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M. BERTRAND-RETALI and L. AIT-HENNANI (Rouen)

UNIFORM CONVERGENCE OF DENSITY ESTIMATORS ON SPHERES

Non-parametric estimation of a probability density for random variables taking values on an s-dimensional unit sphere is studied in [1], [5], [6]. The object of the present paper is to establish new uniform convergence theorems for several estimators: we use successively the histogram method, the spherical cap and the kernel methods. In part D, we present simulation results.

Let $\mathcal{D}$ be the set of continuous densities, defined on the sphere S; we estimate f, an element of $\mathcal{D}$, from a sample of size n, denoted by $X_1, \ldots, X_n$. The density f satisfies $\int_S f(x)\, d\mu(x) = 1$, where μ is the Lebesgue measure on S.

A. The histogram estimator. We are going to describe a partition of the sphere which will allow us to use the main theorem of [4].

This theorem establishes a necessary and sufficient condition for uniform convergence—in probability and almost completely—using the histogram estimator on a metric space, for every f in $\mathcal{D}$. To use it for S, it will be sufficient to construct a sequence $\Delta_{k(n)}$ of partitions $\Delta_k = \{\Delta_{k,r} : r \in R_k\}$, the Borel sets $\Delta_{k,r}$ being such that

$$\lim_{k\to\infty} \sup_{r\in R_k} (\operatorname{diam} \Delta_{k,r}) = 0, \qquad \lim_{k\to\infty} \sup_{r\in R_k} (\operatorname{area} \Delta_{k,r}) = 0,$$

$$\limsup_{k\to\infty} \frac{\sup_{r\in R_k} (\operatorname{area} \Delta_{k,r})}{\inf_{r\in R_k} (\operatorname{area} \Delta_{k,r})} < \infty.$$

We choose the integer $k(n)$ such that $\lim_{n\to\infty} k(n) = +\infty$. For $r \in R_k$, let $\nu_{n,r}$ be the number of X_i's belonging to $\Delta_{k,r}$.

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The *histogram estimator* $\widehat{f}_n$ is given by

$$\forall r \in R_k,\ \forall x \in \Delta_{k,r},\quad \widehat{f}_n(x) = \frac{\nu_{nr}}{n\mu(\Delta_{k,r})},$$

$\mu(\Delta_{k,r})$ denoting the area of $\Delta_{k,r}$. With these notations, the main theorem of [4] states that $\widehat{f}_n$ is uniformly convergent, in probability and almost completely, if and only if

$$[\inf_{r\in R_k} \mu(\Delta_{k,r})]^{-1} = o(n/\log n) \qquad \text{where } k = k(n).$$

First, we are going to construct the partition for $s = 3$. Then we shall explain it for any s.

1. *Partition for* $s = 3$. A parametric representation of S is

$$\begin{aligned} x_1 &= \cos\theta_1, && \theta_1 \in [0,\pi],\\ x_2 &= \sin\theta_1\cos\theta_2, &&\\ x_3 &= \sin\theta_1\sin\theta_2, && \theta_2 \in [0,2\pi[. \end{aligned}$$

The "poles" of S, corresponding to $\theta_1 = 0$ and $\theta_1 = \pi$, must belong to a unique element of the partition, so we define the Borel sets $\Delta_{k,r} = \Delta_{k,r_1,r_2}$ in the following manner:

$$\begin{aligned} \Delta_{k,0} &= [0, \arccos(1-1/k^2)[\times [0,2\pi[,\\ \Delta_{k,1,r_2} &= [\arccos(1-1/k^2), \arccos(1-2/k)[\times [(r_2-1)\pi/k, r_2\pi/k[\\ &\hspace{6cm} \text{for } r_2 = 1,\ldots,2k,\\ \Delta_{k,r_1,r_2} &= [\arccos(1-2(r_1-1)/k), \arccos(1-2r_1/k)[\times [(r_2-1)\pi/k, r_2\pi/k[\\ &\hspace{4cm} \text{for } r_1 = 2,\ldots,k-1;\ r_2 = 1,\ldots,2k,\\ \Delta_{k,k,r_2} &= [\arccos(-1+2/k), \arccos(-1+1/k^2)[\times [(r_2-1)\pi/k, r_2\pi/k[\\ &\hspace{6cm} \text{for } r_2 = 1,\ldots,2k,\\ \Delta_{k,k+1} &= [\arccos(-1+1/k^2), \pi] \times [0,2\pi[, \end{aligned}$$

these intervals being closed when necessary. Then we can easily see that, for each $\Delta_{k,r}$, $\mu(\Delta_{k,r})$ is equivalent to $2\pi/k^2$, and that there are $2k^2+2$ elements in the partition. The necessary and sufficient condition is then

$$k^2 = o(n/\log n).$$

2. *Construction for arbitrary* s. A parametric representation of S is: for $\theta_i \in [0,\pi]$ when $i = 1,\ldots,s-2$ and $\theta_{s-1} \in [0,2\pi[$,

$$\begin{aligned} x_1 &= \cos\theta_1,\\ x_i &= \prod_{j=1}^{i-1} \sin\theta_j \cos\theta_i, \qquad i = 2,\ldots,s-1, \end{aligned}$$

$$x_s = \prod_{j=1}^{s-1} \sin \theta_j.$$

In $\mathbb{R}^s$, the distance between two points M and M' belonging to the sphere, associated with $(\theta_i)_{i=1,\ldots,s-1}$ and $(\theta'_i)_{i=1,\ldots,s-1}$, is

$$d^2(M, M') = 4 \sum_{i=1}^{s-1} \prod_{j=1}^{i-1} \sin \theta_j \sin \theta'_j \sin^2 \frac{\theta_i - \theta'_i}{2}.$$

We notice that, for $i = 1, \ldots, s-2$, $\sin \theta_i = 0$ implies that $\theta_{i+1}, \ldots, \theta_{s-1}$ are arbitrary.

