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SPECTRAL DENSITY ESTIMATION FOR STATIONARY STABLE RANDOM FIELDS

Abstract. We consider a stationary symmetric stable bidimensional process with discrete time, having the spectral representation (1.1). We consider a general case where the spectral measure is assumed to be the sum of an absolutely continuous measure, a discrete measure of finite order and a finite number of absolutely continuous measures on several lines. We estimate the density of the absolutely continuous measure and the density on the lines.

1. Introduction. A complex random variable $X = X_1 + iX_2$ is *symmetric α-stable* (S.α.S) if its characteristic function is of the form

$$\exp\Big\{ -\int\limits_{S_2} |t_1x_1 + t_2x_2|^\alpha \, d\Gamma_{X_1,X_2}(x_1,x_2)\Big\},$$

where $t = t_1 + it_2$ and Γ_{X_1,X_2} is a symmetric measure on the unit sphere S_2 of $\mathbb{R}^2$.

A stochastic process $\{X_t, t \in T\}$ is (S.α.S) if all linear combinations, $a_1X_{t_1} + \ldots + a_nX_{t_n}$, are (S.$\alpha$.S). When $X = X_1 + iX_2$ and $Y = Y_1 + iY_2$ are jointly (S.α.S) and $1 < \alpha \le 2$, the *covariation* of X with Y is defined in [3] by

$$[X,Y]_\alpha = \int\limits_{S_4} (x_1 + ix_2)(y_1 + iy_2)^{\langle \alpha-1\rangle} \, d\Gamma_{X_1,X_2,Y_1,Y_2}(x_1,x_2,y_1,y_2),$$

where by convention, for $b > 0$ and Z a complex number, $Z^{\langle b\rangle} = |Z|^{b-1}Z^*$, Z^* being the complex conjugate of Z. The quantity $\|X\|_\alpha = [X,X]_\alpha^{1/\alpha}$ is a norm [11] on the linear space of (S.α.S) random variables. If $(\xi_t)_{t\in\mathbb{R}}$ is a complex (S.α.S) process with independent increments, then the measure μ

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defined by $\mu(]s,t]) = \|\xi_t - \xi_s\|_\alpha^\alpha$ is a Lebesgue–Stieltjes measure [3], called the *spectral measure* of ξ. When μ is absolutely continuous, its density is called the *spectral density* of ξ.

For stationary complex symmetric α-stable (S.α.S) processes having the spectral representation

$$X(t) = \int_{-\infty}^{\infty} e^{it\lambda}\, d\xi(\lambda),$$

where ξ is a (S.α.S) process with independent isotropic increments, the spectral density function ϕ is estimated in [11].

Let us consider a bidimensional (random field), complex (S.α.S) process in discrete time, having the spectral representation

$$X(n,m) = \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} e^{i(\lambda_1 n + \lambda_2 m)}\, d\xi(\lambda_1,\lambda_2), \tag{1.1}$$

where ξ is a (S.α.S) process with independent isotropic increments; that means ξ is an additive complex function defined on the Borel subsets of $[-\pi,\pi]^2$ such that:

• for any integer k and any Borel sets $B_1,\ldots,B_k$, $(\xi(B_1),\ldots,\xi(B_k))$ is (S.α.S);

• for any integer k and any disjoint Borel sets $B_1,\ldots,B_k$, $\xi(B_1),\ldots,\xi(B_k)$ are complex (S.α.S) independent random variables;

• for all Borel sets B, the distribution of the random variable $e^{i\theta}\xi(B)$ is independent of θ.

For more details about the spectral representation see [8] and [1].

We define the *spectral measure* by

$$[\xi(B),\xi(B)]_\alpha = \int_B d\mu(\lambda_1,\lambda_2) \qquad \text{for any Borel subset } B \text{ of } [-\pi,\pi]^2,$$

where $[X,Y]_\alpha$ denotes the covariation of X with Y [3] when X and Y are jointly (S.α.S).

