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BAYES SEQUENTIAL ESTIMATION PROCEDURES FOR EXPONENTIAL-TYPE PROCESSES

Abstract. The Bayesian sequential estimation problem for an exponential family of processes is considered. Using a weighted square error loss and observing cost involving a linear function of the process, the Bayes sequential procedures are derived.

1. Introduction. The paper deals with Bayesian sequential estimation for continuous time stochastic processes whose likelihood functions have the exponential form $\exp[\vartheta Z(t) + \Phi(\vartheta)S(t)]$, where $(Z(t),\ S(t))$, $t \ge 0$, is a two-dimensional observed process, ϑ is a parameter with values in an open interval $\Theta \subset \mathbb{R}$ and $\Phi(\vartheta)$ is a real deterministic function. It is assumed that $Z(0) = 0, S(0) = 0$ and $S(t)$, which may be nonrandom as well, is strictly increasing and continuous as a function of t and $S(t) \to \infty$ as $t \to \infty$. One has to estimate the mean parameter $\mu = -\Phi'(\vartheta)$. The loss due to estimation error is assumed to be of the form $L(\mu, d) = V^{-1}(\mu)(d-\mu)^2$, where d is the chosen estimate and $V(\mu) = -\Phi''(\vartheta)$ denotes the variance parameter. The cost of observation is defined by a linear function of the observed process. Assuming that $V(\mu)$ is a quadratic function of μ, Bayes sequential procedures are derived explicitly in two cases: when the cost is a linear function of $S(t)$ and when it is a linear function of both $S(t)$ and $Z(t)$ providing that $Z(t)$ is nondecreasing as a function of t.

The problem of finding Bayes sequential procedures has been studied in some special cases of the exponential statistical model considered in this paper. Much attention in the literature is devoted to Bayes sequential estimation of the arrival rate μ of a Poisson process. Shapiro and Wardrop (1978) considered the procedures restricted to rules terminating at arrivals. Using the loss $\mu^{-2}(\mu - d)^2$ and sampling costs involving cost per unit time

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and cost per arrival, they derived a Bayes sequential procedure and studied its large sample properties. El-Sayyad and Freeman (1973) considered the same cost and loss structure in a continuous time approach. Assuming the loss to be of the form $\mu^{-p}(\mu-d)^2$, where $0 \le p \le 3$, Shapiro and Wardrop (1980a) also solved the problem in continuous time applying the notion of "monotone case" for continuous time problems and employing Dynkin's formula. Novic (1980) considered the problem for the same total loss using a discrete time approach. For a class of loss functions, Rasmussen (1980) studied the Bayes sequential estimation problem for the gamma process as a continuous time analogue of the problem of sequential estimation of the mean of a normal distribution when the variance is unknown. The class of loss functions considered by her does not include the loss considered in the present paper. In finding Bayes sequential procedures for sampling from a one-parameter exponential family of distributions Shapiro and Wardrop (1980b) considered the cost function involving only one component—the sample size cost.

The exponential statistical model considered in the present paper includes a large class of stochastic processes. If the cost involves a linear function of $Z(t)$ the results of the paper are applicable, in particular, to Poisson-type processes (comprising the Poisson, Weibull, pure birth and some other counting processes), and to gamma and negative binomial processes. If the cost involves only a linear function of $S(t)$, then the derived results apply, moreover, to some diffusion processes (Wiener processes with linear drift and Ornstein–Uhlenbeck processes, for example).

In the sense of Barndorff-Nielsen (1980), the exponential statistical model considered in the present paper is a (2,1)-curved exponential family. If the cost involves a linear function of $Z(t)$, then the optimal stopping time $\tau^*_{r,\alpha}$ has a form (see formula (14)) such that the canonical statistics $Z(\tau^*_{r,\alpha})$ and $S(\tau^*_{r,\alpha})$ are not affinely dependent. This implies that the optimal stopping times derived do not reduce the model to a noncurved exponential family—in contrast to efficient stopping times. For the problem of reducing curved exponential families of stochastic processes to noncurved ones, see Stefanov (1988).