The area of a part $S' \subset S$ is

$$\mu(S') = \int_{S'} \prod_{i=1}^{s-1} \sin^{m_i} \theta_i \, d\theta_i \qquad \text{with } m_i = s-1-i;\ i = 1, \ldots, s-1.$$

For positive integers $q \geq 0$, define

$$I_q = \int_0^{\pi/2} \sin^{2q+1} \theta \, d\theta, \qquad J_q = \int_0^{\pi} \sin^{2q} \theta \, d\theta.$$

First, let us construct the elements which do not contain the poles—i.e. the points such that, for one index $i = 1, \ldots, s-2$, $\sin \theta_i = 0$. These elements can be written as

$$\Delta_{k,r} = \prod_{i=1}^{s-1} [\alpha_{r_i - 1}, \alpha_{r_i}[, \qquad r \in R'_k.$$

We choose the values α_{r_i}, $i = 1, \ldots, s-1$, in the following manner. Consider the integral

$$\int_{\alpha_{r_i-1}}^{\alpha_{r_i}} \sin^{m_i} \theta_i \, d\theta_i.$$

If $m_i = 2q_i + 1$ with $q_i \in \mathbb{N}$, then define

$$F_{q_i}(\alpha) = \int_0^{\alpha} \sin^{2q_i+1} \theta_i \, d\theta_i \qquad \text{for } \alpha \in [0, \pi].$$

Then $F_{q_i}(\alpha)$ is increasing from 0 to $F_{q_i}(\pi) = 2I_{q_i}$; we define α_{r_i} from

$$F_{q_i}(\alpha_{r_i}) = \frac{2r_i}{k} I_{q_i} \qquad \text{for } r_i = 1, \ldots, k.$$

Then

$$\int_{\alpha_{r_i-1}}^{\alpha_{r_i}} \sin^{2q_i+1} \theta_i \, d\theta_i = \frac{2}{k} I_{q_i}.$$

If $m_i = 2q_i$ with $q_i \in \mathbb{N}^*$, then define

$$G_{q_i}(\alpha) = \int_0^{\alpha} \sin^{2q_i} \theta_i \, d\theta_i \qquad \text{for } \alpha \in [0, \pi].$$

Then $G_{q_i}(\alpha)$ is increasing from 0 to J_{q_i}; we define α_{r_i} from

$$G_{q_i}(\alpha_{r_i}) = \frac{r_i}{k} J_{q_i} \qquad \text{for } r_i = 1, \ldots, k.$$

Then

$$\int_{\alpha_{r_i-1}}^{\alpha_{r_i}} \sin^{2q_i} \theta_i \, d\theta_i = \frac{1}{k} J_{q_i}.$$

For $m_i = 0$, i.e. $i = s-1$, we choose

$$[\alpha_{r_{s-1}-1}, \alpha_{r_{s-1}}[\, = [(r_{s-1}-1)\pi/k, r_{s-1}\pi/k[, \qquad r_{s-1} = 1, \ldots, 2k.$$

Using the values of I_{q_i} and J_{q_i}, we can easily see that for $r_i = 2, \ldots, k-1$; $i = 1, \ldots, s-2$; and $r_{s-1} = 1, \ldots, 2k$,

$$\mu(\Delta_{k,r}) = \frac{C(s)}{k^{s-1}},$$

where $C(s)$ is a constant; its value follows from the preceding formulations. The whole partition is constructed by generalization of the method explained for $s = 3$. When, for an index $i = 1, \ldots s-2$, $\sin\theta_i = 0$, the associated element of the partition satisfies: $\theta_{i+1}, \ldots, \theta_{s-2}$ are in $[0, \pi]$, and θ_{s-1} in $[0, 2\pi[$; the intervals for $\theta_1, \ldots, \theta_i$ are chosen to make the area of $\Delta_{k,r}$ equivalent to the preceding expression.

Example (for $s = 4$). For $r_1 = 2, \ldots, k-1; r_2 = 2, \ldots, k-1$; and $r_3 = 1, \ldots, 2k$,

$$\begin{aligned}\Delta_{k,r} = {} & [\alpha_{r_1-1}, \alpha_{r_1}[\\ & \times [\arccos(1 - 2(r_2-1)/k), \arccos(1 - 2r_2/k)[\times [(r_3-1)\pi/k, r_3\pi/k[, \\ & \mu(\Delta_{k,r}) = \pi^2/k^3,\end{aligned}$$

α_{r_1} being given from

$$\frac{1}{2}\alpha_{r_1} - \frac{1}{4}\sin^2 \alpha_{r_1} = \frac{r_1}{2k}\pi,$$

and

$$\begin{aligned}\Delta_{k,0} &= [0, (3\pi/4)^{1/3}/k[\times [0, \pi] \times [0, 2\pi[, \\ \Delta_{k,k+1} &= [\pi - (3\pi/4)^{1/3}/k, \pi] \times [0, \pi] \times [0, 2\pi[, \\ \Delta_{k,1,0} &= [(3\pi/4)^{1/3}/k, \alpha_1[\times [0, \sqrt{2}/k[\times [0, 2\pi[, \\ \Delta_{k,1,k+1} &= [(3\pi/4)^{1/3}/k, \alpha_1[\times [\pi - \sqrt{2}/k, \pi] \times [0, 2\pi[,\end{aligned}$$

$$\Delta_{k,1,1,r_3} = [(3\pi/4)^{1/3}/k, \alpha_1[\times [\sqrt{2}/k, \arccos(1-1/k)[$$
$$\times [(r_3-1)\pi/k, r_3\pi/k[, \qquad r_3 = 1, \ldots, 2k,$$

and so on.

The number of elements in the partition is

$$K_{n,4} = 2 + k(2k^2+2) = 2k^3 + 2k + 2.$$

Coming back to the general case, we have

$$K_{n,s} = 2k^{s-1} + 2\frac{k^{s-2}-1}{k-1}.$$

The necessary and sufficient condition is then

$$k^{s-1} = o(n/\log n).$$

B. The spherical cap estimator. For the sphere S in $\mathbb{R}^s$, the spherical cap estimator is defined as in [6].

With each $x \in S$, we associate the spherical cap with pole x and radius h_n, denoted by $B_{n,x}$; here h_n is a sequence of positive real numbers such that

$$\lim_{n\to\infty} h_n = 0.$$

The area of $B_{n,x}$ is

$$\mu(B_{n,x}) = C_s h_n^{s-1} + o(h_n^{s-1}), \qquad \text{where } C_s = \frac{2\pi^{(s-1)/2}}{(s-1)\Gamma((s-1)/2)}.$$

We estimate the density f in the following manner. Let $\nu_{n,x}$ be the number of X_i's belonging to $B_{n,x}$. Define

$$\forall x \in S, \quad \widetilde{f}_n(x) = \frac{\nu_{nx}}{nC_s h_n^{s-1}}.$$

We are going to prove the following theorem:

For each element $f \in \mathcal{D}$, $\widetilde{f}_n$ is uniformly convergent—in probability and almost completely—if and only if

$$h_n^{1-s} = o(n/\log n).$$

Proof of the "if" part. We suppose that

$$h_n^{1-s} = o(n/\log n),$$

and we are going to prove that, for every f in $\mathcal{D}$, $\widetilde{f}_n$ converges almost completely to f, uniformly on S.