To our knowledge the case we deal with is more general than those considered in [11], [6]: we suppose that the spectral measure of the process is the sum of an absolutely continuous measure, a discrete measure of finite order and a finite number of absolutely continuous measures on several lines:

$$d\mu(\lambda_1,\lambda_2) = \phi(\lambda_1,\lambda_2)d\lambda_1 d\lambda_2 + \sum_{j=1}^{q} a'_j \delta_{(w_{1j},w_{2j})} + \sum_{i=1}^{q'} \phi_i(u_1)\delta_{u_2 = a_i u_1 + b_i}, \tag{1.2}$$

where ϕ and ϕ_i for $i = 1, \dots, q'$ are nonnegative continuous functions, $a'_j \in \mathbb{R}^+$, $a_i, b_i \in \mathbb{R}$ and $w_{1j}, w_{2j} \in [-\pi, \pi]$, $j = 1, \dots, q$, $i = 1, \dots, q'$.

Using Jackson polynomial kernels and the double kernel method, we estimate the function $\phi(\lambda_1, \lambda_2)$ for every (λ_1, λ_2) in $]-\pi, \pi[^2$. Under appropriate conditions on the function ϕ, we obtain rates of convergence. Let us also indicate that our methods allow us to estimate the positive numbers a'_j $(j = 1, \dots, q)$ and the functions ϕ_i $(i = 1, \dots, q')$. For brevity, we consider here only the estimation of ϕ; for details, see [13].

2. The periodogram and its statistical properties. Given NM observations of the process $X : (X(n, m))_{\substack{0 \le n \le N-1 \\ 0 \le m \le M-1}}$, where N and M satisfy:

- $N - 1 = 2k(n - 1)$ with $n \in \mathbb{N}$, $k \in \mathbb{N} \cup \{1/2\}$; if $k = 1/2$ then $n = 2n_1 - 1$, $n_1 \in \mathbb{N}$.
- $M - 1 = 2k(m - 1)$ with $m \in \mathbb{N}$, $k \in \mathbb{N} \cup \{1/2\}$; if $k = 1/2$ then $m = 2m_1 - 1$, $m_1 \in \mathbb{N}$.

We consider the function

$$H^{(N)}(\lambda) = \sum_{m'=-k(n-1)}^{k(n-1)} h_k(m'/n) \cos(\lambda m'),$$

where h_k is defined in [7] such that

$$H^{(N)}(\lambda) = \frac{1}{q_{k,n}} \left(\frac{\sin \frac{n\lambda}{2}}{\sin \frac{\lambda}{2}} \right)^{2k} \quad \text{with} \quad q_{k,n} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\frac{\sin \frac{n\lambda}{2}}{\sin \frac{\lambda}{2}} \right)^{2k} d\lambda.$$

The *Jackson polynomial kernel* is $|H_N(\lambda)|^\alpha = |A_N H^{(N)}(\lambda)|^\alpha$, where $A_N = B_{\alpha,N}^{-1/\alpha}$ with $B_{\alpha,N} = \int_{-\pi}^{\pi} |H^{(N)}(\lambda)|^\alpha \, d\lambda$. We define $H_M(\lambda)$ in the same way.

The following lemma, proved in [6], will be frequently used in the sequel.

LEMMA 2.1 [6]. *Let*

$$B'_{\alpha,N} = \int_{-\pi}^{\pi} \left| \frac{\sin \frac{n\lambda}{2}}{\sin \frac{\lambda}{2}} \right|^{2k\alpha} d\lambda \quad \textit{and} \quad J_{N,\alpha} = \int_{-\pi}^{\pi} |u|^\gamma |H_N(\lambda)|^\alpha \, d\lambda,$$