2. An exponential family of processes and conjugate priors. Let $X(t)$, $t \ge 0$, be a continuous or discrete time stochastic process defined on a probability space $(\Omega, \mathcal{F}, P_\vartheta)$, where ϑ is a parameter with values in an open interval $\Theta \subset \mathbb{R}$. Denote by $P_{\vartheta,t}$ the restriction of P_ϑ to the σ-algebra $\mathcal{F}_t = \sigma\{X(s) : s \le t\}$. Suppose that for each t the family $P_{\vartheta,t}$, $\vartheta \in \Theta$, is dominated by a measure Q_t which is the restriction of a probability measure Q to $\mathcal{F}_t$. Moreover, assume that the density functions (likelihood functions) have the form

$$\frac{dP_{\vartheta,t}}{dQ_t} = \exp[\vartheta Z(t) + \Phi(\vartheta)S(t)], \tag{1}$$

where $Z(t)$, $t \geq 0$, and $S(t), t \geq 0$, are real-valued stochastic processes adapted to the filtration $\mathcal{F}_t, t \geq 0$, and $\Phi(\vartheta)$ is a twice continuously differentiable real-valued function with $-\Phi''(\vartheta) > 0$ for all $\vartheta \in \Theta$. It is also assumed that $Z(0) = 0$, $S(0) = 0$; that $Z(t)$ is right continuous as a function of t, P_ϑ-a.s.; and that $S(t)$ is strictly increasing and continuous as a function of t, and $S(t) \to \infty$ as $t \to \infty$, P_ϑ-a.s.

Define $\mu = -\Phi'(\vartheta)$ and $V(\mu) = -\Phi''(\vartheta)$. The problem of estimating the parameter μ will be considered using a Bayes sequential approach. Let $(\mathcal{M}, \mathcal{B}_{\mathcal{M}})$ denote the parameter space for $\mu \in \mathcal{M}$, where $\mathcal{M}$ is an open interval $(\underline{\mu}, \overline{\mu})$ ($\underline{\mu}$ and/or $\overline{\mu}$ possibly infinite). An exponential family of conjugate prior distributions on $(\mathcal{M}, \mathcal{B}_{\mathcal{M}})$ will be considered. Denote by $\mathcal{Y}$ the interior of the convex hull of the set of all possible values of the process $(Z(t), S(t))$, $t \geq 0$. Let $\vartheta(\mu)$ be the inverse function of $\mu = -\Phi'(\vartheta)$. Suppose that the following condition is satisfied:

(i) there exists a constant γ such that

$$\int_{\mathcal{M}} \exp[r\vartheta(\mu) + \alpha\Phi(\vartheta(\mu))]\, d\mu < \infty$$

for every $(r, \alpha) \in \mathcal{Y}$ and $\alpha > \gamma$.

Let us define a family $\pi_{r,\alpha}$, $(r, \alpha) \in \mathcal{Y}$, $\alpha > \gamma$, of prior distributions of the parameter μ on $(\mathcal{M}, \mathcal{B}_{\mathcal{M}})$ according to the following form of densities (with respect to the Lebesgue measure $d\mu$):

$$g(\mu; r, \alpha) = C(r, \alpha) \exp[r\vartheta(\mu) + \alpha\Phi(\vartheta(\mu))]. \tag{2}$$

From condition (i) it follows that there exists a norming constant $C(r, \alpha)$ such that $\pi_{r,\alpha}$ is a probability distribution on $(\mathcal{M}, \mathcal{B}_{\mathcal{M}})$. The expectation evaluated with respect to this distribution will be denoted by E.

The following two lemmas, which follow from the paper of Magiera (1992), will be used in the next sections.

LEMMA 1. *Suppose that* $EV^{-1}(\mu)$ *and* $E\mu^2V^{-1}(\mu)$ *exist for every* $(r, \alpha) \in \mathcal{Y}$, $\alpha > \gamma$, *and both*

$$\exp[r\vartheta(\mu) + \alpha\Phi(\vartheta(\mu))] \quad \textit{and} \quad \mu \exp[r\vartheta(\mu) + \alpha\Phi(\vartheta(\mu))]$$

tend to zero as $\mu \to \underline{\mu}$ *or* $\overline{\mu}$. *Then*

$$\alpha E\mu V^{-1}(\mu) = rEV^{-1}(\mu) \tag{3}$$

and

$$E(\alpha\mu - r)^2 V^{-1}(\mu) = \alpha. \ \blacksquare \tag{4}$$

Lemma 2. *Suppose that in the exponential statistical model for stochastic processes, defined by* (1), *the function* $V(\mu)$ *has the quadratic form*