Let x be an element of S, and $\widetilde{f}_n(x)$ the associated estimator. We choose $0x_1 = 0x$. Let $k_n = [1/h_n]$. Then

$$\frac{k_n}{k_n+1} < k_n h_n \le 1.$$

Now, k_n being chosen, we construct the partition as in part A; x belongs to $\Delta_{k_n,0}$, and the corresponding histogram estimator is

$$\widehat{f}_{n,k_n}(x) = \frac{\nu_{n,0}(k_n)}{n\mu(\Delta_{k_n,0})}, \qquad \text{where } \nu_{n,0}(k_n) \text{ is the number of } X_i\text{'s in } \Delta_{k_n,0}.$$

We do the same construction with the integer $k_n + 1$:

$$\widehat{f}_{n,k_n+1}(x) = \frac{\nu_{n,0}(k_n+1)}{n\mu(\Delta_{k_n+1,0})}.$$

Since $\Delta_{k_n,0}$ (resp. $\Delta_{k_n+1,0}$) is (by part A) the spherical cap of pole x and radius $1/k_n$ (resp. $1/(k_n+1)$), we can write

$$\frac{\nu_{n,0}(k_n+1)}{n\mu(\Delta_{k_n,0})} \le \widetilde{f}_n(x) \le \frac{\nu_{n,0}(k_n)}{n\mu(\Delta_{k_n+1,0})},$$

or

$$\frac{\mu(\Delta_{k_n+1,0})}{\mu(\Delta_{k_n,0})}\widehat{f}_{n,k_n+1}(x) \le \widetilde{f}_n(x) \le \frac{\mu(\Delta_{k_n,0})}{\mu(\Delta_{k_n+1,0})}\widehat{f}_{n,k_n}(x).$$

From the choices of h_n and k_n, we claim that $\widehat{f}_{n,k_n}$ and $\widehat{f}_{n,k_n+1}$ converge to f uniformly almost completely.

Choosing a positive η, we suppose that the events $\{d(\widehat{f}_{n,k_n}, f) < \eta\}$ and $\{d(\widehat{f}_{n,k_n+1}, f) < \eta\}$ are realized. For large n

$$-\eta + \left[\frac{\mu(\Delta_{k_n+1,0})}{\mu(\Delta_{k_n,0})} - 1\right] f(x) \le \widetilde{f}_n(x) - f(x) \le \left[\frac{\mu(\Delta_{k_n,0})}{\mu(\Delta_{k_n+1,0})} - 1\right] f(x) + 2\eta.$$

Let H be such that $f < H$. Then, for large n,

$$\left|\frac{\mu(\Delta_{k_n+1,0})}{\mu(\Delta_{k_n,0})} - 1\right| H < \eta.$$

Thus, for large n,

$$P[d(\widetilde{f}_n, f) > 3\eta] \le P[d(\widehat{f}_{n,k_n}, f) > \eta] + P[d(\widehat{f}_{n,k_n+1}, f) > \eta].$$

The choices of h_n and k_n imply the convergence of the series on the right-hand side.

The uniform and almost complete convergence of $\widetilde{f}_n$ to f follows immediately.

Proof of the "only if" part. We suppose that, for every f in $\mathcal{D}$, $\widetilde{f}_n$ converges to f uniformly in probability. First, we show $h_n^{1-s} = o(n)$.

We choose a coordinate system and we consider the spherical cap with radius $1/4$ and pole x ($\theta_1 = 0$); we choose f to be an element of $\mathcal{D}$ such that, on this cap, f is an arbitrary positive number α.

From this choice of f, and from the hypothesis, we get

$$\lim_{n\to\infty} P[\nu_{nx} = 0] = 0,$$

that is,

$$\lim_{n\to\infty}(1-\alpha C_s h_n^{s-1})^n = 0,$$

so that $\lim_{n\to\infty} n\log(1-\alpha C_s h_n^{s-1}) = -\infty$ and thus $h_n^{1-s} = o(n)$.

Now, we show that $h_n^{1-s} = o(n/\log n)$. Let β be fixed in $]0,\pi/2[$, and let S' be the part of S defined by $\beta \le \theta_i \le \pi-\beta$ for $i=1,\dots,s-2$, and $0 \le \theta_{s-1} < 2\pi$.

Let k_n be an integer to be defined later; we construct the corresponding partition (as in part A), and let $\{\Delta_{k_n,r} : r \in R'_{k_n}\}$ be the set of its elements included in S'.

For each $\Delta_{k_n,r}$, we define its center $\overline{x}_{k_n,r}$ as follows. For large n, $\Delta_{k_n,r}$ can be written as $\prod_{i=1}^{s-1}[\alpha_{r_i-1},\alpha_{r_i}[$ for every $r\in R'_{k_n}$. Then $\overline{x}_{k_n,r} = \alpha_{r_i-1/2}$, $i=1,\dots,s-1$, with

$$\int_0^{\alpha_{r_i-1/2}} \sin^{m_i}\theta_i\,d\theta_i = \begin{cases} \dfrac{2r_i-1}{k} I_{q_i} \\ \text{or} \\ \dfrac{2r_i-1}{2k} J_{q_i} \end{cases} \qquad \text{for } i=1,\dots,s-2,$$

and

$$\alpha_{r_{s-1}-1/2} = \frac{2r_{s-1}-1}{2k}\pi.$$

Consider the distance (in $\mathbb{R}^s$) from $\overline{x}_{k_n,r}$ to the boundary of $\Delta_{k_n,r}$. Using the expression for $d(M,M')$ (part A), we can easily see that there exists a positive constant $C(s,\beta)$ such that

$$\inf_{r\in R'_{k_n}} d(\overline{x}_{k_n,r},\ \text{boundary of } \Delta_{k_n,r}) \ge C(s,\beta)^{1/2}/k_n.$$

This implies that, for each r in R'_{k_n}, $\Delta_{k_n,r}$ contains the spherical cap with pole $\overline{x}_{k_n,r}$ and radius $C(s,\beta)^{1/2}/k_n$. Choose $k_n = [C(s,\beta)^{1/2}/h_n]$. Then, for each r in R'_{k_n}, $\Delta_{k_n,r}$ contains the spherical cap with pole $\overline{x}_{k_n,r}$ and radius h_n, i.e. $B_{n,\bar{x}_{k_n,r}}$.

Moreover, by definition of S', R'_{k_n} has $[C'(s,\beta)k_n^{s-1}]$ elements, where $C'(s,\beta)$ is a positive number depending only on s and β.