where $\gamma \in \,]0, 2]$. *Then*

$$B'_{\alpha,N} \begin{cases} \ge 2\pi \left(\dfrac{2}{\pi} \right)^{2k\alpha} n^{2k\alpha - 1} & \textit{if } 0 < \alpha < 2, \\ \le \dfrac{4\pi k\alpha}{2k\alpha - 1} n^{2k\alpha - 1} & \textit{if } \dfrac{1}{2k} < \alpha < 2, \end{cases}$$

and

$$J_{N,\alpha} \le \begin{cases} \dfrac{\pi^{\gamma+2k\alpha}}{2^{2k\alpha}(\gamma-2k\alpha+1)}\cdot\dfrac{1}{n^{2k\alpha-1}} & \textit{if } \dfrac{1}{2k}<\alpha<\dfrac{\gamma+1}{2k}, \\ \dfrac{2k\alpha\pi^{\gamma+2k\alpha}}{2^{2k\alpha}(\gamma+1)(2k\alpha-\gamma-1)}\cdot\dfrac{1}{n^{\gamma}} & \textit{if } \dfrac{\gamma+1}{2k}<\alpha<2. \end{cases}$$

The proof of the following lemma is the same as in the one-dimensional case in [3].

LEMMA 2.2. *If ξ is a $(S.\alpha.S)$ process with independent and isotropic increments*, *then*

$$\mathbf{E}\exp\Big(i\,\mathrm{Re}\Big[\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(u_1,u_2)\,d\xi(u_1,u_2)\Big]\Big) = \exp\Big(-C_\alpha\int_{-\pi}^{\pi}|f(u_1,u_2)|^\alpha\,d\mu(u_1,u_2)\Big)$$

for every $f\in L_\alpha(\mu)$, where $C_\alpha=(2\pi\alpha)^{-1}\int_{-\pi}^{\pi}|\cos\theta|^\alpha\,d\theta$.

Now we denote by $\mathcal{A}$ the set of points where there are no atoms:

$$\mathcal{A}=\Big\{(\lambda_1,\lambda_2)\in\,]-\pi,\pi[^2\colon \lambda_1\ne w_{1j},\ \lambda_2\ne w_{2j} \text{ and } \frac{\lambda_2-b_i-a_i\lambda_1}{2\pi}\notin\mathbb{Z} \text{ for } 1\le i\le q' \text{ and } 1\le j\le q\Big\}.$$

In the sequel we choose large k such that $k\alpha>1$ and we consider the following periodogram:

$$d_{N,M}(\lambda_1,\lambda_2)=A_NA_M\,\mathrm{Re}\Big[\sum_{n'=-k(n-1)}^{k(n-1)}\sum_{m'=-k(m-1)}^{k(m-1)} h_k\Big(\frac{n'}{n}\Big)h_k\Big(\frac{m'}{m}\Big) \times e^{-i(\lambda_1n'+\lambda_2m')}X(n'+k(n-1),m'+k(m-1))\Big].$$

Following the study realized in the one-dimensional case by Masry and Cambanis in [11], one can show easily that, for $(\lambda_1,\lambda_2)\in\mathcal{A}$,

$$\lim_{N\to\infty}\mathbf{E}\{\exp[ird_{N,M}(\lambda_1,\lambda_2)]\}=\exp[-C_\alpha|r|^\alpha\phi(\lambda_1,\lambda_2)]. \tag{2.1}$$

We modify this periodogram taking the power p, with $0<p<\alpha/2$, and multiplying by a normalization constant:

$$\widehat{I}_{N,M}(\lambda_1,\lambda_2)=C_{p,\alpha}|d_{N,M}(\lambda_1,\lambda_2)|^p.$$

The normalization constant $C_{p,\alpha}$ is given by

$$C_{p,\alpha} = \frac{D_p}{F_{p,\alpha} C_\alpha^{p/\alpha}},$$

where

$$D_p = \int_{-\infty}^{\infty} \frac{1-\cos u}{|u|^{1+p}}\, du \quad \text{and} \quad F_{p,\alpha} = \int_{-\infty}^{\infty} \frac{1-e^{-|u|^\alpha}}{|u|^{1+p}}\, du, \quad 0 < p < \frac{\alpha}{2}.$$

