 < l_1 + n_1 = l_2 + n_2\}\,,\\
R_5 &:= \{(l_1, l_2) \in R : l_1 < l_2 < l_2 + n_2 < l_1 + n_1\}\,,\\
R_6 &:= \{(l_1, l_2) \in R : l_1 = l_2 < l_1 + n_1 = l_2 + n_2\}\,,\\
R_7 &:= \{(l_1, l_2) \in R : l_2 < l_2 + n_2 \le l_1 < l_1 + n_1\}\,,\\
R_8 &:= \{(l_1, l_2) \in R : l_2 < l_1 < l_2 + n_2 < l_1 + n_1\}\,,\\
R_9 &:= \{(l_1, l_2) \in R : l_2 = l_1 < l_2 + n_2 < l_1 + n_1\}\,,\\
R_{10} &:= \{(l_1, l_2) \in R : l_2 < l_1 < l_2 + n_2 < l_1 + n_1\}\,,\\
R_{11} &:= \{(l_1, l_2) \in R : l_2 < l_1 < l_1 + n_1 < l_2 + n_2\}\,.
\end{aligned}$$

Then $R = \bigcup_{i=1}^{11} R_i$ and $R_i \cap R_j = \emptyset$ for $i \ne j$.

Put $U_{N,i}(k,\lambda,\mu) = \sum_{r=1}^{11} U_{N,i}^{(r)}(k,\lambda,\mu)$, $i = 1,2,3,4$, where $U_{N,i}^{(r)}(k,\lambda,\mu)$ has the same form as $U_{N,i}(k,\lambda,\mu)$ with $\sum_R$ replaced by $\sum_{R_r}$. By symmetry of pairs of indices (l_1, n_1) and (l_2, n_2), we have for $i = 1,2,3$,

$$U_{N,i}^{(r)}(k,\lambda,\mu) = U_{N,i}^{(r+6)}(k,\lambda,\mu)\,, \qquad r = 1,\ldots,5\,.$$

Lemma 7.1. *If b_n is a sequence of complex numbers, then*

$$\sum_{l_1=1}^{N-n_1}\sum_{l_2=1}^{N-n_2} b_{l_2-l_1} = \sum_{s=-N}^{N} b_s V(s,n_1,n_2)$$

where

$$V(s,n_1,n_2) = \begin{cases} 0 & \text{if } -(N-n_1) \ge s,\\ N-n_1-s & \text{if } -(N-n_1) < s \le \min(0, n_1-n_2),\\ N-\max(n_1,n_2) & \text{if } \min(0,n_1-n_2) \le s \\ & \qquad \le \max(0, n_1-n_2),\\ N-n_2-s & \text{if } \max(0,n_1-n_2) \le s < N-n_2,\\ 0 & \text{if } N-n_2 \le s. \end{cases}$$

Proof. See [7], p. 95.

LEMMA 7.2. *Let a_s, b_s, c_s be sequences of nonnegative numbers such that $\sum_{s=0}^{\infty} a_s < \infty$, $\sum_{s=0}^{\infty} b_s < \infty$ and $\sum_{s=0}^{\infty} c_s < \infty$. Then for $i = 2, 3, 4, 6$,*

$$Q_i = \frac{1}{N^2} \sum_{n_1,n_2=1}^{N-1} \sum_{(l_1,l_2)\in R_i} a_{l_1} b_{l_2-l_1} c_{l_2-l_1+n_2-n_1} = O\Big(\frac{1}{N}\Big) \sum_{s=0}^{\infty} a_s$$

and

$$Q_5 = \frac{1}{N^2} \sum_{n_1,n_2=1}^{N-1} \sum_{(l_1,l_2)\in R_5} a_{l_1} b_{l_2-l_1} c_{l_1-l_2+n_1-n_2} = O\Big(\frac{1}{N}\Big) \sum_{s=0}^{\infty} a_s \,.$$

Proof. We give the proof for $i = 2$; the other cases can be treated in the same way.

$$\begin{aligned} Q_2 &= \frac{1}{N^2} \sum_{n_1,n_2=1}^{N-1} 1_{\{n_1 \ge n_2\}} \sum_{(l_1,l_2)\in R_2} a_{l_1} b_{l_2-l_1} c_{l_2-l_1+n_2-n_1} \\ &\quad + \frac{1}{N^2} \sum_{n_1,n_2=1}^{N-1} 1_{\{n_1 < n_2\}} \sum_{(l_1,l_2)\in R_2} a_{l_1} b_{l_2-l_1} c_{l_2-l_1+n_2-n_1} \\ &= Q_{21} + Q_{22} \,. \end{aligned}$$

We have

$$Q_{21} = \frac{1}{N^2} \sum_{n_1,n_2=1}^{N-1} 1_{\{n_1 \ge n_2\}} \sum_{l_1=1}^{N-n_1} \sum_{s=1+n_1-n_2}^{N-n_2-l_1} a_{l_1} b_s c_{s+n_2-n_1} \,,$$

by putting $s = l_2 - l_1$ and observing that $l_2 - l_1 > n_1 - n_2$ in R_2. Hence

$$\begin{aligned} Q_{21} &= \frac{1}{N^2} \sum_{n_2=1}^{N-1} \sum_{p=0}^{N-n_2-1} \sum_{l_1=1}^{N-p-n_2} \sum_{r=1}^{N-n_1-p-l_1} a_{l_1} b_{r+p} c_r \\ &\le \frac{1}{N^2} \sum_{n_2=1}^{N-1} \sum_{p=0}^{N-2} \sum_{l_1=1}^{N-1} \sum_{r=1}^{N-2} a_{l_1} b_{r+p} c_r \\ &\le \frac{1}{N} \Big(\sum_{n=0}^{\infty} a_n\Big)\Big(\sum_{p=0}^{\infty} b_p\Big)\Big(\sum_{r=0}^{\infty} c_r\Big) = O\Big(\frac{1}{N}\Big) \sum_{s=0}^{\infty} a_s \,. \end{aligned}$$

We treat Q_{22} in the same way and we obtain the result.

