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TWO MUTUALLY RAREFIED RENEWAL PROCESSES

Abstract. Let us consider two independent renewal processes generated by appropriate sequences of life times. We say that a renewal time is accepted if in the time between a signal and the preceding one, some signal of the second process occurs. Our purpose is to analyze the sequences of accepted renewals. For simplicity we consider continuous and discrete time separately. In the first case we mainly consider the renewal process rarefied by the Poisson process, in the second we analyze the process generated by the motion of draughtsmen moved by die tossing.

1. The problem. Let us consider two independent renewal processes generated by sequences $\{X_1^{(i)}, X_2^{(i)}, \ldots\}$, $i = 1, 2$, of nonnegative, uniformly distributed, independent random variables.

Introduce the usual notation for the renewal times, the renewal processes and the residual time processes:

$$\begin{aligned} S_0^{(i)} &= 0, \qquad S_n^{(i)} = S_{n-1}^{(i)} + X_n^{(i)}, \qquad n = 1, 2, \ldots, \\ N_i(t) &= \max\{n \ge 0 : S_n^{(i)} < t\}, \\ \gamma_i(t) &= S_{N_i(t)+1}^{(i)} - t, \qquad t \ge 0,\ i = 1, 2. \end{aligned}$$

We say that the renewal time $S_n^{(1)}$ of process 1 is *accepted* by process 2 if in the interval $(S_{n-1}^{(1)}, S_n^{(1)}]$ a renewal in process 2 occurs. Similarly, renewal times of process 2 are accepted by process 1.

Let $Z_0'' = 0, Z_1'', Z_2'', \ldots$ denote the accepted times of process 2 and let $Z_1', Z_2', \ldots$ denote the accepted times of process 1. Notice that $Z_0'' = 0 < Z_1' \le Z_1'' \le Z_2' \le \ldots$, i.e. these times alternate or sometimes coincide (our notations and the coincidence concept are derived from [1]). Our purpose is to analyze the probability distribution function of the sequence of random

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variables

$$V_n = Z'_n - Z''_{n-1}, \qquad U_n = Z''_n - Z'_n, \qquad n = 1, 2, \ldots \tag{1}$$

The residual time processes, the renewal times and the accepted times (dotted lines) are shown in Fig. 1.

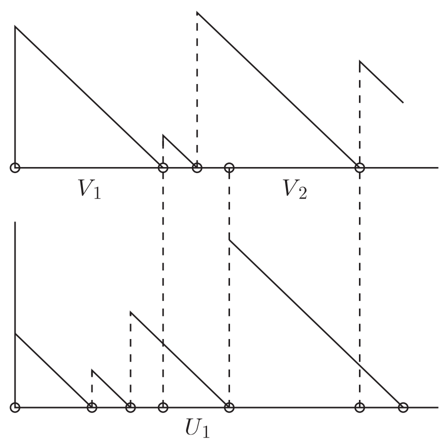

Fig. 1

Let us introduce the standard notation for the probability distribution functions of the life times of the renewal processes, the renewal functions and the probability distribution function of the residual time process:

$$\begin{aligned} F_i(x) &= P(X_1^{(i)} < x)\,, \\ H_i(t) &= EN_i(t)\,, \\ G_i(t,x) &= P(\gamma_i(t) < x), \qquad x \geq 0,\ t \geq 0,\ i = 1, 2\,. \end{aligned}$$

Recall that

$$H_i(t) = \sum_{n=1}^{\infty} P(S_n^{(i)} < t)\,,$$

$$G_i(t,x) = F_i(t+x) - \int_0^t (1 - F_i(t+x-u))\, dH_i(u)\,.$$

By X_1° and X_2° we denote some probabilistic copies of the random variables $X_1^{(1)}$ and $X_1^{(2)}$, independent of the random variables used before. We also mark by $^\circ$ probabilistic copies of other random variables and processes. We deal with the relationship between the above-defined probability distribution functions. The relationships may be simply expressed by using the random variables and their copies.

PROPOSITION 1. *The random sequence* (1) *is a Markov chain for which*

$$V_1 \stackrel{\mathrm{d}}{=} X_1^\circ\,, \tag{2}$$

$$U_n \stackrel{\mathrm{d}}{=} \begin{cases} X_2^\circ & \text{if } V_n = 0, \\ \gamma_2^\circ(V_n) & \text{if } V_n > 0, \end{cases} \tag{3}$$

$$V_{n+1} \stackrel{\mathrm{d}}{=} \begin{cases} X_1^\circ & \text{if } U_n = 0, \\ \gamma_1^\circ(U_n) & \text{if } U_n > 0, \end{cases} \qquad n = 1, 2, \ldots \tag{4}$$

The distribution functions are as follows ($n = 1, 2, \ldots$):

$$P(V_1 < x) = F_1(x)\,,$$

$$P(U_n < x \mid V_n = y) = \begin{cases} F_2(x) & \text{if } y = 0, \\ G_2(y, x) & \text{if } y > 0, \end{cases}$$

$$P_1(x \mid y) = P(V_{n+1} < x \mid U_n = y) = \begin{cases} F_1(x) & \text{if } y = 0, \\ G_1(y, x) & \text{if } y > 0. \end{cases}$$