We show that

$$\mathbf{E}\widehat{I}_{N,M}(\lambda_1,\lambda_2) = [\psi_{N,M}(\lambda_1,\lambda_2)]^{p/\alpha}, \tag{2.2}$$

$$\mathbf{Var}[\widehat{I}_{N,M}(\lambda_1,\lambda_2)] = V_{\alpha,p}[\psi_{N,M}(\lambda_1,\lambda_2)]^{2p/\alpha}, \tag{2.3}$$

where $V_{\alpha,p} = C_{p,\alpha}^2 C_{2p,\alpha}^{-1} - 1$ and

$$\psi_{N,M}(\lambda_1,\lambda_2) = I_{N,M}(\lambda_1,\lambda_2) + J_{N,M}(\lambda_1,\lambda_2) + K_{N,M}(\lambda_1,\lambda_2)$$

with

$$I_{N,M}(\lambda_1,\lambda_2) = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} |H_N(\lambda_1-u_1)|^\alpha |H_M(\lambda_2-u_2)|^\alpha \phi(u_1,u_2)\, du_1\, du_2,$$

$$J_{N,M}(\lambda_1,\lambda_2) = \sum_{j=1}^{q} a'_j |H_N(\lambda_1-w_{1j})|^\alpha |H_M(\lambda_2-w_{2j})|^\alpha,$$

$$K_{N,M}(\lambda_1,\lambda_2) = \sum_{i=1}^{q'} \int_{-\pi}^{\pi} |H_N(\lambda_1-v_1)|^\alpha |H_M(\lambda_2-a_iv_1-b_i)|^\alpha \phi_i(v_1)\, dv_1.$$

As in [11] one can show that for $(\lambda_1,\lambda_2) \in \mathcal{A}$, the function $\psi_{N,M}(\lambda_1,\lambda_2)$ converges to $\phi(\lambda_1,\lambda_2)$. Therefore $\widehat{I}_{N,M}(\lambda_1,\lambda_2)$ is an asymptotically unbiased estimator of $[\phi(\lambda_1,\lambda_2)]^{p/\alpha}$ but it is not consistent because the variance is proportional to $[\phi(\lambda_1,\lambda_2)]^{2p/\alpha}$.

3. The smoothed periodogram. In this section, using two spectral windows, we smooth the periodogram $\widehat{I}_{N,M}$ and we obtain consistent estimators of $[\phi(\lambda_1,\lambda_2)]^{p/\alpha}$ at the points (λ_1,λ_2) where there are no atoms. Let W be a nonnegative, even, continuous function vanishing for $|\lambda| > 1$, with $\int_{-1}^{1} W(\lambda)\, d\lambda = 1$. The spectral windows W_N, W_M are defined by

$$W_N(\lambda) = M_N W(M_N\lambda) \quad \text{and} \quad W_M(\lambda) = L_M W(L_M\lambda)$$

where M_N and L_M satisfy

$$\lim_{N\to\infty} M_N = \infty, \quad \lim_{N\to\infty} \frac{M_N}{N} = 0,$$
$$\lim_{M\to\infty} L_M = \infty, \quad \lim_{M\to\infty} \frac{L_M}{M} = 0.$$

We consider the following estimator:

$$f_{N,M}(\lambda_1,\lambda_2)=\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}W_N(\lambda_1-u_1)W_M(\lambda_2-u_2)\widehat{I}_{N,M}(u_1,u_2)\,du_1\,du_2.$$