LEMMA 7.3. *Let a_n and b_n be sequences of complex numbers such that $\sum_{n=0}^{\infty} |a_n| < \infty$ and $\sum_{n=0}^{\infty} |b_n| < \infty$ and d a complex number such that $|d| \le 1$. Put*

$$P_i = \frac{1}{N^2} \sum_{n_1,n_2=1}^{N-1} \sum_{l_1,l_2\in R_i} a_{l_2-l_1} d^{n_1-(l_2-l_1)} b_{l_2-l_1+n_2-n_1} \qquad \text{for } i = 2, 3, 4, 6 \,,$$

$$P_5 = \frac{1}{N^2} \sum_{n_1,n_2=1}^{N-1} \sum_{l_1,l_2 \in R_5} a_{l_2-l_1} d^{n_2} b_{l_1-l_2+n_1-n_2} \,.$$

Then as $N \to \infty$, *we have*

$$P_2 = D_N(d)\Big(\sum_{r=1}^{\infty} a_r\Big)\Big(\sum_{s=1}^{\infty} b_s\Big) + O\Big(\frac{1}{N}\Big),$$

$$P_3 = D_N(d) a_0 \Big(\sum_{s=1}^{\infty} b_s\Big) + O\Big(\frac{1}{N}\Big),$$

$$P_4 = D_N(d) b_0 \Big(\sum_{s=1}^{\infty} a_s\Big) + O\Big(\frac{1}{N}\Big),$$

$$P_5 = D_N(d)\Big(\sum_{r=1}^{\infty} a_r\Big)\Big(\sum_{s=1}^{\infty} b_s\Big) + O\Big(\frac{1}{N}\Big),$$

$$P_6 = D_N(d) a_0 b_0 + O\Big(\frac{1}{N}\Big)$$

where

$$D_N(d) = \frac{1}{N} \sum_{n=1}^{N} \Big(1 - \frac{n}{N}\Big) d^n \,.$$

Proof. See [5], p. 181.

LEMMA 7.4. *Let*

$$P_N(\lambda) = \frac{1}{N} \sum_{n=2}^{N} \Big(1 - \frac{n}{N}\Big) \phi(\lambda)^n \,.$$

Then

(i) $$|P_N(\lambda)| \le 1 \qquad \textit{for all } \lambda \in \mathbb{R}\,,$$

(ii) $$|P_N(\lambda)| \le \frac{\beta}{N\lambda^2} + \frac{4\beta^2}{N^2\lambda^2} \qquad \textit{for all } \lambda \ne 0\,,$$

(iii) $$|P_N(\lambda)| \le \frac{1}{N} + \frac{\beta}{N|\lambda|} + \frac{4\beta^2}{N^2\lambda^2} \qquad \textit{for all } \lambda \ne 0\,.$$

Proof. (i) We have

$$|P_N(\lambda)| \le \frac{1}{N} \sum_{n=2}^{N} \Big(1 - \frac{n}{N}\Big) \le 1\,.$$

(ii) If $\lambda \ne 0$, then

$$P_N(\lambda) = \frac{1}{N}\bigg\{\frac{\phi(\lambda)^2}{1-\phi(\lambda)} - \frac{\phi(\lambda)^2}{N[1-\phi(\lambda)]^2}(2 - \phi(\lambda) - \phi(\lambda)^{N-1})\bigg\}\,,$$

which gives the result.

(iii) We write

$$\frac{\phi(\lambda)^2}{1-\phi(\lambda)} = \phi(\lambda) + \frac{\phi(\lambda)}{1-\phi(\lambda)}\,.$$

Lemma 7.5. *Assuming that H is a bounded function with a bounded derivative, let*

$$S_N(u,a) = \int\limits_{-\infty}^{\infty} W_N(u-v)P_N(v)\exp(-iva)\,dv - \frac{2\pi}{N}W_N(u)\sum_{n=2}^{N}\left(1-\frac{n}{N}\right)f_n(a).$$

Then

$$S_N(u,a) = O(M_N/N)\,.$$

Proof. As H is a bounded function with a bounded derivative, we have $|H(u)| \le A_1$ and $|H(u-v)-H(u)| \le A_2|u|$. For $n \ge 2$, $f_n(u)$ and $\phi(\lambda)^n$ belong to $L^1 \cap L^2$, therefore $\int_{-\infty}^{\infty} \phi(\lambda)^n \exp(-i\lambda a)\,d\lambda = 2\pi f_n(a)$ a.s. Thus

$$\int\limits_{-\infty}^{\infty} P_N(\lambda)\exp(-i\lambda a)\,d\lambda = \frac{2\pi}{N}\sum_{n=2}^{N}\left(1-\frac{n}{N}\right)f_n(a) \quad \text{a.s.,}$$

and

$$\begin{aligned}
|S_N(u,a)| &= \Big|\int\limits_{-\infty}^{\infty}(W_N(u-v)-W_N(u))P_N(v)\,dv\Big| \\
&\le \int\limits_{-\infty}^{\infty}|H(M_Nu-v)-H(M_Nu)|\left|P_N\left(\frac{v}{M_N}\right)\right|dv \\
&\le I_1+I_2+I_3
\end{aligned}$$

where

$$\begin{aligned}
I_1 &\le A_2 \int\limits_{|v|\le M_N/N} |v|\,dv = O\left(\frac{M_N^2}{N^2}\right), \\
I_2 &\le A_2\beta \int\limits_{M_N/N<|v|<1} \left(\frac{|v|}{\beta N}+\frac{M_N}{N}+\frac{4M_N^2}{N^2|v|}\right)dv = O\left(\frac{M_N}{N}\right), \\
I_3 &\le 2A_1\beta^2 \int\limits_{|v|\ge 1}\left(\frac{1}{v^2}+\frac{4}{Nv^2}\right)dv = O\left(\frac{1}{N}\right).
\end{aligned}$$

Lemma 7.6. *Let h and l belong to $L^1[0,\infty[$ and suppose $uh(u)$ and $ul(u)$ belong to $L^1[0,\infty[$. Then*

(i) $$\sum_{n=1}^{\infty}(c_nd_n)^{1/2} \le \beta\Big(\int\limits_0^{\infty}\int\limits_0^{\infty} h(u)l(v)\,du\,dv\Big)^{1/2},$$

(ii) $$\sum_{n=1}^{\infty} n(c_n d_n)^{1/2} \le \Big(\int_0^{\infty} \int_0^{\infty} (\beta^2 u+\beta)(\beta^2 v+\beta) h(u) l(v)\,du\,dv\Big)^{1/2}$$

where $c_n = \int_0^\infty h(u) f_n(u)\,du$ *and* $d_n = \int_0^\infty l(u) f_n(u)\,du$.