We choose f in $\mathcal{D}$ with $f=\alpha$ on S', α being an arbitrarily small positive number. From the hypothesis, $\widetilde{f}_n$ converges to f uniformly in probability, so

$$\lim_{n\to\infty} P[d(\widetilde{f}_n, f) > \alpha/2] = 0.$$

If one of the $\Delta_{k_n,r}$ included in S' contains no X_i, then neither does the cap $B_{n,\bar{x}_{k_n,r}}$ and $\widetilde{f}_n(\overline{x}_{k_n,r}) = 0$, so $d(\widetilde{f}_n, f) \ge \alpha$. The convergence hypothesis implies

$$\lim_{n\to\infty} P\Big[\bigcup_{r\in R'_{k_n}} \{\nu_{n,r}(k_n)=0\}\Big] = 0,$$

$\nu_{n,r}(k_n)$ being the number of X_i's belonging to $\Delta_{k_n,r}$. That is,

$$\lim_{n\to\infty} P\Big[\bigcap_{r\in R'_{k_n}} \{\nu_{n,r}(k_n) \geq 1\}\Big] = 1.$$

Here, we remind that two events A and B of positive probability are *in negative correlation* if

$$P(A|B) \leq P(A), \quad \text{that is,} \quad P(A\cap B) \leq P(A)P(B).$$

More generally, the events $A_1, \ldots, A_n$ of positive probability are *in negative correlation* if

$$\forall I \subset \{1,\ldots,n\}, \quad P\Big[\bigcap_{i\in I} A_i\Big] \leq \prod_{i\in I} P(A_i),$$

that is, the realization of one of the A_i diminishes the probability that the others are realized.

The events in the intersection several lines above are in negative correlation, thus

$$\lim_{n\to\infty} \prod_{r\in R'_{k_n}} P[\nu_{n,r}(k_n) \geq 1] = 1.$$

Then, remembering that $f = \alpha$ on S', we have

$$\lim_{n\to\infty} \prod_{r\in R'_{k_n}} [1 - (1 - \alpha\mu(\Delta_{k_n,r}))^n] = 1.$$

From part A, $\mu(\Delta_{k_n,r}) = C(s)/k_n^{s-1}$; taking the logarithm, we obtain, for large n,

$$\forall \alpha > 0, \quad 1 - \frac{n\alpha C(s)}{k_n^{s-2}\log[C'(s,\beta)k_n^{s-1}]} < 0,$$

thus

$$\lim_{n\to\infty} \frac{k_n^{s-1}\log[C'(s,\beta)k_n^{s-1}]}{n} = 0.$$

Using the definition of k_n from h_n, and $h_n^{1-s} = o(n)$, we obtain the desired result.

C. The kernel estimator. Let K be a positive function, defined on $\mathbb{R}^+$, such that

$$\int_0^\infty K(u)u^{(s-3)/2}\,du < \infty.$$

For this function K and for a sequence of positive numbers h_n with $\lim_{n\to\infty} h_n = 0$ the *kernel estimator* of f is

$$\widetilde{f}_n(x) = \frac{1}{nh_n^{s-1}C_{K,s}(h_n)} \sum_{i=1}^n K\bigg(\frac{1-\langle x, X_i\rangle}{h_n^2}\bigg),$$

where $\langle x, X_i\rangle$ is the scalar product and

$$C_{K,s}(h_n) = h_n^{1-s} \int_S K\left(\frac{1-\langle x,y\rangle}{h_n^2}\right) d\mu(y),$$

$d\mu(y)$ being the area element on S.

The constant $C_{K,s}(h_n)$ does not depend on x and can be written as

$$C_{K,s}(h_n) = \frac{2\pi^{(s-1)/2}}{\Gamma((s-1)/2)} \int_0^{2/h_n^2} (2u - u^2 h_n^2)^{(s-3)/2} K(u)\, du$$

with

$$\lim_{n\to\infty} C_{K,s}(h_n) = \frac{2\pi^{(s-1)/2}}{\Gamma((s-1)/2)} \int_0^{\infty} (2u)^{(s-3)/2} K(u)\, du.$$

Notice first that if we choose

$$K(u) = \mathbf{1}_{[0,1/2]}(u),$$

then

$$C_{K,s}(h_n) = \frac{2\pi^{(s-1)/2}}{\Gamma((s-1)/2)} \int_0^{1/2} (2u - u^2 h_n^2)^{(s-3)/2}\, du,$$

that is,

$$C_{K,s}(h_n) = h_n^{1-s} \frac{2\pi^{(s-1)/2}}{\Gamma((s-1)/2)} \int_0^{2\arcsin h_n/2} \sin^{s-2}\theta\, d\theta.$$

From part B, we see that $h_n^{s-1} C_{K,s}(h_n)$ is the area of the cap $B_{n,x}$, and thus the estimator $\widetilde{f}_n$ defined from that function K is the spherical cap estimator.

We are going to prove two uniform convergence theorems for the kernel estimator: a necessary condition for convergence in probability, and a sufficient condition for almost complete convergence. In the proofs, we will follow the method used in [3]. Thus, we do not give all the details; we just indicate how these methods can be adapted for S.

1. *Necessary condition for convergence*. The theorem is:

Suppose that

$$\lim_{y\to\infty} y \int_y^{\infty} K(u)(2u)^{(s-3)/2}\, du = 0.$$

Then, *for every* f *in* $\mathcal{D}$, *if* $\widetilde{f}_n$ *converges to* f *uniformly in probability*, *then* $h_n^{1-s} = o(n/\log n)$.

First, we show that

$$h_n^{1-s} = o(n).$$

As in [3], we suppose that this condition is not satisfied, and we show that, for an element f in $\mathcal{D}$, $\widetilde{f}_n$ does not converge in probability.

If h_n^{1-s} is not $o(n)$, there exists a positive α and an infinite subset N_1 of $\mathbb{N}$ such that

$$\forall n \in N_1, \quad h_n^{1-s} > \alpha n.$$

We define a parametric representation of S, and we choose f in $\mathcal{D}$ equal to α on C defined by

$$C = \{x \in S : 0 \le \theta_1 \le \pi/4;\ \theta_i \in [0,\pi],\ i = 1,\ldots,s-2;\ \theta_{s-1} \in [0,2\pi]\}.$$

Let H be an upper bound of f.

We choose a positive number M such that

$$\int_M^\infty K(u)(2u)^{(s-3)/2}\,du < \inf\left(\frac{1}{4},\frac{\alpha}{4H}\right)\int_0^\infty K(u)(2u)^{(s-3)/2}\,du.$$

Let

$$\varrho_n = h_n\sqrt{2M}$$

and let Q_n be the cap with pole ξ ($\theta_1 = 0$) and radius ϱ_n. Let H_n be the event: no one of the X_i's belongs to Q_n.