As in [6], for giving the best rate of convergence of this estimator we introduce on ϕ two hypotheses (h_1) and (h_2) called regularity hypotheses:

$$(h_1)\qquad \phi(\lambda_1-u_1,\lambda_2-u_2)=\phi(\lambda_1,\lambda_2)+R_1(\lambda_1,\lambda_2,u_1,u_2)$$
$$\text{with } |R_1(\lambda_1,\lambda_2,u_1,u_2)|\le C_1\|(u_1,u_2)\|^{\gamma},\quad \text{where } 0<\gamma\le 1,$$

$$(h_2)\qquad \phi(\lambda_1-u_1,\lambda_2-u_2)=\phi(\lambda_1,\lambda_2)+\frac{\partial\phi}{\partial x}(\lambda_1,\lambda_2)u_1+\frac{\partial\phi}{\partial y}(\lambda_1,\lambda_2)u_2$$
$$+R_2(\lambda_1,\lambda_2,u_1,u_2)$$
$$\text{with } |R_2(\lambda_1,\lambda_2,u_1,u_2)|\le C_2\|(u_1,u_2)\|^{\gamma},\quad \text{where } 1\le\gamma<2,$$

C_1 and C_2 being nonnegative constants.

THEOREM 3.1. *Let $\lambda_1,\lambda_2\in\mathcal{A}$. Then*:

(i) $f_{N,M}(\lambda_1,\lambda_2)$ *is an asymptotically unbiased estimator of the quantity* $[\phi(\lambda_1,\lambda_2)]^{p/\alpha}$.

(ii) *Choosing k so large that $\gamma+1<2k\alpha$, we have*

$$\mathbf{E}[f_{N,M}(\lambda_1,\lambda_2)]-[\phi(\lambda_1,\lambda_2)]^{p/\alpha}
=\begin{cases} O\Big(\dfrac{1}{M_N^{\gamma}}+\dfrac{1}{(L_M)^{\gamma}}\Big) & \text{if } \phi \text{ satisfies } (h_1),\\[2mm] O\Big(\dfrac{1}{M_N}+\dfrac{1}{L_M}\Big) & \text{if } \phi \text{ satisfies } (h_2).\end{cases}$$

Proof. Using (2.2), we have

$$\mathbf{E}[f_{N,M}(\lambda_1,\lambda_2)]=\int_{M_N(\lambda_1-\pi)}^{M_N(\lambda_1+\pi)}\int_{L_M(\lambda_2-\pi)}^{L_M(\lambda_2+\pi)}W(u)W(v)$$
$$\times[\psi_{N,M}(\lambda_1-u/M_N,\lambda_2-v/L_M)]^{p/\alpha}\,du\,dv.$$

Since W is vanishing for $|\lambda|>1$ and $p/\alpha<1$, for N and M large enough we obtain

$$|\mathbf{E}[f_{N,M}(\lambda_1,\lambda_2)]-[\phi(\lambda_1,\lambda_2)]^{p/\alpha}|$$
$$\le\int_{-1}^{1}\int_{-1}^{1}W(u)W(v)$$
$$\times|\psi_{N,M}(\lambda_1-u/M_N,\lambda_2-v/L_M)-\phi(\lambda_1,\lambda_2)|^{p/\alpha}\,du\,dv.$$

Using the facts that H_N and H_M are 2π-periodic, $|H_N|^\alpha$ and $|H_M|^\alpha$ are two kernels, and ϕ is uniformly continuous on $[-\pi,\pi]^2$, we find that

$$\lim_{\substack{N\to\infty\\ M\to\infty}} I_{N,M}(\lambda_1-u/M_N,\lambda_2-v/L_M)-\phi(\lambda_1,\lambda_2)=0. \tag{3.1}$$

On the other hand,

$$\begin{aligned}&J_{N,M}(\lambda_1-u/M_N,\lambda_2-v/L_M)\\ &\le \sum_{j=1}^{q}\frac{a'_j}{B'_{\alpha,N}B'_{\alpha,M}}\\ &\quad\times\frac{1}{\big|\sin\big[\frac12(\lambda_1-u/M_N-w_{1j})\big]\big|^{2k\alpha}\big|\sin\big[\frac12(\lambda_2-v/L_M-w_{2j})\big]\big|^{2k\alpha}}.\end{aligned}$$