Proof of Proposition 5.3. We have

$$U_{N,1}(k,\lambda,\mu) - E\{J_k(N,\lambda)\}\overline{E\{J_k(N,\mu)\}} = W'_{N,1}(k,\lambda,\mu) + W''_{N,1}(k,\lambda,\mu)$$

where

$$W'_{N,1}(k,\lambda,\mu) = 2U^{(1)}_{N,1}(k,\lambda,\mu) - E\{J_k(N,\lambda)\}\overline{E\{J_k(N,\mu)\}}\,,$$
$$W''_{N,1}(k,\lambda,\mu) = 2\sum_{r=2}^{5} U^{(r)}_{N,1}(k,\lambda,\mu) + U^{(6)}_{N,1}(k,\lambda,\mu)\,.$$

Assumption $(\mathcal{H}_1)$ and Theorem 5.1 ensure that

$$E\{J_k(N,\lambda)\} = \frac{1}{2\pi\beta N}\sum_{n=1}^{N-1}(N-n)e_n^k(\lambda) + O\Big(\frac{1}{N}\Big).$$

As $|e_n^k(\lambda)| \le 2\int_{-\infty}^{\infty} |B_k(u)| f_n(u)\,du = 2d_n^k$ with $\sum_{n\ge1} d_n^k < \infty$ and $\sum_{n\ge1} n d_n^k < \infty$, we have

$$E\{J_k(N,\lambda)\}\overline{E\{J_k(N,\mu)\}} = \frac{1}{(2\pi\beta N)^2}\sum_{n_1,n_2=1}^{N-1}(N-n_1)(N-n_2)e^k_{n_1}\overline{e^k_{n_2}} + O\Big(\frac{1}{N}\Big)$$

where the $O(1/N)$ is uniform in λ and μ. For $(l_1,l_2)\in R_1$, t_{l_1}, $t_{l_1+n_1}-t_{l_1}$, $t_{l_2}-t_{l_1+n_1}$, $t_{l_2+n_2}-t_{l_2}$ are independent, therefore

$$\begin{aligned}
U^{(1)}_{N,1}&(k,\lambda,\mu)(2\pi\beta N)^2 \\
&= \sum_{n_1,n_2=1}^{N-1}\sum_{(l_1,l_2)\in R_1}\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty B(t_1,u_1)B(t_2+u_1+t_1,u_2) \\
&\quad\times\Big\{\exp\Big(i\Big(\lambda-\frac{2\pi}{T}k\Big)u_1\Big)+\exp(-i\lambda u_1)\Big\} \\
&\quad\times\Big\{\exp\Big(-i\Big(\mu-\frac{2\pi}{T}k\Big)u_2\Big)+\exp(i\mu u_2)\Big\} \\
&\quad\times\exp\Big\{i\frac{2\pi}{T}k(t_2+u_1)\Big\} \\
&\quad\times f_{l_1}(t_1)f_{n_1}(u_1)f_{l_2-l_1-n_1}(t_2)f_{n_2}(u_2)\,dt_1\,dt_2\,du_1\,du_2
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{n_1,n_2=1}^{N-1} \sum_{(l_1,l_2)\in R_1} \sum_{j_1+j_2=0} \int_0^\infty \int_0^\infty \left\{ \exp\left(i\left(\lambda - \frac{2\pi}{T}k\right)u_1\right) \right.\\
&\quad \left. + \exp(-i\lambda u_1)\right\} \times \left\{ \exp\left(-i\left(\mu - \frac{2\pi}{T}k\right)u_2\right) + \exp(i\mu u_2)\right\} \\
&\quad \times \phi\left(\frac{2\pi}{T}(j_2+k)\right)^{l_2-l_1-n_1} B_{j_1}(u_1)B_{j_2}(u_2) f_{n_1}(u_1) f_{n_2}(u_2)\, du_1\, du_2 \\
&\quad + \sum_{n_1,n_2=1}^{N-1} \sum_{j_1+j_2\neq 0} \int_0^\infty \int_0^\infty B_{j_1}(u_1)B_{j_2}(u_2) \exp\left(i\frac{2\pi}{T}(j_2+k)u_1\right) \\
&\quad \times \left\{ \exp\left(i\left(\lambda - \frac{2\pi}{T}k\right)u_1\right) + \exp(-i\lambda u_1)\right\} \\
&\quad \times \left\{ \exp\left(-i\left(\mu - \frac{2\pi}{T}k\right)u_2\right) + \exp(i\mu u_2)\right\} \\
&\quad \times \sum_{(l_1,l_2)\in R_1} \phi\left(\frac{2\pi}{T}(j_1+j_2)\right)^{l_1} \phi\left(\frac{2\pi}{T}(j_2+k)\right)^{l_2-l_1-n_1} \\
&\quad \times f_{n_1}(u_1) f_{n_2}(u_2)\, du_1\, du_2 \\
&= (Y_1+Y_2+Y_3)(2\pi\beta N)^2 .
\end{aligned}
$$

We have

$$
\begin{aligned}
|Y_3| &\le \frac{1}{(\pi\beta N)^2} \sum_{n_1,n_2=1}^{N-1} \sum_{j_1+j_2\neq 0} \int_0^\infty \int_0^\infty |B_{j_1}(u_1)B_{j_2}(u_2)| \\
&\qquad \times \sum_{(l_1,l_2)\in R_1} \left|\phi\left(\frac{2\pi}{T}(j_1+j_2)\right)\right|^{l_1} f_{n_1}(u_1) f_{n_2}(u_2)\, du_1\, du_2 \\
&\le \frac{1}{(\pi\beta N)^2} \sum_{n_1,n_2=1}^{N-1} \sum_{j_1+j_2\neq 0} \int_0^\infty \int_0^\infty |B_{j_1}(u_1)B_{j_2}(u_2)| \\
&\quad \times \sum_{l_1=1}^{N-n_1} \sum_{s=n_1}^{N-n_2-l_1} \left|\phi\left(\frac{2\pi}{T}(j_1+j_2)\right)\right|^{l_1} f_{n_1}(u_1) f_{n_2}(u_2)\, du_1\, du_2 .
\end{aligned}
$$