We get

$$P(H_n) = [1 - \alpha\mu(Q_n)]^n.$$

We use the hypothesis on h_n and the choice of ϱ_n to obtain

$$P(H_n) > e^{-2(2M)^{(s-1)/2}C_s} > 0 \qquad \text{for large } n \text{ in } N_1.$$

Let f^{H_n} be the density of X conditioned by H_n:

$$f^{H_n}(x) = \begin{cases} 0 & \text{on } Q_n,\\ \dfrac{f(x)}{1-\alpha C_s(2M)^{(s-1)/2}h_n^{s-1}} & \text{on } S - Q_n.\end{cases}$$

Then we bound the mean of $\widetilde{f}_n(\xi)$ conditioned by H_n; as in [3], we obtain

$$E[\widetilde{f}_n(\xi) \mid H_n] \le \frac{[2\pi^{(s-1)/2}/\Gamma((s-1)/2)]H}{(1-\alpha C_s\varrho_n^{s-1})C_{K,s}(h_n)}\int_{\varrho_n^2/(2h_n^2)}^{2/h_n^2} K(u)(2u-u^2h_n^2)^{(s-3)/2}\,du.$$

For large n, using $\varrho_n^2/(2h_n^2) = M$, we get

$$E[\widetilde{f}_n(\xi) \mid H_n] \le \frac{[2\pi^{(s-1)/2}/\Gamma((s-1)/2)]H}{(1-\alpha C_s\varrho_n^{s-1})C_{K,s}(h_n)}\int_M^\infty K(u)(2u)^{(s-3)/2}\,du,$$

and, from the definition of M,

$$E[\widetilde{f}_n(\xi) \mid H_n] \le \frac{\alpha}{4(1-\alpha C_s \varrho_n^{s-1})} \frac{2\pi^{(s-1)/2}/\Gamma((s-1)/2)] \int_0^\infty K(u)(2u)^{(s-3)/2}\,du}{C_{K,s}(h_n)}.$$

Remembering that $\lim_{n\to\infty} \varrho_n = 0$ and

$$\lim_{n\to\infty} C_{K,s}(h_n) = \frac{2\pi^{(s-1)/2}}{\Gamma((s-1)/2)} \int_0^\infty K(u)(2u)^{(s-3)/2}\,du$$

we obtain, for large n, $E[\widetilde{f}_n(\xi) \mid H_n] \le \frac{1}{4}\alpha(1+\varepsilon)$.

The proof is then as in [3], using the Markov inequality, and the fact that, for large n in N_1, $P(H_n)$ is strictly positive.

Now we show

$$h_n^{1-s} = o(n/\log n).$$

We suppose that $h_n^{1-s} = o(n)$, but that the condition $h_n^{1-s} = o(n/\log n)$ is not satisfied. Then there exists a positive β and an infinite subset N_1 of $\mathbb{N}$ such that

$$\forall n \in N_1, \quad h_n^{1-s} > \beta n/\log n.$$

Let α be a positive number, to be made precise further, and let us choose f:

$$f(x) = \begin{cases} f(\theta_1,\ldots,\theta_{s-1}) = \alpha & \text{on } C = [0,\pi/2]\times[0,\pi]^{s-3}\times[0,2\pi[, \\ a\sin\theta_1 + b & \text{on } [\pi/2, 2\pi/3]\times[0,\pi]^{s-3}\times[0,2\pi[, \\ H & \text{elsewhere.} \end{cases}$$

The constants a, b, H are well known from α, using the continuity condition, and $\int_S f\,d\mu = 1$. More precisely, we get

$$H = \frac{d_s - a_s\alpha}{b_s},$$

d_s, a_s, b_s being positive numbers, known from the choice of s.

We choose $\beta_0 = \beta/(12C_s)$, decreasing the value of β if necessary to get $\beta_0 < d_s/a_s$. Using the hypothesis on K:

$$\lim_{y\to\infty} y \int_y^\infty K(u)(2u)^{(s-3)/2}\,du = 0,$$

that is, $\forall\varepsilon > 0$, $\exists M_0$, $\forall M > M_0$,

$$M \int_M^\infty K(u)(2u)^{(s-3)/2}\,du < \varepsilon \int_0^\infty K(u)(2u)^{(s-3)/2}\,du,$$

we choose $\varepsilon = \inf(\beta_0 b_s/(4d_s), 1/4)$; then M_0 is known.

Next, we choose a positive M such that

$$M > \max(M_0, a_s\beta_0/d_s, \beta_0, 1)$$

and

$$\alpha = \frac{\beta_0}{M}.$$

Then H is known and

$$\frac{\alpha}{4H} = \frac{\beta_0 b_s}{4(d_s M - a_s \beta_0)};$$

thus,

$$\frac{\alpha}{4H} > \frac{\beta_0 b_s}{4M d_s},$$

and from the choices of ε and M,

$$\int_M^\infty K(u)(2u)^{(s-3)/2}\, du < \frac{\alpha}{4H} \int_0^\infty K(u)(2u)^{(s-3)/2}\, du.$$

We shall use this inequality at the end of the proof.

We choose the integer

$$k_n = \left[\frac{h_n^{-1}}{\sqrt{2M}(C_s 2^s)^{1/(s-1)}}\right]$$

and let

$$\varrho_n = \frac{k_n^{-1}}{(C_s 2^s)^{1/(s-1)}}.$$

Then, for large n, $2^s k_n^{s-1} > 1/(3MC_s)$.

For large n in N_1, we have $k_n^{s-1} > \beta' n/\log n$, where $\beta' = \beta/(3MC_s) = 4\alpha$. This inequality is valid if β is chosen small enough.

We make a partition of C, similar to the partition defining $\widehat{f}_n$ on S: without going into details, we simply note that we divide $[0, \pi/2]$ for θ_1 and the partition is associated with the integer $2k_n$.

Let K_n be the number of elements in this partition; K_n is equivalent to $2^s k_n^{s-1}$. For each element, the area is equivalent to $C_s \varrho_n^{s-1}$.

We obtain a similar result to Proposition 1 of [3]:

Let J_n be the exact number of $\Delta_{n,t}$, $t = 1, \ldots, K_n$, containing no element of the sample. Then for every $\varepsilon > 0$,

$$\lim_{n\to\infty} P[1 \le J_n \le \varepsilon K_n] = 1.$$

We can also state (cf. [3]):

Let j an integer in $\{1, \ldots, K_n\}$ and integers $t_1, \ldots, t_j$ be such that

$$1 \le t_1 < \ldots < t_j \le K_n.$$

Let $V_n(t_1, \ldots, t_j)$ be the event: each $\Delta_{n,t}$, $t = t_1, \ldots, t_j$, is empty, while each among the others contains at least a point of the sample; the hypothesis $h_n^{1-s} = o(n)$ implies $K_n = o(n)$. Let α' and α'' be the positive numbers defined in [3]; suppose n is so large that $K_n < \alpha' n$, and let ν be an integer such that $[\alpha' n] + 1 \le \nu \le \alpha'' n$; let ν_n be the number of X_i's belonging to C. Then the distribution of each X_i $(i = 1, \ldots, n)$ conditioned by the event

$$\mathcal{E}_n(\nu; t_1, \ldots, t_j) = \{\nu_n = \nu\} \cap V_n(t_1, \ldots, t_j)$$

admits the density

$$f^*(x) = \begin{cases} \dfrac{n-\nu}{n} \dfrac{f(x)}{1 - \alpha K_n C_s \varrho_n^{s-1}} & \text{if } x \in S - C, \\ \dfrac{\nu}{n\alpha} \dfrac{f(x)}{(K_n - j)\varrho_n^{s-1} C_s} & \text{if } x \in C - \bigcup_{r=1}^{j} \Delta_{n,t_r}, \\ 0 & \text{if } x \in \bigcup_{r=1}^{j} \Delta_{n,t_r}. \end{cases}$$