Since $\lambda_1\ne w_{1j}$ and $\lambda_2\ne w_{2j}$ for $j\in\{1,\dots,q\}$, using Lemma 2.1 we obtain

$$J_{N,M}(\lambda_1-u/M_N,\lambda_2-v/L_M)=O\bigg(\frac{1}{n^{2k\alpha-1}m^{2k\alpha-1}}\bigg). \tag{3.2}$$

Considering all possible cases, and partitioning the integrals, we show easily (see [13]) that

$$\begin{aligned}K_{N,M}&(\lambda_1-u/M_N,\lambda_2-v/L_M)\\ &=O\bigg(\frac{1}{m^{2k\alpha-1}}+\frac{1}{n^{2k\alpha-1}}+\frac{1}{n^{2k\alpha-1}m^{2k\alpha-1}}\bigg).\end{aligned} \tag{3.3}$$

Using the inequality

$$|x^r-y^r|\le\frac{r}{2}(x^{r-1}+y^{r-1})|x-y|,\qquad x,y\in\mathbb{R}^{+*},\ r\in[0,1]\cup[2,\infty[,$$

and the equalities (3.2), (3.3) we can show that

$$[\psi_{N,M}(\lambda_1-u/M_N,\lambda_2-v/L_M)]^{p/\alpha-1}+[\phi(\lambda_1,\lambda_2)]^{p/\alpha-1}$$

is bounded by a constant, for N and M large enough. We get

$$\begin{aligned}|\mathbf{E}[f_{N,M}(\lambda_1,\lambda_2)]-[\phi(\lambda_1,\lambda_2)]^{p/\alpha}|&\le \text{const}\int_{-1}^{1}\int_{-1}^{1}W(u)W(v)\\ &\quad\times|\psi_{N,M}(\lambda_1-u/M_N,\lambda_2-v/L_M)-\phi(\lambda_1,\lambda_2)|\,du\,dv.\end{aligned}$$

1) If ϕ satisfies the hypothesis (h_1) then we have

$$\begin{aligned}&|\psi_{N,M}(\lambda_1-u/M_N,\lambda_2-v/L_M)-\phi(\lambda_1,\lambda_2)|\\ &\le C_1\int_{\lambda_1-u/M_N-\pi}^{\lambda_1-u/M_N+\pi}\int_{\lambda_2-v/L_M-\pi}^{\lambda_2-v/L_M+\pi}|H_N(t)H_M(t')|^\alpha\\ &\quad\times(|u/M_N+t|+|v/L_M+t'|)^\gamma\,dt\,dt'\\ &\quad+J_{N,M}(\lambda_1-u/M_N,\lambda_2-v/L_M)+K_{N,M}(\lambda_1-u/M_N,\lambda_2-v/L_M).\end{aligned}$$

Using twice the inequality

$$|x+y|^r \le \begin{cases} |x|^r + |y|^r & \text{if } 0 < r \le 1, \\ 2^r(|x|^r + |y|^r) & \text{if } r \ge 1, \end{cases} \tag{3.4}$$

we show that

$$\begin{aligned}
\frac{1}{\mathrm{const}} &|\mathbf{E}[f_{N,M}(\lambda_1,\lambda_2)][\phi(\lambda_1,\lambda_2)]^{p/\alpha}| \\
&\le \frac{C_1}{M_N^\gamma} \int_{-1}^{1} W(u)|u|^\gamma\,du \\
&\quad + C_1 \int_{-1}^{1} W(u) \int_{\lambda_1-u/M_N-\pi}^{\lambda_1-u/M_N+\pi} |H_N(t)|^\alpha |t|^\gamma\,dt\,du \\
&\quad + \frac{C_1}{L_M^\gamma} \int_{-1}^{1} W(v)|v|^\gamma\,dv \\
&\quad + C_1 \int_{-1}^{1} W(v) \int_{\lambda_2-v/L_M-\pi}^{\lambda_2-v/L_M+\pi} |H_M(t')|^\alpha |t'|^\gamma\,dt'\,dv \\
&\quad + \int_{-1}^{1}\int_{-1}^{1} W(u)W(v) \\
&\quad \times [J_{N,M}(\lambda_1-u/M_N,\lambda_2-v/L_M) \\
&\quad + K_{N,M}(\lambda_1-u/M_N,\lambda_2-v/L_M)]\,du\,dv.
\end{aligned}$$