PROPOSITION 2. *The stationary Markov chain* (1) *satisfies the system of equations*

$$U \stackrel{\mathrm{d}}{=} \begin{cases} X_2^\circ & \text{if } V^\circ = 0, \\ \gamma_2^\circ(V^\circ) & \text{if } V^\circ > 0, \end{cases}$$

$$V \stackrel{\mathrm{d}}{=} \begin{cases} X_1 & \text{if } U = 0, \\ \gamma_1^\circ(U) & \text{if } U > 0. \end{cases}$$

The stationary distribution function $W(u, v) = P(U < u, V < v)$ *and the boundary distributions* $W_1(u) = W(u, \infty)$, $W_2(v) = W(\infty, v)$ *satisfy the system of equations*

$$W(x, y) = \int_0^y P_1(x \mid u)\, dW_1(u)\,,$$

$$W_2(v) = W_1(0+) F_1(v) + \int_{0+}^{\infty} G_1(u, v)\, dW_1(u)\,,$$

$$W_1(u) = W_2(0+) F_2(u) + \int_{0+}^{\infty} G_2(v, u)\, dW_2(v)\,.$$

Consider the case of two renewal processes with the same generating distribution function $F_2 = F_1$. In this case consider the sequences of random variables

$$W_{2n} = U_n, \qquad W_{2n-1} = V_n, \qquad n = 1, 2, \ldots, \tag{5}$$

which are the life times for the superposition of accepted times.

Corollary 1. *The sequence of random variables* (5) *is a homogeneous Markov chain for which*

$$W_1 \stackrel{\mathrm{d}}{=} X_1\,,$$
$$W_{n+1} \stackrel{\mathrm{d}}{=} \begin{cases} X_1^\circ & \text{if } W_n = 0,\\ \gamma_1^\circ(W_n) & \text{if } W_n > 0, \end{cases} \qquad n = 1, 2, \ldots \tag{6}$$

The probability transition functions are as follows:

$$P(W_{n+1} < x \mid W_n = y) = P_1(x \mid y)\,.$$

The stationary sequence of random variables (5) *satisfies the equation*

$$W \stackrel{\mathrm{d}}{=} \begin{cases} X_1 & \text{if } W^\circ = 0,\\ \gamma_1^\circ(W^\circ) & \text{if } W^\circ > 0, \end{cases}$$

and the stationary probability distribution function satisfies the equation

$$W(x) = W(0+)F_1(x) + \int\limits_{0+}^{\infty} G_1(u, x)\, dW(u)\,.$$

2. Special case. Assume that process 1 is a Poisson process with parameter λ. Then $F_1(x) = 1 - \exp(-\lambda x) = G_1(t, x)$. This implies that $V_n, n = 1, 2, \ldots$, are independent random variables, exponentially distributed with parameter λ. Thus $U_n, n = 1, 2, \ldots$, are independent and equidistributed. Consider the distances $Z_n = Z_n'' - Z_{n-1}'' = U_n + V_n$, $n = 1, 2, \ldots$, between the accepted times in the renewal process 2.

Proposition 3. *The random variables* $Z_n, n = 1, 2, \ldots$, *are independent and equidistributed with probability distribution function*

$$Z \stackrel{\mathrm{d}}{=} \begin{cases} X_2^\circ & \text{if } N_1(X_2^\circ) > 0,\\ X_2^\circ + Z^\circ & \text{otherwise.} \end{cases} \tag{7}$$

P r o o f. There are two possibilities: in the interval $(0, X_1^{(2)})$ there occurs a signal of the Poisson process and then $Z = X_1^{(2)}$, otherwise $Z \stackrel{\mathrm{d}}{=} X_1^{(2)} + Z^\circ$.

Equation (7) immediately implies the following corollary (it can also be derived from (2)–(4) after suitable analytic transformations).

Corollary 1. *The Laplace–Stieltjes transform of the distribution function of the random variable* Z *has the form*

$$E\exp(-sZ) = z^*(s) = \frac{f^*(s) - f^*(s+\lambda)}{1 - f^*(s+\lambda)}\,,$$

where $f^*(s) = E\exp(-sX_2^\circ)$,

$$EZ = \frac{EX_2^\circ}{1 - f^*(\lambda)}\,,$$

provided that the expected value EX_2° *is finite.*

It is easy to verify that we deal with rarefied renewal processes with probability of acceptance depending upon the preceding life time.

If $\lambda \to 0$ then process 2 is rarefied with probability of nonacceptance which converges to 1. For the distribution function of the random variable Z we have a limit theorem similar to the theorem of Rényi [2].