$$V(\mu) = \eta_2\mu^2 + \eta_1\mu + \eta_0, \tag{5}$$

where η_2, η_1, η_0 *are some constants. Then* (i) *and all the conditions of Lemma* 1 *hold for* $\gamma = \eta_2$ *and, moreover,*

$$EV^{-1}(\mu) = \frac{\alpha - \eta_2}{\alpha V(r/\alpha)}$$

for $(r, \alpha) \in \mathcal{Y}$ *and* $\alpha > \eta_2$, *and*

$$E\mu = \frac{r + \eta_1}{\alpha - 2\eta_2} \tag{6}$$

for $(r + \eta_1, \alpha - 2\eta_2) \in \mathcal{Y}$. ■

3. The (r_t, α_t) process and statement of the problem. Let us consider the process $(r_t, \alpha_t), t \geq 0$, with $r_t = r + Z(t), \alpha_t = \alpha + S(t)$, where $(Z(t), S(t))$, $t \geq 0$, is a continuous time process belonging to the exponential family considered. The process $(r_t, \alpha_t), t \geq 0$, is a right continuous Markov process with Euclidean topological space $(E, \mathcal{T}, \mathcal{B}_E)$, $E \subset \mathbb{R} \times \mathbb{R}_+$, with the natural Euclidean topology $\mathcal{T}$ on E. Moreover, the process (r_t, α_t), $t \geq 0$, is a Feller process (see Shiryaev (1973), p. 18, for the definition), since for every bounded, $\mathcal{B}_E$-measurable, continuous function f on $(E, \mathcal{B}_E)$ and for every $t \geq 0$, the function $E_{r,\alpha}[f(r_t, \alpha_t)]$ is continuous in (r, α). $E_{r,\alpha}$ denotes expectation when the prior distribution on μ is the $\pi_{r,\alpha}$, defined by (2). Thus, the process (r_t, α_t), $t \geq 0$, is a strongly measurable strong Markov process, since, by Dynkin (1965), pp. 98, 99, so is every right continuous Feller process on the topological space $(E, \mathcal{T}, \mathcal{B}_E)$.

Sequential estimation procedures of the form (τ, d) will be considered where τ is a stopping time with respect to $\mathcal{F}_t$, $t \geq 0$, and $d = d(r_\tau, \alpha_\tau)$ is an $\mathcal{F}_\tau$-measurable function.

The loss associated with estimation, when μ is the true value of the parameter and d is the chosen estimate, is of the form

$$L(\mu, d) = V^{-1}(\mu)(d - \mu)^2.$$

If the prior distribution over μ has the density $g(\mu; r, \alpha)$, given by (2), then the posterior density of μ given $\mathcal{F}_\tau$ is $g(\mu; r_\tau, \alpha_\tau)$. The formulae for expectations in Lemmas 1 and 2 also hold for this posterior distribution upon substituting r_τ and α_τ for r and α.

It follows from Lemma 1 that for any stopping time τ, the Bayes estimator of μ given $\mathcal{F}_\tau$ is

$$d^* = \frac{r_\tau}{\alpha_\tau},$$

by (3), and the posterior expected loss is

$$E[L(\mu, d^*)/\mathcal{F}_\tau] = \frac{1}{\alpha_\tau}$$

(from (4)), which is independent of r_τ. Thus, the procedure is identified with the stopping time. The Bayes estimation problem then reduces to the following optimal stopping problem. The total loss (cost) of observing the process up to time τ is defined to be

$$\mathcal{L}(r_\tau, \alpha_\tau) = \frac{1}{\alpha_\tau} + c_1\alpha_\tau + c_2 r_\tau, \tag{7}$$

where c_1 and c_2 are nonnegative constants. One has to find an optimal stopping time which will determine the moment when to stop observing, so as to minimize the expected total loss $E_{r,\alpha}\mathcal{L}(r_\tau, \alpha_\tau)$ over all stopping times τ. Such a stopping time is called a Bayes sequential procedure or *optimal stopping time*.