Putting $s = l_2 - l_1$ and observing that $l_2 - l_1 \ge n_1$ in R_1, we get

$$
\begin{aligned}
|Y_3| \le \frac{1}{(\pi\beta)^2 N} \frac{T}{2\pi} \sum_{n_1,n_2=1}^{N-1} \sum_{j_1+j_2\neq 0} \int_0^\infty \int_0^\infty \frac{|B_{j_1}(u_1)B_{j_2}(u_2)|}{|j_1+j_2|} \\
\times f_{n_1}(u_1) f_{n_2}(u_2)\, du_1\, du_2 = O(1/N) .
\end{aligned}
$$

Next,

$$|Y_2| \le \frac{1}{(\pi\beta)^2 N}\left(\frac{\beta T}{2\pi}+1\right)\sum_{n_1,n_2=1}^{N-1}\sum_{j\ne k}\int_0^\infty\int_0^\infty |B_j(u_1)B_j(u_2)| \\ \times f_{n_1}(u_1)f_{n_2}(u_2)\,du_1\,du_2 = O(1/N)\,,$$

therefore

$$W'_{N,1}(k,\lambda,\mu) = \frac{1}{(2\pi\beta)^2}\sum_{n_1,n_2=1}^{N-1} e^k_{n_1}(\lambda)\overline{e^k_{n_2}(\mu)} \\ \times\frac{1}{N^2}\Big\{2\sum_{R_1}1-(N-n_1)(N-n_2)\Big\}+O\bigg(\frac{1}{N}\bigg).$$

On the other hand, by Lemma 7.1,

$$2\sum_{R_1}1 = 2\sum_{s=n_1}^{N-n_2}(N-n_2-s)1_{\{N-n_1-n_2-1\ge 0\}} \\ = (N-n_2-n_1)(N-n_2-n_1+1)1_{\{N-n_1-n_2-1\ge 0\}}\,.$$

Hence

$$\frac{1}{N^2}\Big\{2\sum_{R_1}1-(N-n_1)(N-n_2)\Big\} \\ = \bigg\{\bigg(1-\frac{n_1+n_2}{N}\bigg)\bigg(1-\frac{n_1+n_2-1}{N}\bigg) \\ -\bigg(1-\frac{n_1}{N}\bigg)\bigg(1-\frac{n_2}{N}\bigg)\bigg\}1_{\{N-n_1-n_2-1\ge 0\}} \\ +\bigg(1-\frac{n_1}{N}\bigg)\bigg(1-\frac{n_2}{N}\bigg)\{1-1_{\{N-n_1-n_2-1\ge 0\}}\} \\ = A(n_1,n_2)+B(n_1,n_2)\,.$$

We have

$$|A(n_1,n_2)| \le \frac{3}{N}\bigg(\frac{4}{3}n_1+n_2+1\bigg) \quad\text{and}\quad |B(n_1,n_2)| \le \frac{n_1+1}{N}\,,$$

therefore

$$|W'_{N,1}(k,\lambda,\mu)| \le \frac{1}{(\pi\beta)^2 N}\sum_{n_1,n_2=1}^{N-1}\int_0^\infty\int_0^\infty |B_k(u_1)B_k(u_2)| \\ \times f_{n_1}(u_1)f_{n_2}(u_2)\{4n_1+3n_2+3+n_1+1\}\,du_1\,du_2 \\ \le \frac{1}{(2\pi\beta)^2 N}\int_0^\infty\int_0^\infty |B_k(u_1)B_k(u_2)| \\ \times\{(\beta^2u_1+\beta)\beta+(\beta^2u_2+\beta)\beta+\beta^2\}\,du_1\,du_2 = O(1/N)$$

using assumption $(\mathcal{H}_2)$.

Let us now evaluate $W''_{N,1}(k,\lambda,\mu)$. For $r=2,\ldots,6$, t_{l_1} and $t_{l_1+n_1}-t_{l_1}$, t_{l_1} and $t_{l_2+n_2}-t_{l_2}$, t_{l_1} and $t_{l_2}-t_{l_1}$ are independent pairs of r.v.'s; therefore

$$|U^{(r)}_{N,1}(k,\lambda,\mu)| \le \frac{4}{(2\pi\beta N)^2}\sum_{n_1,n_2=1}^{N-1}\sum_{(l_1,l_2)\in R_r}\sum_{(j_1,j_2)\in\mathbb{Z}^2}\left|\phi\left(\frac{2\pi}{T}(j_1+j_2)\right)^{l_1}\right| \times E[h_{j_1}(t_{l_1+n_1}-t_{l_1})h_{j_2}(t_{l_2+n_2}-t_{l_2})]\,.$$

Let $\gamma=t_{l_1+n_1}-t_{l_1}$ and $\delta=t_{l_2+n_2}-t_{l_2}$, and choose $\gamma'\le\gamma$ and $\delta'\le\delta$, γ' and δ' being independent for $r=2,3,4,5,6$:

$$\gamma \ge \begin{cases} t_{l_2}-t_{l_1}, & r=2,4,\\ t_{l_1+n_1}-t_{l_1}, & r=3,6,\\ (t_{l_2}-t_{l_1})+(t_{l_1+n_1}-t_{l_2+n_2}), & r=5,\end{cases}$$

and

$$\delta \ge \begin{cases} t_{l_2+n_2}-t_{l_1+n_1}, & r=3,\\ t_{l_2+n_2}-t_{l_2}, & r=2,4,5,\\ 0, & r=6.\end{cases}$$

Since $h_j(u)$ is decreasing on $[0,\infty[$, we have

$$E\{h_{j_1}(\gamma)h_{j_2}(\delta)\} \le E\{h_{j_1}(\gamma')h_{j_2}(\delta')\} \le E\{h_{j_1}(\gamma')\}E\{h_{j_2}(\delta')\}\,.$$