We now conclude as in [3]. Let

$$\psi(x) = E[\widetilde{f}_n(x) \mid \mathcal{E}_n(\nu; t_1, \ldots, t_j)].$$

Then

$$\begin{aligned} \psi(x) = {} & \frac{1}{h_n^{s-1} C_{K,s}(h_n)} \int_{S-C} K\left(\frac{1-(x,u)}{h_n^2}\right) \frac{n-\nu}{n(1-\alpha)} f(u)\, d\mu(u) \\ & + \frac{1}{h_n^{s-1} C_{K,s}(h_n)} \\ & \times \int_{C - \bigcup_{r=1}^{j} \Delta_{n,t_r}} K\left(\frac{1-\langle x,u\rangle}{h_n^2}\right) \frac{\nu}{n\alpha(K_n - j)\varrho_n^{s-1} C_s} f(u)\, d\mu(u). \end{aligned}$$

Let ε be in $]0,1[$, and suppose $1 \le j \le \varepsilon K_n$. Then, for large n,

$$(K_n - j)\varrho_n^{s-1} C_s > 1 - \varepsilon,$$

and we can bound

$$\begin{aligned} \psi(x) \le {} & \frac{1}{h_n^{s-1} C_{K,s}(h_n)} \int_{S-C} K\left(\frac{1-\langle x,u\rangle}{h_n^2}\right) \frac{1-\alpha'}{1-\alpha} f(u)\, d\mu(u) \\ & + \frac{1}{h_n^{s-1} C_{K,s}(h_n)} \int_{C - \bigcup_{r=1}^{j} \Delta_{n,t_r}} K\left(\frac{1-\langle x,u\rangle}{h_n^2}\right) \frac{\alpha''}{\alpha(1-\varepsilon)} f(u)\, d\mu(u). \end{aligned}$$

If α' and α'' are chosen such that

$$\frac{1-\alpha'}{1-\alpha} < 1 + 2\varepsilon \quad \text{and} \quad \frac{\alpha''}{\alpha(1-\varepsilon)} < 1 + 2\varepsilon$$

then

$$\psi(x) \le \int_{S - \bigcup_{r=1}^{j} \Delta_{n,t_r}} \frac{1+2\varepsilon}{h_n^{s-1} C_{K,s}(h_n)} K\left(\frac{1-\langle x,u\rangle}{h_n^2}\right) f(u)\, d\mu(u).$$

Let us choose $x = \xi$, corresponding to $\theta_1 = 0$, a pole of $\Delta_{n,t_1} = \Delta_{k_n,0}$. We obtain

$$\psi(\xi) \le \frac{1+2\varepsilon}{h_n^{s-1} C_{K,s}(h_n)} \int\limits_{S-\bigcup_{r=1}^{j} \Delta_{n,t_r}} K\left(\frac{1-\langle \xi, u\rangle}{h_n^2}\right) f(u)\, d\mu(u),$$

that is,

$$\psi(\xi) \le \frac{1+2\varepsilon}{h_n^{s-1} C_{K,s}(h_n)} \int\limits_{S-\bigcup_{r=1}^{j} \Delta_{n,t_r}} K\left(\frac{1-\cos\theta_1}{h_n^2}\right) f(\theta_1,\ldots,\theta_{s-1})\, d\mu(\theta).$$

Let D'' be the image of the integration domain under the change of variable $u = (1-\cos\theta_1)/h_n^2$. Then

$$\psi(\xi) \le \frac{1+2\varepsilon}{C_{K,s}(h_n)} \frac{2\pi^{(s-1)/2}}{\Gamma((s-1)/2)} (\sup f) \int\limits_{D''} K(u)(2u)^{(s-3)/2}\, du.$$

The image of the cap Δ_{n,t_1} has no common point with D'' and is the interval $[0, \varrho_n^2/(2h_n^2)]$. Thus

$$\psi(\xi) \le (1+2\varepsilon)(\sup f) \frac{2\pi^{(s-1)/2}}{\Gamma((s-1)/2) C_{K,s}(h_n)} \int\limits_{\varrho_n^2/(2h_n^2)}^{\infty} K(u)(2u)^{(s-3)/2}\, du.$$

Remembering that

$$\frac{\varrho_n^2}{2h_n^2} = \frac{1}{(C_s 2^s)^{2/(s-1)} 2h_n^2 k_n^2} \quad \text{and} \quad k_n^2 \le \frac{1}{2Mh_n^2 (C_s 2^s)^{2/(s-1)}}$$

we have $\varrho_n^2/2h_n^2 \ge M$ and

$$\psi(\xi) \le (1+2\varepsilon)(\sup f) \frac{2\pi^{(s-1)/2}}{\Gamma((s-1)/2) C_{K,s}(h_n)} \int\limits_{M}^{\infty} K(u)(2u)^{(s-3)/2}\, du.$$

Recall also that

$$\int\limits_{M}^{\infty} K(u)(2u)^{(s-3)/2}\, du < \inf\left(\frac{\alpha}{4H}, \frac{\alpha}{4M}\right) \int\limits_{0}^{\infty} K(u)(2u)^{(s-3)/2}\, du$$

and, from the definition of M,

$$\int\limits_{M}^{\infty} K(u)(2u)^{(s-3)/2}\, du < \inf\left(\frac{\alpha}{4H}, \frac{1}{4}\right) \int\limits_{0}^{\infty} K(u)(2u)^{(s-3)/2}\, du.$$

But $\sup f = \sup(\alpha, H)$ and thus

$$\psi(\xi) < \left(\frac{1}{2} + \varepsilon\right) \frac{[2\pi^{(s-1)/2}/\Gamma((s-1)/2)] \int_0^\infty K(u)(2u)^{(s-3)/2}\, du}{C_{K,s}(h_n)} \alpha$$

and for large n,

$$\psi(\xi) < \left(\frac{1}{2} + \varepsilon\right)\frac{\alpha}{1-\varepsilon'}.$$

Choosing $\varepsilon = \varepsilon' = 1/10$, for large n, we get $\psi(\xi) < \frac{2}{3}\alpha$, and the end of the proof is similar to [3].

2. *Sufficient condition for convergence*. In this part, too, we proceed as in [3].