The problem of finding Bayes sequential procedures for our processes will be solved in two steps. First, the solution will be given for an exponential class of processes with stationary independent increments. Next, the general case will be reduced to that special case by using a random time transformation.

4. Bayes sequential procedures for an exponential class of processes with stationary independent increments. In this section we consider the exponential statistical model defined by (1) with continuous time parameter and with $S(t) \equiv t$. It is well known that in that case the exponential family of processes reduces to processes with stationary independent increments.

We take for Bayes sequential procedures the infinitesimal look-ahead procedures which are derived from the infinitesimal operator at $\mathcal{L}(r, \alpha)$ for the process (r_t, α_t), $t \geq 0$, where $r_t = r + Z(t)$, $\alpha_t = \alpha + t$.

Let $f(r, \alpha)$ be a measurable real-valued function defined on E and continuous in α. The infinitesimal operator of the process $(r_t, \alpha_t), t \geq 0$, is defined by (see Shiryaev (1973), p. 19, for a general definition)

$$\mathcal{A}f(r, \alpha) = \lim_{t\to 0} \frac{E_{r,\alpha}[f(r_t, \alpha_t)] - f(r, \alpha)}{t} \tag{8}$$

provided this limit exists. Of particular interest is the infinitesimal operator at $\mathcal{L}(r, \alpha)$ where $\mathcal{L}(r, \alpha)$ is defined by (7).

In the sequel it will be assumed that the exponential-type processes have a quadratic function $V(\mu)$ given by (5). Contrary to appearances, this assumption is practically not restrictive. The class considered contains all known processes with stationary independent increments. It follows from the results of Morris (1982) for natural exponential families of distributions that there are exactly six processes with stationary independent increments and $V(\mu)$ quadratic, namely, the Poisson process ($\eta_2 = \eta_0 = 0, \eta_1 = 1$), negative binomial process ($\eta_2 = \eta_1 = 1, \eta_0 = 0$), gamma process ($\eta_2 = 1$, $\eta_1 = \eta_0 = 0$), Wiener process with linear drift ($\eta_2 = \eta_1 = 0$, $\eta_0 = 1$), generalized hyperbolic secant process ($\eta_2 = \eta_0 = 1$, $\eta_1 = 0$), and binomial process ($\eta_2 = -1$, $\eta_1 = 1$, $\eta_0 = 0$).

LEMMA 3. *The infinitesimal operator at* $\mathcal{L}(r,\alpha)$ *is*

$$\mathcal{A}\mathcal{L}(r,\alpha) = -\frac{1}{\alpha^2} + c_1 + c_2\frac{r+\eta_1}{\alpha-2\eta_2}, \tag{9}$$

for $\alpha > 2\eta_2$. *If* $c_2 = 0$, *then* $\mathcal{A}\mathcal{L}(r,\alpha) = -\alpha^{-2} + c_1$ *is defined for* $\alpha > \eta_2$.

Proof. Since $\mathcal{L}(r,\alpha) = \alpha^{-1} + c_1\alpha + c_2 r$, taking into account (6) yields

$$E_{r,\alpha}\mathcal{L}(r_t,\alpha_t) - \mathcal{L}(r,\alpha) = \frac{1}{\alpha+t} - \frac{1}{\alpha} + \left(c_1 + c_2\frac{r+\eta_1}{\alpha-2\eta_2}\right)t,$$

which, in view of (8), gives formula (9). ∎

Just as in Shapiro and Wardrop (1980a), the loss $\mathcal{L}$ is said to be in the *monotone case* if and only if $\mathcal{A}\mathcal{L}(r_s,\alpha_s) \ge 0$ for some $s \ge 0$, and $\mathcal{A}\mathcal{L}(r_s,\alpha_s) \ge 0$ implies $\mathcal{A}\mathcal{L}(r_t,\alpha_t) \ge 0$ for all $t > s$. This is a modified definition of the monotone case given by Chow, Robbins and Siegmund (1971) for discrete time problems and it is interpreted in a similar way. Namely, if $\mathcal{A}\mathcal{L}(r_t,\alpha_t) > 0$, then the "infinitesimal" prospect for the future (proceeding from state (r_t,α_t)) is bad since, in view of (8), the expected value of the incremental change in $\mathcal{L}$ is positive. If the loss is in the monotone case, then once the infinitesimal prospect becomes bad, it remains bad. Thus, if $\mathcal{L}$ is well behaved, the infinitesimal look-ahead procedure which stops the first time $\mathcal{A}\mathcal{L}(r_t,\alpha_t)$ is nonnegative, should be optimal.