Notice that the probability density of $(t_{l_2}-t_{l_1})+(t_{l_1+n_1}-t_{l_2+n_2})$ is $f_{n_2-n_1}$, so for $r=2,\ldots,6$,

$$|U^{(r)}_{N,1}(k,\lambda,\mu)| \le G^{(r)}_{N,1}$$
$$= \frac{1}{(\pi\beta N)^2}\sum_{n_1,n_2=1}^{N-1}\sum_{(l_1,l_2)\in R_r}\sum_{(j_1,j_2)\in\mathbb{Z}^2}\left|\phi\left(\frac{2\pi}{T}(j_1+j_2)^{l_1}\right)^{l_1}\right| g^{(r)}(l_1,l_2,n_1,n_2)\,,$$

where

$$g^{(r)}(l_1,l_2,n_1,n_2) = \begin{cases} a_{j_1}(l_2-l_1)a_{j_2}(n_2), & r=2,\\ a_{j_1}(n_1)a_{j_2}(n_2-n_1+l_2-l_1), & r=3,\\ a_{j_1}(l_2-l_1)a_{j_2}(n_2), & r=4,\\ a_{j_1}(n_1-n_2)a_{j_2}(n_2), & r=5,\\ a_{j_1}(n_1), & r=6,\end{cases}$$

with $a_j(n)=\int_0^\infty h_j(u)f_n(u)\,du$. R_5 being characterized by $l_2-l_1>0$ and $l_2-l_1<n_1-n_2$, putting $s=l_2-l_1$, we have

$$G^{(5)}_{N,1} = \frac{1}{(\pi\beta N)^2}\sum_{n_1,n_2=1}^{N-1}\sum_{j_1+j_2\ne 0}\sum_{l_1=1}^{N-n_1}\sum_{s=1}^{n_1-n_2-1} a_{j_1}(n_1-n_2)a_{j_2}(n_2) \times\left|\phi\left(\frac{2\pi}{T}(j_1+j_2)\right)^{l_1}\right| 1_{\{n_1-n_2-2\ge 0\}}$$

$$+\frac{1}{(\pi\beta N)^2}\sum_{n_1,n_2=1}^{N-1}\sum_{j_1+j_2=0}\sum_{s=1}^{n_1-n_2-1}(N-n_1)$$
$$\times a_{j_1}(n_1-n_2)a_{j_2}(n_2)1_{\{n_1-n_2-2\geq 0\}}$$
$$\leq \frac{T}{2\pi(\pi\beta)^2 N}\sum_{n_2=1}^{N-1}\sum_{p=2}^{N-1-n_2}\sum_{j_1+j_2\neq 0}\frac{a_{j_1}(p)a_{j_2}(n_2)}{|j_1+j_2|}$$
$$+\frac{1}{(\pi\beta)^2 N}\sum_{n_2=1}^{N-1}\sum_{p=2}^{N-n_2-1}\sum_{j_1+j_2=0}(p-1)a_{j_1}(p)a_{j_2}(n_2)$$
$$\leq \frac{T}{2\pi^3 N}\sum_{j_1+j_2\neq 0}\int_0^\infty\int_0^\infty \frac{h_{j_1}(u_1)h_{j_2}(u_2)}{|j_1+j_2|}\,du_1\,du_2$$
$$+\frac{1}{(\pi\beta)^2 N}\sum_{j_1+j_2=0}\int_0^\infty\int_0^\infty(\beta^2u_1+2\beta)h_{j_1}(u_1)h_{j_2}(u_2)\,du_1\,du_2$$
$$=O\left(\frac{1}{N}\right).$$

In the same way $G_{N,1}^{(r)}=O(1/N)$ for $r=2,3,4,6$, therefore $W''_{N,1}(k,\lambda,\mu)=O(1/N)$ uniformly in λ and μ.

To show that $U_{N,3}(k,\lambda,\mu)=O(1/N)$ and $U_{N,4}(k,\lambda,\mu)=O(1/N)$, we use the method employed to evaluate $W''_{N,1}(k,\lambda,\mu)$.

Proof of Proposition 5.4. As for $W''_{N,1}(k,\lambda,\mu)$, we have

$$U_{N,2}^{(1)}(k,\lambda,\mu)=O(1/N)$$

uniformly in λ and μ.

For $r=2,3,4,6$, writing

$$t_{l_2}=(t_{l_2}-t_{l_1})+t_{l_1}\,,\qquad t_{l_1+n_1}-t_{l_1}=(t_{l_1+n_1}-t_{l_2})+(t_{l_2}-t_{l_1})$$

and

$$t_{l_2+n_2}-t_{l_2}=(t_{l_2+n_2}-t_{l_1+n_1})+(t_{l_1+n_1}-t_{l_2})\,,$$

we have

$$U_{N,2}^{(r)}(k,\lambda,\mu)=\Omega(\lambda,\mu)+\Omega\left(\lambda,-\mu+\frac{2\pi}{T}k\right)$$
$$+\Omega\left(-\lambda+\frac{2\pi}{T}k,\mu\right)$$
$$+\Omega\left(-\lambda+\frac{2\pi}{T}k,-\mu+\frac{2\pi}{T}k\right)$$

where

$$\begin{aligned}
\Omega(\lambda,\mu) &= \frac{1}{(2\pi\beta N)^2} \\
&\times \sum_{n_1,n_2=1}^{N-1} \sum_{(l_1,l_2)\in R_r} \sum_{(j_1,j_2)\in\mathbb{Z}^2} E\biggl(B_{j_1}(t_{l_2}-t_{l_1}) B_{j_2}(t_{l_2+n_2}-t_{l_1+n_1}) \\
&\times \exp\left(i\frac{2\pi}{T} j_2 (t_{l_1+n_2}-t_{l_2})\right) \\
&\times \exp\left(i\frac{2\pi}{T}(j_1+j_2)t_{l_1}\right) \\
&\times \exp\left(i\frac{2\pi}{T}(j_2+k)(t_{l_2}-t_{l_1})\right) \\
&\times \exp\left(i\biggl(\lambda-\frac{2\pi}{T}k\biggr)(t_{l_2}-t_{l_1})\right) \exp(i(\lambda-\mu)(t_{l_1+n_1}-t_{l_2})) \\
&\times \exp\left(-i\biggl(\mu-\frac{2\pi}{T}k\biggr)(t_{l_2+n_2}-t_{l_1+n_1})\right) \\
&\times \exp\left(i\frac{2\pi}{T}(j_1+j_2)t_{l_1}\right)\biggr).
\end{aligned}$$