PROPOSITION 4. *If* $\lambda \to 0$ *and* $EX_2^\circ < \infty$, *then* $\lambda EZ \to 1$ *and* Z/EZ *has an exponential limit distribution function with parameter* 1:

$$\lim_{\lambda\to 0} P\left(\frac{Z}{EZ} > x\right) \to \exp(-x), \qquad x > 0\,.$$

Proof. Using the Laplace–Stieltjes transform, we get

$$E\exp\left(\frac{-sZ}{EZ}\right) = \frac{1}{\lambda}\left(f^*\left(\frac{s}{EZ}\right) - f^*\left(\frac{s}{EZ}+\lambda\right)\right)\frac{\frac{s}{EZ}+\lambda}{1-f^*\left(\frac{s}{EZ}+\lambda\right)}$$
$$\times\left[\frac{s}{EX_2^\circ}\,\frac{1-f^*(\lambda)}{\lambda}+1\right]^{-1} \underset{\lambda\to 0}{\longrightarrow} \frac{1}{s+1}\,.$$

3. Discrete life time. Equations (3), (4) or equation (6) can be solved approximately by passing to discrete approximation of probability distribution functions generating the renewal processes. The approximation $F(x) \cong F(\frac{1}{h}[hx])$, where $h > 0$ and $[x]$ denotes the integer part of x, leads to considering our problem in integer random variables. Here we assume that the support of $F(x)$ is finite. Now we limit our considerations to the case of renewal processes generated by the same distribution function. Extension to the asymmetrical case is easy.

Let us introduce the standard notation for the probability distribution functions, the renewal equation, the residual time and the residual distribution function:

$$p_n = P(X^\circ = n), \qquad n = 1,\ldots,N,$$
$$u_0 = 1\,,$$
$$u_n = \sum_{k=(n-N)_+}^{n-1} u_k p_{n-k}\,,$$
$$\gamma(n) = S_{N(n)+1} - n, \qquad n = 1,2,\ldots,$$
$$g(n,j) = P(\gamma(n) = j) = \begin{cases} u_n & \text{if } j = 0,\\ \sum_{k=(n-N+j)_+}^{n-1} u_k p_{n-k+j} & \text{if } j = 1,\ldots,N-1,\end{cases}$$

where $(x)_+ = \max(0,x)$.

Now we pass to the discrete version of equation (6). Obviously, we analyze the probability of integer values instead of the probability distribution functions.

Proposition 5. *The probabilities* $w_n(j) = P(W_n = j)$, $j = 0, 1, \ldots, N$, $n = 1, 2, \ldots$, *of the Markov chain* (5) *satisfy the recursive system of equations*

$$w_1(j) = p_j, \qquad j = 1, \ldots, N,$$
$$w_n(j) = \begin{cases} w_{n-1}(0)p_N & \text{if } j = N, \\ \sum_{k=1}^{N} w_{n-1}(k)g(k,j) + w_{n-1}(0)p_j & \text{if } j = 0, 1, \ldots, N-1, \end{cases} \qquad n = 2, 3, \ldots$$

The stationary probabilities $w(j) = P(W = j)$, $j = 0, 1, \ldots, N$, *satisfy the homogeneous system of linear equations*

$$w(j) = \begin{cases} w(0)p_N & \text{if } j = N, \\ \sum_{k=1}^{N} w(k)g(k,j) + w(0)p_j & \text{if } j = 0, 1, \ldots, N-1, \end{cases}$$

with the condition $w(0) + \ldots + w(N) = 1$.

We obtain the probability distribution function of Z from the equation $Z \stackrel{\mathrm{d}}{=} W + \gamma_1^\circ(W)$. Thus

$$v(z) = P(Z = z) = \sum_{j=(z-N+1)_+}^{\min(N,z)} w(j)g(j, z-j), \qquad z = 1, \ldots, 2N-1.$$

4. Example. Let us consider a game of two players moving two draughtsmen on the real line using die tossing. The first player tosses the die and places his draughtsman on the resulting position. Each player tosses the die and moves his draughtsman according to the result as long as he reaches or exceeds the position of the opponent. The game reminds the process of camping by emigrants in competition.

The considered game is an example of mutually rarefied renewal processes generated by the uniform probability distribution function $p_n = 1/6$, $n = 1, 2, \ldots, 6$.

The stationary probability distribution function of the random distance W between the draughtsmen equals

$$w(0) = 0.1827, \quad w(1) = 0.2081, \quad w(2) = 0.1929, \quad w(3) = 0.1675,$$
$$w(4) = 0.1320, \quad w(5) = 0.0863, \quad w(6) = 0.0305,$$

with expected value 2.2386. The probability distribution function of the distance between the consecutive positions of each player equals

$$v(1) = 0.0651, \quad v(2) = 0.1027, \quad v(3) = 0.1407, \quad v(4) = 0.1756\,,$$
$$v(5) = 0.2022, \quad v(6) = 0.2132, \quad v(7) = 0.0465, \quad v(8) = 0.0296\,,$$
$$v(9) = 0.0161, \quad v(10) = 0.0067, \quad v(11) = 0.0016,$$

with expected value twice as great.

References

[1] I. Kopocińska and B. Kopociński, *On coincidences in renewal streams*, Zastos. Mat. 19 (1987), 169–180.

[2] A. Rényi, *A characterization of the Poisson process*, MTA Mat. Kut. Int. Kőzl. 1 (1956), 519–527.

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