For $\mathcal{L}$ in the monotone case, define the stopping time

$$\tau_{r,\alpha} = \inf\{t \ge 0 : \mathcal{A}\mathcal{L}(r_t,\alpha_t) \ge 0\}. \tag{10}$$

THEOREM 1. *If* $c_2 = 0$, *then the Bayes sequential procedure is the fixed time procedure*

$$\tau_{r,\alpha} = \tau_\alpha^0 = \max\{0, c_1^{-1/2} - \alpha\}$$

for $\alpha > \eta_2$.

Proof. If $c_2 = 0$ and $\alpha > \eta_2$, then by Lemma 3, $\mathcal{A}\mathcal{L}(r_t,\alpha_t) = -(\alpha+t)^{-2} + c_1$ is independent of r_t. Thus, $\mathcal{A}\mathcal{L}(r_t,\alpha_t) \ge 0$ if and only if $c_1(\alpha+t)^2 \ge 1$. It is then obvious that the loss $\mathcal{L}$ is in the monotone case and the Bayes sequential procedure reduces to the fixed time procedure determined by τ_α^0. ∎

For $c_2 > 0$ the process $Z(t), t \geq 0$, will be assumed to have nondecreasing sample functions. Note that η_1 is nonnegative (and equals 1 or 0) for all the processes considered (and not only the ones with nondecreasing sample functions).

LEMMA 4. *If $c_2 > 0$, then $\mathcal{L}$ is in the monotone case for all $r > 0$ and $\alpha > 4\eta_2$.*

P r o o f. Note that $\mathcal{AL}(r_t, \alpha_t) \geq 0$ if and only if

$$r + Z(t) + \eta_1 \geq \frac{\alpha + t - 2\eta_2}{c_2}\left[\frac{1}{(\alpha+t)^2} - c_1\right].$$

The left hand side is nondecreasing in t (by the assumption that $Z(t)$ has nondecreasing sample functions) and increasing in r. The right hand side is decreasing in both t and α, for t and α satisfying $(\alpha+t)[c_1(\alpha+t)^2+1] > 4\eta_2$, and consequently for $t \geq 0$ and $\alpha > 4\eta_2$. Hence the result follows. ■

In view of Lemma 4 and formula (10), if $c_1 \geq 0$ and $c_2 > 0$ the following stopping time will be considered:

$$\tau_{r,\alpha} = \inf\left\{t \geq 0 : r + Z(t) + \eta_1 \geq \frac{\alpha + t - 2\eta_2}{c_2}\left[\frac{1}{(\alpha+t)^2} - c_1\right]\right\} \tag{11}$$

for $r > 0$ and $\alpha > 4\eta_2$.

LEMMA 5. *If $c_1 > 0$ and $c_2 > 0$, then*

$$\tau_{r,\alpha} \leq c_1^{-1/2} \tag{12}$$

and

$$Z(\tau_{r,\alpha}) \leq \frac{\alpha - 2\eta_2}{c_2\alpha^2}. \tag{13}$$

If $c_1 = 0$, then the bound on $\tau_{r,\alpha}$ is infinite.

P r o o f. The form of the infinitesimal operator $\mathcal{AL}$ implies that any stopping time $\tau_{r,\alpha}$ with both costs c_1 and c_2 positive is bounded by the corresponding $\tau_{r,\alpha}$ with only one cost positive. Thus it suffices to bound the $\tau_{r,\alpha}$ and $Z(\tau_{r,\alpha})$ defined with one nonzero cost. Theorem 1 yields the bound on $\tau_{r,\alpha}$. Assuming $c_1 = 0, \mathcal{AL}(r_t, \alpha_t) < 0$ implies that

$$Z(t) < r + Z(t) + \eta_1 < \frac{\alpha + t - 2\eta_2}{c_2(\alpha+t)^2} \leq \frac{\alpha - 2\eta_2}{c_2\alpha^2}$$

for $r > 0$ and $\alpha > 4\eta_2$. This implies the bound on $Z(\tau_{r,\alpha})$. ■

Now (7), (12) and (13) yield the following corollary.