Since for $r = 2, 3, 4, 6$, $t_{l_2+n_2}-t_{l_1+n_1}$ and $t_{l_1+n_1}-t_{l_2}$, $t_{l_2}-t_{l_1}$ are independent,

$$\begin{aligned}
\Omega(\lambda,\mu) = \frac{1}{(2\pi\beta N)^2} & \sum_{n_1,n_2=1}^{N-1} \sum_{(l_1,l_2)\in R_r} \sum_{(j_1,j_2)\in\mathbb{Z}^2} \phi\biggl(\frac{2\pi}{T}(j_1+j_2)\biggr)^{l_1} \\
& \times z_{j_1}\left(l_2-l_1, \lambda+\frac{2\pi}{T}j_2\right) \phi\biggl(\lambda-\mu+\frac{2\pi}{T}j_2\biggr)^{n_1-(l_2-l_1)} \\
& \times z_{j_2}\left(n_2-n_1+l_2-l_1, -\mu+\frac{2\pi}{T}k\right)
\end{aligned}$$

where $z_j(n,\lambda) = \int_0^\infty B_j(u) f_n(u) \exp(i\lambda u)\, du$ for $n \ge 1$ and $z_j(0,\lambda) = B_j(0)$. Hence

$$\begin{aligned}
\Omega(\lambda,\mu) = \frac{1}{(2\pi\beta N)^2} & \sum_{n_1,n_2=1}^{N-1} \sum_{(l_1,l_2)\in R_r} \sum_{j_1+j_2=0} z_{j_1}\left(l_2-l_1, \lambda-\frac{2\pi}{T}j_1\right) \\
& \times \phi\biggl(\lambda-\mu-\frac{2\pi}{T}j_1\biggr)^{n_1-(l_2-l_1)} z_{j_2}\left(n_2-n_1+l_2-l_1, -\mu+\frac{2\pi}{T}k\right) \\
& + Q(\lambda,\mu)
\end{aligned}$$

where

$$|Q(\lambda,\mu)| \le \frac{1}{(2\pi\beta N)^2}\sum_{n_1,n_2=1}^{N-1}\sum_{j_1+j_2\neq 0}\sum_{(l_1,l_2)\in R_r} a_{l_1} b_{j_1}(l_2-l_1) b_{j_2}(n_2-n_1+l_2-l_1)$$

with $a_l = |\phi(\frac{2\pi}{T}(j_1+j_2))^l|$, $b_j(l)=\int_0^\infty |B_j(u)| f_l(u)\,du,\ l\ge 1$, and $b_j(0) = |B_j(0)|$. Lemma 7.2 implies that

$$\begin{aligned}|Q(\lambda,\mu)| &\le \frac{1}{(2\pi\beta)^2 N}\sum_{j_1+j_2\neq 0}\sum_{l=1}^{\infty}\sum_{p=1}^{\infty}\sum_{s=1}^{\infty} a_l b_{j_1}(s) b_{j_2}(p)\\ &\le \frac{1}{(2\pi\beta)^2 N}\sum_{j_1+j_2\neq 0}\int_0^\infty\int_0^\infty \frac{|B_{j_1}(u_1)B_{j_2}(u_2)|}{|j_1+j_2|}\,du_1\,du_2 = O\left(\frac{1}{N}\right)\end{aligned}$$

where the $O(1/N)$ term is uniform in λ and μ.

To evaluate $U_{N,2}^{(5)}(k,\lambda,\mu)$, we write $t_{l_2+n_2}-t_{l_1+n_1} = -(t_{l_1+n_1}-t_{l_2+n_2})$ and $t_{l_1+n_1}-t_{l_1} = (t_{l_1+n_1}-t_{l_2+n_2}) + (t_{l_2+n_2}-t_{l_2}) + (t_{l_2}-t_{l_1})$ and we use the formula $B_j(-u)=\exp(-i\frac{2\pi}{T}ju)B_j(u)$; then the same method as above gives us

$$\begin{aligned}U_{N,1}^{(5)}(k,\lambda,\mu) &= \Omega'(\lambda,\mu) + \Omega'\left(\lambda,-\mu+\frac{2\pi}{T}k\right)\\ &\quad + \Omega'\left(-\lambda+\frac{2\pi}{T}k,\mu\right) + \Omega'\left(-\lambda+\frac{2\pi}{T}k,-\mu+\frac{2\pi}{T}k\right)\end{aligned}$$

where

$$\begin{aligned}\Omega'(\lambda,\mu) &= \frac{1}{(2\pi\beta N)^2}\sum_{n_1,n_2=1}^{N-1}\sum_{(l_1,l_2)\in R_r}\sum_{j_1+j_2=0} z_{j_1}\left(l_2-l_1,\lambda-\frac{2\pi}{T}j_1\right)\\ &\qquad\times\phi\left(\lambda-\mu-\frac{2\pi}{T}j_1\right)^{n_2} z_{j_2}\left(n_1-n_2+l_1-l_2,-\mu+\frac{2\pi}{T}k\right)\\ &\quad + O\left(\frac{1}{N}\right).\end{aligned}$$

Put $\overline{D}_N(\lambda) = D_N(\phi(\lambda))$. Then Lemma 7.3 gives us

$$\begin{aligned}U_{N,2}(k,\lambda,\mu) &= 2\sum_{r=1}^{5} U_{N,2}^{(r)}(k,\lambda,\mu) + U_{N,2}^{(6)}\\ &= S(\lambda,\mu) + S\left(-\lambda+\frac{2\pi}{T}k\right) + S\left(\lambda,-\mu+\frac{2\pi}{T}k\right)\\ &\quad + S\left(-\lambda+\frac{2\pi}{T}k,-\mu+\frac{2\pi}{T}k\right) + O\left(\frac{1}{N}\right)\end{aligned}$$

where

$$S(\lambda,\mu)=\frac{1}{(2\pi\beta)^2}\sum_{j\in\mathbb{Z}}\overline{D}_N\Big(\lambda-\mu-\frac{2\pi}{T}j\Big)\Big(2Z_j\Big(\lambda-\frac{2\pi}{T}j\Big)\overline{Z_j\Big(\mu-\frac{2\pi}{T}k\Big)}$$
$$+2B_j(0)\overline{Z_j\Big(\mu-\frac{2\pi}{T}k\Big)}+2Z_j\Big(\lambda-\frac{2\pi}{T}j\Big)\overline{B_j(0)}$$
$$+B_j(0)\overline{B_j(0)}+2Z_j\Big(\lambda-\frac{2\pi}{T}j\Big)\overline{Z_j\Big(-\lambda+\frac{2\pi}{T}k\Big)}\Big)$$

and $Z_j(\lambda)=\sum_{n=1}^{\infty}z_j(n,\lambda)=\int_0^{\infty}B_j(u)\exp(i\lambda u)\,du$. We have