On the other hand, since H_N is an even function, we have

$$\begin{aligned}
&\int_{\lambda_1-u/M_N-\pi}^{\lambda_1-u/M_N+\pi} |H_N(t)|^\alpha |t|^\gamma\,dt \\
&\qquad \le \int_{-\pi}^{\pi} |H_N(t)|^\alpha |t|^\gamma\,dt + (2\pi)^\gamma \int_{\pi}^{|\lambda_1|+|u/M_N|+\pi} |H_N(t)|^\alpha |t|^\gamma\,dt.
\end{aligned}$$

Using Lemma 2.1, for N large enough we have

$$\int_{\pi}^{|\lambda_1|+|u/M_N|+\pi} |H_N(t)|^\alpha |t|^\gamma\,dt = O(T_N(\lambda_1)), \tag{3.5}$$

where

$$T_N(\lambda_1) = \begin{cases} \dfrac{1}{n^{2k\alpha-1}} & \text{if } \lambda_1 \ne 0, \\ \dfrac{1}{M_N n^{2k\alpha-1}} & \text{if } \lambda_1 = 0. \end{cases}$$

We gather the results (3.2), (3.3) and (3.5) to obtain

$$|\mathbf{E}[f_{N,M}(\lambda_1,\lambda_2)]-[\phi(\lambda_1,\lambda_2)]^{p/\alpha}| = O\bigg(T_N(\lambda_1)+T_M(\lambda_2)+\frac{1}{M_N^\gamma}+\frac{1}{L_M^\gamma} + \frac{1}{n^\gamma}+\frac{1}{m^\gamma} + \frac{1}{n^{2k\alpha-1}}+\frac{1}{m^{2k\alpha-1}}+\frac{1}{n^{2k\alpha-1}m^{2k\alpha-1}}\bigg).$$

Since $0<\gamma\le 1$ and $\gamma+1<2k\alpha$, it is clear that

$$|\mathbf{E}[f_{N,M}(\lambda_1,\lambda_2)]-[\phi(\lambda_1,\lambda_2)]^{p/\alpha}| = O\bigg(\frac{1}{M_N^\gamma}+\frac{1}{L_M^\gamma}\bigg).$$

2) If ϕ satisfies the hypothesis (h_2), using the facts that H_N, H_M are 2π-periodic kernels, and (3.5), (3.4) we get

$$\begin{aligned}
\frac{1}{\mathrm{const}}&|\mathbf{E}[f_{N,M}(\lambda_1,\lambda_2)]-[\phi(\lambda_1,\lambda_2)]^{p/\alpha}| \\
&\le \bigg|\frac{\partial\phi}{\partial x}(\lambda_1,\lambda_2)\bigg|\int_{-1}^{1}|u/M_N|W(u)\,du \\
&+\bigg|\frac{\partial\phi}{\partial y}(\lambda_1,\lambda_2)\bigg|T_M(\lambda_2) \\
&+\bigg|\frac{\partial\phi}{\partial x}(\lambda_1,\lambda_2)\bigg|T_N(\lambda_1)+\bigg|\frac{\partial\phi}{\partial y}(\lambda_1,\lambda_2)\bigg|\int_{-1}^{1}|v/L_M|W(v)\,dv \\
&+2^\gamma\frac{C_2}{M_N^\gamma}\int_{-1}^{1}|u|^\gamma W(u)\,du+2^\gamma\frac{C_2}{L_M^\gamma}\int_{-1}^{1}|v|^\gamma W(v)\,dv \\
&+2^\gamma C_2\int_{\lambda_1-u/M_N-\pi}^{\lambda_1-u/M_N+\pi}|H_N(t)|^\alpha|t|^\gamma\,dt \\
&+2^\gamma C_2\int_{\lambda_2-v/L_M-\pi}^{\lambda_2-v/L_M+\pi}|H_M(t')|^\alpha|t'|^\gamma\,dt' \\
&+\int_{-1}^{1}\int_{-1}^{1}W(u)W(v) \\
&\times[J_{N,M}(\lambda_1-u/M_N,\lambda_2-v/L_M) \\
&+K_{N,M}(\lambda_1-u/M_N,\lambda_2-v/L_M)]\,du\,dv.
\end{aligned}$$