COROLLARY. *If $c_1 \geq 0$ and $c_2 > 0$, then*

$$E_{r,\alpha}\mathcal{L}(r_{\tau_{r,\alpha}}, \alpha_{\tau_{r,\alpha}}) \leq \frac{2(\alpha - \eta_2)}{\alpha^2} + c_1(\alpha + c_1^{-1/2}) + c_2 r.$$

THEOREM 2. *If $c_1 \geq 0$ and $c_2 > 0$, then the stopping time $\tau_{r,\alpha}$ defined by* (11) *is optimal*, *i.e.*, $E_{r,\alpha}\mathcal{L}(r_{\tau_{r,\alpha}}, \alpha_{\tau_{r,\alpha}}) \leq E_{r,\alpha}\mathcal{L}(r_\tau, \alpha_\tau)$ *for all stopping times* τ.

Proof. The proof is based on Dynkin's identity for the infinitesimal operator at $\mathcal{L}(r,\alpha)$ of the process $(r_t, \alpha_t), t \geq 0$. Suppose τ is a stopping time such that $E_{r,\alpha}\tau < \infty$. It then follows from Wald's identity that also $E_{r,\alpha}Z(\tau) < \infty$. Observe that if either $E_{r,\alpha}\tau = \infty$ or $E_{r,\alpha}Z(\tau) = \infty$, then the expected loss $E_{r,\alpha}\mathcal{L}(r_\tau, \alpha_\tau)$ is infinite. Moreover, note that the identity

$$E_{r,\alpha} f(r_\tau, \alpha_\tau) - f(r,\alpha) = E_{r,\alpha} \int_0^\tau \mathcal{A} f(r_t, \alpha_t)\, dt,$$

due to Dynkin, holds with $f(r,\alpha) = \mathcal{L}(r,\alpha)$, where $\mathcal{L}(r,\alpha)$ is the loss function considered. It follows from the Corollary that $E_{r,\alpha}\mathcal{L}(r_{\tau_{r,\alpha}}, \alpha_{\tau_{r,\alpha}}) < \infty$. Thus, just as in Shapiro and Wardrop (1980a), Dynkin's identity yields

$$\begin{aligned}
&E_{r,\alpha}\mathcal{L}(r_\tau, \alpha_\tau) - E_{r,\alpha}\mathcal{L}(r_{\tau_{r,\alpha}}, \alpha_{\tau_{r,\alpha}}) \\
&\quad = E_{r,\alpha} \int_0^\tau \mathcal{A}\mathcal{L}(r_t, \alpha_t)\, dt - E_{r,\alpha} \int_0^{\tau_{r,\alpha}} \mathcal{A}\mathcal{L}(r_t, \alpha_t)\, dt \\
&\quad = E_{r,\alpha} \mathbf{1}_{\{\tau \geq \tau_{r,\alpha}\}} \int_{\tau_{r,\alpha}}^{\tau} \mathcal{A}\mathcal{L}(r_t, \alpha_t)\, dt - E_{r,\alpha} \mathbf{1}_{\{\tau < \tau_{r,\alpha}\}} \int_{\tau}^{\tau_{r,\alpha}} \mathcal{A}\mathcal{L}(r_t, \alpha_t)\, dt,
\end{aligned}$$

which is nonnegative by definition of $\tau_{r,\alpha}$ and the monotone property of $\mathcal{L}$. ■

5. Bayes sequential procedures for exponential-type processes. Let $(Z(t), S(t))$, $t \geq 0$, be the exponential-type process defined in Section 2. It is well known (see, for example, Stefanov (1986)) that after the random time transformation $t_s = \inf\{t : S(t) \geq s\}$, $\widetilde{Z}(s) = Z(t_s)$, $s \geq 0$, the process $\widetilde{Z}(s)$, $s \geq 0$, is an exponential-type process with stationary independent increments.

In this section we exhibit Bayes sequential estimation procedures for those continuous time processes $(Z(t), S(t))$, $t \geq 0$, for which $V(\mu)$ has the quadratic form (5).