We recall that a function defined on $\mathbb{R}^+$ is called π_m-*simple* if, for a fixed integer m, it is constant on each element of the partition π_m, where

$$\pi_m = \{I_{m,j} = [j/2^m, (j+1)/2^m[: j \in \mathbb{N}\}.$$

We suppose that K is chosen such that there exist two sequences φ_m^+ and φ_m^- of $\mathbb{R}^+$-integrable π_m-simple functions with

$$\varphi_m^- \le \varphi_{m+1}^- \le K \le \varphi_{m+1}^+ \le \varphi_m^+ \qquad \text{for large } m.$$

For instance, every function K of bounded variation in the neighboorhood of infinity satisfies this condition.

We suppose, moreover, that $u^{(s-1)/2}K(u)$ is decreasing for large u, and that $\int_0^\infty u^{(s-1)/2}K(u)\,du$ exists, with

$$\lim_{m\to\infty}\int_0^\infty u^{(s-3)/2}\varphi_m^+(u)\,du = \lim_{m\to\infty}\int_0^\infty u^{(s-3)/2}\varphi_m^-(u)\,du = \int_0^\infty u^{(s-3)/2}K(u)\,du.$$

We are going to prove the following theorem:

If K satisfies the above hypotheses and if $h_n^{1-s} = o(n/\log n)$, then for each element f of $\mathcal{D}$, $\widetilde{f}_n$ converges to f uniformly almost completely.

We set

$$\varphi_m^+ = \sum_{j=0}^\infty \alpha_{m_j}\mathbf{1}_{I_{m_j}}, \qquad \varphi_m^- = \sum_{j=0}^\infty \alpha'_{m_j}\mathbf{1}_{I_{m_j}}.$$

We can write

$$\frac{1}{nh_n^{s-1}C_{K,s}(h_n)}\sum_{i=1}^n\sum_{j=0}^\infty \alpha'_{m_j}\mathbf{1}_{I_{m_j}}\left(\frac{1-\langle x, X_i\rangle}{h_n^2}\right) \le \widetilde{f}_n(x) \le \frac{1}{nh_n^{s-1}C_{K,s}(h_n)}\sum_{i=1}^n\sum_{j=0}^\infty \alpha_{m_j}\mathbf{1}_{I_{m_j}}\left(\frac{1-\langle x, X_i\rangle}{h_n^2}\right).$$

Consider the event

$$\left\{ \mathbf{1}_{I_{m_j}}\left(\frac{1-\langle x, X_i\rangle}{h_n^2}\right) = 1 \right\},$$

that is,

$$\left\{ \frac{j}{2^m} \le \frac{1-\langle x, X_i\rangle}{h_n^2} < \frac{j+1}{2^m} \right\},$$

or

$$\{X_i \in B_{n,m,j+1,x} - B_{n,m,j,x} = C_{n,m,j,x}\},$$

where $B_{n,m,j,x}$ (resp. $B_{n,m,j+1,x}$) is the spherical cap with pole x and radius $a_n = (j/2^{m-1})^{1/2}h_n$ (resp. $b_n = ((j+1)/2^{m-1})^{1/2}h_n$). Let

$$\widetilde{f}_{n,m,j}(x) = \frac{\nu_{n,m,j,x}}{nC_s a_n^{s-1}} \quad \text{and} \quad \widetilde{f}_{n,m,j+1}(x) = \frac{\nu_{n,m,j+1,x}}{nC_s b_n^{s-1}}$$

be the spherical cap estimators corresponding to these two caps. When j and m are chosen, the hypothesis about h_n implies the uniform almost complete convergence of these two estimators. For the chosen j and m,

$$\begin{aligned}
&\frac{(2^{m-1})^{(s-1)/2}}{nh_n^{s-1}C_{K,s}(h_n)} \sum_{i=1}^{m} \mathbf{1}_{I_{m_j}}\left(\frac{1-\langle x, X_i\rangle}{h_n^2}\right) \\
&\qquad = \frac{(2^{m-1})^{(s-1)/2}}{nh_n^{s-1}C_{K,s}(h_n)}(\nu_{n,m,j+1,x} - \nu_{n,m,j,x}) \\
&\qquad = \frac{C_s}{C_{K,s}(h_n)}[(j+1)^{(s-1)/2}\widetilde{f}_{n,m,j+1}(x) - j^{(s-1)/2}\widetilde{f}_{n,m,j}(x)].
\end{aligned}$$

So the preceding bounds allow us to write

$$\begin{aligned}
&\frac{C_s}{C_{K,s}(h_n)} \sum_{j=0}^{\infty} \alpha'_{mj} \frac{1}{(2^{m-1})^{(s-1)/2}} \\
&\qquad\qquad \times [(j+1)^{(s-1)/2}\widetilde{f}_{n,m,j+1}(x) - j^{(s-1)/2}\widetilde{f}_{n,m,j}(x)] - f(x) \\
&\qquad \le \widetilde{f}_n(x) - f(x) \\
&\qquad \le \frac{C_s}{C_{K,s}(h_n)} \sum_{j=0}^{\infty} \alpha_{mj} \frac{1}{(2^{m-1})^{(s-1)/2}} \\
&\qquad\qquad \times [(j+1)^{(s-1)/2}\widetilde{f}_{n,m,j+1}(x) - j^{(s-1)/2}\widetilde{f}_{n,m,j}(x)] - f(x).
\end{aligned}$$

Consider, first, the upper bound of $\widetilde{f}_n(x) - f(x)$. We can write it as

$$\frac{C_s}{C_{K,s}(h_n)} \sum_{j=0}^{\infty} \frac{\alpha_{m,j}}{2^{(m-1)(s-1)/2}}$$
$$\times \{(j+1)^{(s-1)/2}[\widetilde{f}_{n,m,j+1}(x) - f(x)] - j^{(s-1)/2}[f_{n,m,j}(x) - f(x)]\}$$
$$+ f(x)\bigg\{\frac{C_s}{C_{K,s}(h_n)} \sum_{j=0}^{\infty} \frac{\alpha_{m,j}}{2^{(m-1)(s-1)/2}}[(j+1)^{(s-1)/2} - j^{(s-1)/2}] - 1\bigg\}.$$

Recall that

$$\lim_{n\to\infty} C_{K,s}(h_n) = \frac{2\pi^{(s-1)/2}}{\Gamma((s-1)/2)} \int_0^{\infty} (2u)^{(s-3)/2} K(u)\, du,$$
$$C_s = \frac{2\pi^{(s-1)/2}}{(s-1)\Gamma((s-1)/2)}.$$

Moreover,

$$(s-1) \int_0^{\infty} \varphi_m^{+}(u)(2u)^{(s-3)/2}\, du = \sum_{j=0}^{\infty} \frac{\alpha_{mj}}{2^{(m-1)(s-1)/2}}[(j+1)^{(s-1)/2} - j^{(s-1)/2}].$$

From the hypotheses about K, there exists an integer m_0 such that, for $m > m_0$,

$$\int_0^{\infty} \varphi_m^{+}(u)(2u)^{(s-3)/2}\, du < (1+\varepsilon) \int_0^{\infty} K(u)(2u)^{(s-3)/2}\, du.$$

Thus, for $n > n_0$ and $m > m_0$, the coefficient of $f(x)$ is smaller than an arbitrary positive number η.