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}W_N(u-\lambda)W_N(u-\mu)S(\lambda,\mu)\,d\lambda\,d\mu=\sum_{i=1}^{5}T_i\,.$$

For example, writing $\overline{D}_N(\lambda)=\frac{N-1}{N^2}\phi(\lambda)+P_N(\lambda)$, we have $T_1=T_{11}+T_{12}$ where

$$|T_{12}|\le\frac{A_4}{N}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|W_N(u-\lambda)W_N(u-\mu)|\,d\lambda\,d\mu$$
$$\times\sum_{j\in\mathbb{Z}}\int_0^{\infty}\int_0^{\infty}|B_j(s)B_j(t)|\,ds\,dt=O\Big(\frac{1}{N}\Big)$$

by (iii) of assumption $(\mathcal{H}_2)$, and

$$T_{11}=\frac{1}{(2\pi\beta)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\sum_{j\in\mathbb{Z}}W_N(u-\lambda)W_N(u-\mu)P_N\Big(\lambda-\mu-\frac{2\pi}{T}j\Big)$$
$$\times\Big(\int_0^{\infty}\int_0^{\infty}B_j(s)B_{-j}(t)\exp\Big(i\Big(\lambda-\frac{2\pi}{T}j\Big)s\Big)$$
$$\times\exp\Big(-i\Big(\mu-\frac{2\pi}{T}k\Big)t\Big)\,ds\,dt\Big)\,d\lambda\,d\mu$$
$$=\frac{1}{(2\pi\beta)^2}\int_{-\infty}^{\infty}\int_0^{\infty}\int_0^{\infty}W_N(u-\lambda)\sum_{j\in\mathbb{Z}}B_j(s)B_{-j}(t)$$
$$\times\exp\Big(i\Big(\lambda-\frac{2\pi}{T}j\Big)s\Big)\exp\Big(-i\Big(\lambda-\frac{2\pi}{T}j-\frac{2\pi}{T}k\Big)t\Big)\,d\lambda\,ds\,dt$$
$$\times\Big(\int_{-\infty}^{\infty}W_N\Big(u-\lambda+\frac{2\pi}{T}j+v\Big)P_N(v)\exp(ivt)\,dv\Big)\,.$$

Lemma 7.5 gives us

$$T_{11} = \frac{1}{2\pi\beta^2} \int_{-\infty}^{\infty} \int_0^{\infty} \int_0^{\infty} W_N(u-\lambda) \sum_{j\in\mathbb{Z}} B_j(s) B_{-j}(t)$$
$$\times \exp\Big(i\Big(\lambda - \frac{2\pi}{T}j\Big)s\Big) \exp\Big(-i\Big(\lambda - \frac{2\pi}{T}j - \frac{2\pi}{T}k\Big)t\Big)$$
$$\times \Big(\frac{2\pi}{N} W_N\Big(u - \lambda + \frac{2\pi}{T}j\Big) \sum_{n=2}^{N} \Big(1 - \frac{n}{N}\Big) f_n(-t)\Big)\, d\lambda\, ds\, dt$$
$$+ O\Big(\frac{M_N}{N}\Big)$$

and therefore $T_{11} = 0 + O(M_N/N)$.

We treat the other T_i in the same way and we obtain the result.

Proof of Proposition 5.5. We have

$$\operatorname{var}\{\Gamma_k(n,N,\omega)\} = \sum_{i=1}^{4} U_{n,N,i}(k,\omega) - |E\{\Gamma_k(n,N,\omega)\}|^2$$

where

$$U_{n,N,1}(k,\omega) = \frac{1}{N^2} \sum_{(l_1,l_2)\in R'} E(B(t_{l_1}, t_{l_1+n} - t_{l_1}) B(t_{l_2}, t_{l_2+n} - t_{l_2}) \times p(l_1,n,k,\omega)\overline{p(l_2,n,k,\omega)})\,,$$

$$U_{n,N,2}(k,\omega) = \frac{1}{N^2} \sum_{(l_1,l_2)\in R'} E(B(t_{l_1}, t_{l_2} - t_{l_1}) B(t_{l_1+n}, t_{l_2+n} - t_{l_1+n}) \times p(l_1,n,k,\omega)\overline{p(l_2,n,k,\omega)})\,,$$

$$U_{n,N,3}(k,\omega) = \frac{1}{N^2} \sum_{(l_1,l_2)\in R'} E(B(t_{l_1}, t_{l_2+n} - t_{l_1}) B(t_{l_2}, t_{l_1+n} - t_{l_2}) \times p(l_1,n,k,\omega)\overline{p(l_2,n,k,\omega)})\,,$$

$$U_{n,N,4}(k,\omega) = \frac{1}{N^2} \sum_{(l_1,l_2)\in R'} E(K(t_{l_1}, t_{l_1+n} - t_{l_1}, t_{l_2+n} - t_{l_1}, t_{l_2} - t_{l_1}) \times p(l_1,n,k,\omega)\overline{p(l_2,n,k,\omega)})$$

with $R' = \{(l_1,l_2) \in \mathbb{N} :\ 1 \le l_1, l_2 \le N - n\}$. Put

$$R'_1 = \{(l_1,l_2) \in R' : l_1 < l_1 + n \le l_2 < l_2 + n\}\,,$$
$$R'_2 = \{(l_1,l_2) \in R' : l_1 < l_2 < l_1 + n < l_2 + n\}\,,$$
$$R'_3 = \{(l_1,l_2) \in R' : l_2 < l_2 + n \le l_1 < l_1 + n\}\,,$$
$$R'_4 = \{(l_1,l_2) \in R' : l_2 < l_1 < l_2 + n < l_1 + n\}\,,$$

$$R'_5 = \{(l_1, l_2) \in R' : l_1 = l_2 < l_1 + n = l_2 + n\}\,.$$

Then $R' = \bigcup_{r=1}^{5} R'_r$ and $R'_i \cap R'_j = \emptyset$ for $i \neq j$, therefore $U_{n,N,1}(k,\omega) = \sum_{r=1}^{5} U^{(r)}_{n,N,1}(k,\omega)$ where $U^{(r)}_{n,N,1}(k,\omega)$ has the same form as $U_{n,N,1}(k,\omega)$ with $\sum_{R'}$ replaced by $\sum_{R'_r}$. Moreover, by symmetry of l_1 and l_2, we have $U^{(1)}_{n,N,1}(k,\omega) = U^{(3)}_{n,N,1}(k,\omega)$ and $U^{(2)}_{n,N,1}(k,\omega) = U^{(4)}_{n,N,1}(k,\omega)$.