Define

$$\tau^*_{r,\alpha} = \begin{cases} \tau^*_\alpha = \inf\{t \geq 0 : S(t) \geq c^{-1/2} - \alpha\}, \qquad \alpha > \eta_2, \text{ if } c_2 = 0, \\ \inf\left\{t \geq 0 : r + Z(t) + \eta_1 \geq \dfrac{\alpha + S(t) - 2\eta_2}{c_2}\left[\dfrac{1}{(\alpha + S(t))^2} - c_1\right]\right\}, \\ \hfill r > 0,\ \alpha > 4\eta_2, \text{ if } c_2 > 0. \end{cases} \tag{14}$$

Since $S(t)$ is strictly increasing and continuous we do not lose any part of the sample functions of the original process after the random time transformation given above. Thus,

$$c_2[r + Z(t) + \eta_1] \geq [\alpha + S(t) - 2\eta_2][(\alpha + S(t))^{-2} - c_1]$$

if and only if

$$c_2[r + \widetilde{Z}(t) + \eta_1] \geq [\alpha + t - 2\eta_2][(\alpha + t)^{-2} - c_1].$$

This implies the equivalence of the optimal stopping problems we are interested in for both processes $(Z(t), S(t))$, $t \geq 0$, and $(\widetilde{Z}(t), t), t \geq 0$. Thus, the results of the previous section yield the following theorems.

THEOREM 3. *The following bounds hold*:

$$S(\tau^*_{r,\alpha}) \leq c_1^{-1/2}, \qquad Z(\tau^*_{r,\alpha}) \leq \frac{\alpha - 2\eta_2}{c_2\alpha^2},$$

where the bounds are infinite if the cost involved is zero, *and*

$$E_{r,\alpha}\mathcal{L}(r_{\tau^*_{r,\alpha}}, \alpha_{\tau^*_{r,\alpha}}) \leq \begin{cases} \alpha^{-1} + c_1(\alpha + c_1^{-1/2}) & \text{if } c_2 = 0; \\ 2(\alpha - \eta_2)\alpha^{-2} + c_1(\alpha + c_1^{-1/2}) + c_2 r & \text{if } c_2 > 0. \end{cases} \blacksquare$$

THEOREM 4. *The stopping time $\tau^*_{r,\alpha}$ is optimal.* ■

EXAMPLE (a family of counting processes). Let $X(t)$, $t \geq 0$, be a counting process and let $X(t) = M(t) + A(t)$ denote its Doob–Meyer decomposition, where $M(t)$ is the martingale part and $A(t)$ is the compensator. Assume that $A(t) = \mu B(t)$, where $\mu > 0$ and $B(t)$ is continuous. It is well known (Liptser and Shiryaev (1978)) that under certain conditions the likelihood function is given by

$$\frac{dP_{\vartheta,t}}{dQ_t} = \exp[\vartheta(X(t) - x_0) + \Phi(\vartheta)B(t)],$$

where $X(0) = x_0$, $\vartheta = \log \mu$ and $\Phi(\vartheta) = -\exp \vartheta$. In this case $V(\mu) = \mu$.

An example is obtained by taking $B(t) = \int_0^t H(s)\,ds$, where $H(t)$ is a positive, predictable stochastic process. In particular, $H(t) \equiv 1$ for the Poisson process, $H(t) = bt^{b-1}$ for the Weibull process (b being a known value), $H(t) = X(t-)$ for the pure birth process and $H(t) = X(t-)[M - X(t-)]^+$ for the logistic birth process, where M is a known constant.

According to Theorem 4, for $c_1 \geq 0$ and $c_2 > 0$, the Bayes sequential estimation procedure is $(\tau^*_{r,\alpha}, d(\tau^*_{r,\alpha}))$, where

$$\tau^*_{r,\alpha} = \inf\left\{t \geq 0 : r + X(t) - x_0 + 1 \geq \frac{\alpha + B(t)}{c_2}\left[\frac{1}{(\alpha + B(t))^2} - c_1\right]\right\}$$

for $r > 0$ and $\alpha > 0$, and

$$d(\tau^*_{r,\alpha}) = \frac{r + X(\tau^*_{r,\alpha}) - x_0}{\alpha + B(\tau^*_{r,\alpha})}.$$

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