Let us choose $m > m_0$. The hypotheses about $u^{(s-1)/2}K(u)$ imply that, for each $\varepsilon > 0$, there exists a finite subset J of $\mathbb{N}$ such that

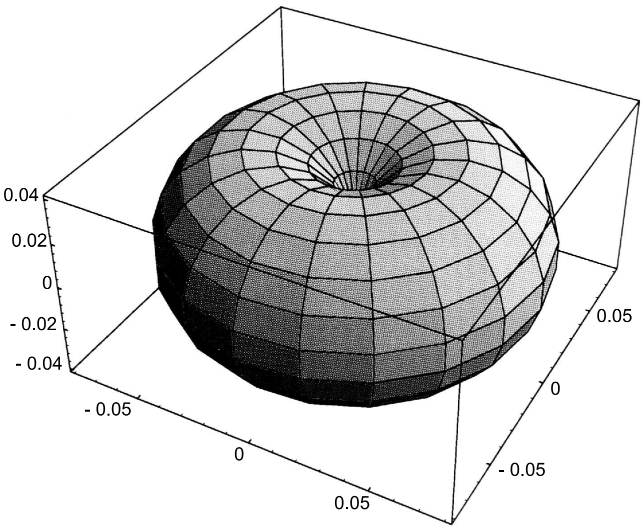
$$\sum_{j\notin J} \alpha_{mj}[j^{(s-1)/2} + (j+1)^{(s-1)/2}] < \varepsilon.$$

Let H be an upper bound for f. For $n > n_0$, $\widetilde{f}_n(x) - f(x)$ is smaller than

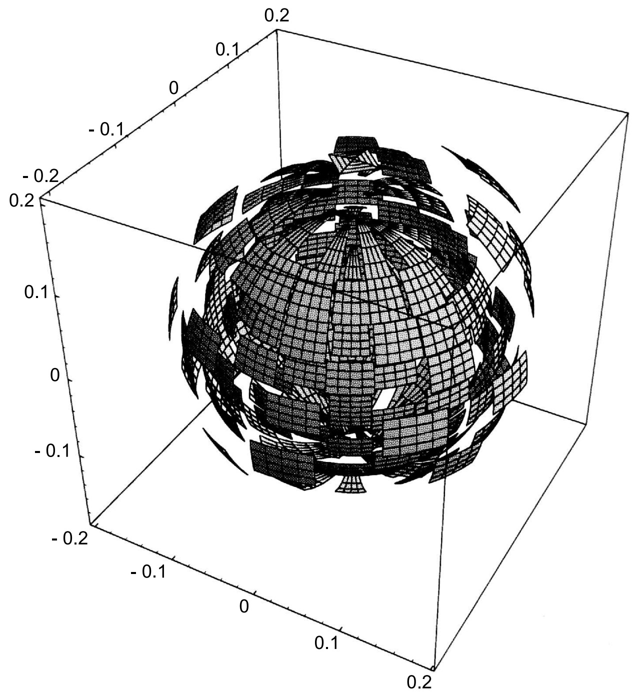
$$\frac{C_s}{C_{K,s}(h_n)} \sum_{j\in J} \frac{\alpha_{mj}}{2^{(m-1)(s-1)/2}}(j+1)^{(s-1)/2}|\widetilde{f}_{n,m,j+1}(x) - f(x)|$$
$$+\frac{C_s}{C_{K,s}(h_n)} \sum_{j\in J} \frac{\alpha_{mj}}{2^{(m-1)(s-1)/2}} j^{(s-1)/2}|\widetilde{f}_{n,m,j}(x) - f(x)| + 2H\varepsilon \frac{C_s}{C_{K,s}(h_n)} + H.$$

The end of proof is similar to [3].

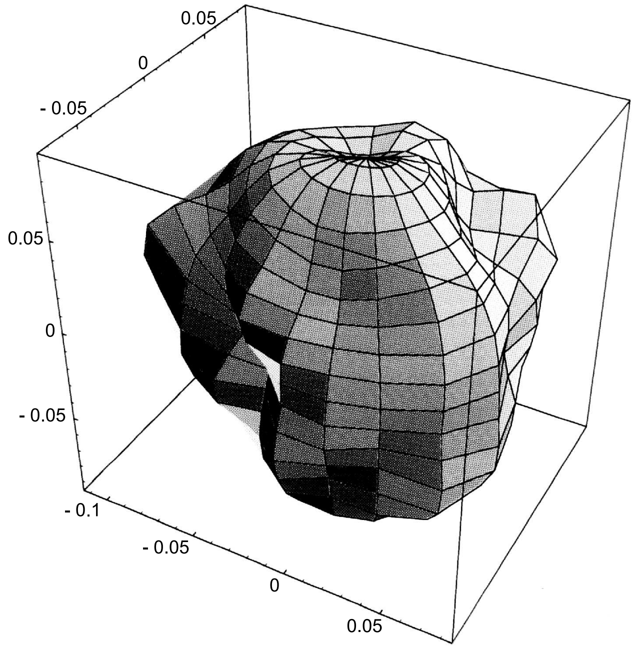
The lower bound for $\widetilde{f}_n(x) - f(x)$ is obtained analogously.



$$f(\theta_1, \theta_2) = \frac{1}{\pi^2} \sin \theta_1$$



The histogram estimator



The kernel estimator

D. Simulation results. We now study the performance of these estimators by simulation methods, for the density

$$f(\theta_1, \theta_2) = \frac{1}{\pi^2} \sin \theta_1$$

with $s = 3$.

The histogram estimate is calculated from a sample of size $n = 5000$, with $k = \sqrt{n}/\log n$.

The kernel estimate is calculated from a sample of size $n = 1000$, with $K(u) = \frac{1}{2}e^{-u}$ $(u \geq 0)$ and $h_n = (\log n)/\sqrt{n}$.

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MONIQUE BERTRAND-RETALI
LARBI AIT-HENNANI
UNIVERSITÉ DE ROUEN
UFR DES SCIENCES MATHÉMATIQUES
ANALYSE ET MODÈLES STOCHASTIQUES
URA CNRS 1378
76821 MONT SAINT AIGNAN CEDEX, FRANCE

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