As in the proof of Proposition 5.3, we have $2U^{(1)}_{n,N,1}(k,\omega) - |E\{\Gamma_k(n,N,\omega)\}|^2 = O(1/N)$, with the $O(1/N)$ term uniform in ω and k. Let

$$\begin{aligned}
|w_2| &= 2|U^{(2)}_{n,N,1}(k,\omega)| \\
&\leq \frac{8}{N^2} \sum_{(l_1,l_2)\in R'_2} \sum_{j_1,j_2\in\mathbb{Z}} E\{|B_{j_1}(t_{l_1+n} - t_{l_1}) B_{j_2}(t_{l_2+n} - t_{l_2})|\} \\
&\quad \times \left|\phi\left(\frac{2\pi}{T}(j_1+j_2)\right)\right|^{l_1} \\
&\leq \frac{8}{N^2} \sum_{(l_1,l_2)\in R'_2} \left(\sum_{j_1+j_2\neq 0} \frac{(c_{j_1}(n) c_{j_2}(n))^{1/2}}{|j_1+j_2|} + \sum_{j\in\mathbb{Z}} c_j(n) \right)
\end{aligned}$$

where $c_j(n) = \int_0^\infty |B_j(u)|^2 f_n(u)\,du$; on the other hand,

$$\begin{aligned}
\sum_{(l_1,l_2)\in R'_2} 1 &= \sum_{s=1}^{n-1} (N-n-s) 1_{\{N-n>n-1\}} \\
&\quad + \sum_{s=1}^{N-n-1} (N-n-s) 1_{\{N-n\leq n-1\}} \\
&\leq |(n-1)(2N-3n)| + (N-n)(N-n-1) 1_{\{N-n\leq n-1\}} \\
&\leq 20nN\,,
\end{aligned}$$

therefore $w_2 = O(1/N)$ uniformly in ω and k, by Lemma 7.6. In the same way $U^{(5)}_{n,N,1}(k,\omega) = O(1/N)$.

Now,

$$\begin{aligned}
&|U_{n,N,2}(k,\omega)| \\
&\quad \leq \frac{4}{N^2} \sum_{(l_1,l_2)\in R'} \sum_{j_1,j_2\in\mathbb{Z}} E\{|B_{j_1}(t_{l_2} - t_{l_1}) B_{j_2}(t_{l_2+n} - t_{l_1+n})|\} \\
&\qquad\qquad\qquad\qquad \times \left|\phi\left(\frac{2\pi}{T}(j_1+j_2)\right)\right|^{l_1}
\end{aligned}$$

$$\le \frac{4}{N^2} \sum_{(l_1,l_2)\in R_2'} \Bigg(\sum_{j_1+j_2\neq 0} \frac{(c_{j_1}(|l_2-l_1|)c_{j_2}(|l_2-l_1|))^{1/2}}{|j_1+j_2|} + \sum_{j\in\mathbb{Z}} c_j(|l_2-l_1|) \Bigg)$$

$$\le \frac{8}{N^2} \sum_{s=0}^{N-n-1} (N-n-s) \Bigg(\sum_{j_1+j_2\neq 0} \frac{(c_{j_1}(s)c_{j_2}(s))^{1/2}}{|j_1+j_2|} + \sum_{j\in\mathbb{Z}} c_j(s) \Bigg)$$

$$\le \frac{8}{N} \sum_{c=0}^{\infty} \Bigg(\sum_{j_1+j_2\neq 0} \frac{(c_{j_1}(s)c_{j_2}(s))^{1/2}}{|j_1+j_2|} + \sum_{j\in\mathbb{Z}} c_j(s) \Bigg) = O\bigg(\frac{1}{N}\bigg).$$

Next,

$$U_{n,N,3}(k,\omega) \le \frac{4}{N^2} \sum_{(l_1,l_2)\in R'} \sum_{j_1+j_2\neq 0} \frac{(c_{j_1}(|l_2-l_1+n|)c_{j_2}(|l_2-l_1-n|))^{1/2}}{|j_1+j_2|} + \frac{4}{N^2} \sum_{(l_1,l_2)\in R'} \sum_{j\in\mathbb{Z}} (c_j(|l_2-l_1+n|)c_j(|l_2-l_1-n|))^{1/2}$$

$$\le \frac{8}{N^2} \sum_{s=0}^{N-n-1} \Bigg(\sum_{j_1+j_2\neq 0} \frac{(c_{j_1}(|s+n|)c_{j_2}(|s-n|))^{1/2}}{|j_1+j_2|} + \sum_{j\in\mathbb{Z}} (c_j(|s+n|)c_j(|s-n|))^{1/2} \Bigg)$$

$$\le \frac{8}{N} \sum_{c=0}^{\infty} \Bigg(\sum_{j_1+j_2\neq 0} \frac{(c_{j_1}(s)c_{j_2}(s))^{1/2}}{|j_1+j_2|} + \sum_{j\in\mathbb{Z}} c_j(s) \Bigg) = O\bigg(\frac{1}{N}\bigg).$$

Finally,

$$|U_{n,N,4}(k,\omega)| \le \frac{8}{N^2} \sum_{(l_1,l_2)\in R'} \sum_{j\in\mathbb{Z}} E\{h_j(|t_{l_1+n}-t_{l_1}|, |t_{l_2}-t_{l_1}|, |t_{l_2+n}-t_{l_1}|)t\} \times |\phi(j-k)^{l_1}|$$

$$\le \frac{8}{N^2} \sum_{(l_1,l_2)\in R'} \Bigg(\sum_{j\neq k} \frac{s_j(|l_2-l_1|)}{|j-k|} + s_k(|l_2-l_1|) \Bigg)$$

where

$$s_j(n) = \int_0^\infty h_j(0,u,0) f_n(u)\, du \quad \text{and} \quad s_j(0) = h_j(0,0,0).$$

As for $U_{n,N,2}(k,\omega)$, we have $U_{n,N,4}(k,\omega) = O(1/N)$ uniformly in ω and k.

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