We use again the results (3.2), (3.3) and (3.5) to obtain

$$
\begin{aligned}
&|\mathbf{E}[f_{N,M}(\lambda_1,\lambda_2)] - [\phi(\lambda_1,\lambda_2)]^{p/\alpha}| \\
&\qquad = O\bigg(\frac{1}{M_N} + \frac{1}{L_M} + \frac{1}{L_M^{\gamma}} + \frac{1}{M_N^{\gamma}} + T_N(\lambda_1) + T_M(\lambda_2) \\
&\qquad\qquad + \frac{1}{n^{\gamma}} + \frac{1}{m^{\gamma}} + \frac{1}{n^{2k\alpha-1}} + \frac{1}{m^{2k\alpha-1}} + \frac{1}{n^{2k\alpha-1}m^{2k\alpha-1}}\bigg).
\end{aligned}
$$

Since $1 < \gamma \le 2$, we have $1/k < (\gamma+1)/(2k) < \alpha$. Thus, we obtain

$$
|\mathbf{E}[f_{N,M}(\lambda_1,\lambda_2)] - [\phi(\lambda_1,\lambda_2)]^{p/\alpha}| = O\bigg(\frac{1}{M_N} + \frac{1}{L_M}\bigg).
$$

Now we show the following lemma which will be used in the sequel.

LEMMA 3.2. *Let* $(\lambda_{1,1},\lambda_{2,1})$ *and* $(\lambda_{1,2},\lambda_{2,2})$ *belong to* $\mathcal{A}$. *Write*

$$
\begin{aligned}
&Q_{N,M}(\lambda_{1,1},\lambda_{1,2},\lambda_{2,1},\lambda_{2,2}) \\
&\qquad = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} |H_N(\lambda_{1,1}-v_1)H_M(\lambda_{2,1}-v_2) \\
&\qquad\qquad \times H_N(\lambda_{1,2}-v_1)H_M(\lambda_{2,2}-v_2)|^{\alpha/2}\,d\mu(v_1,v_2).
\end{aligned}
$$

Then

$$
\begin{aligned}
&Q_{N,M}(\lambda_{1,1},\lambda_{1,2},\lambda_{2,1},\lambda_{2,2}) \\
&\quad \le O\bigg(\frac{1}{n^{2k\alpha-1}} + \frac{1}{m^{2k\alpha-1}} + \frac{1}{n^{2k\alpha-1}m^{2k\alpha-1}}\bigg) \\
&\qquad + \bigg(\frac{\sup(\phi)}{\pi^2}\Big(\frac{\pi}{2}\Big)^{4k\alpha}\frac{1}{(nm)^{2k\alpha-1}}\bigg)\Re(\delta,\lambda_{1,2},\lambda_{1,1})\Re(\delta',\lambda_{2,2},\lambda_{2,1}),
\end{aligned}
$$

where the function $\Re$ *is defined by*

$$
\Re(x,y,z) = \frac{\pi}{\big(\sin\frac{x}{2}\big)^{2k\alpha}} + \frac{2x}{\inf\big[\big(\sin\frac{|y-z|+x}{2}\big)^{k\alpha}, \big(\sin\frac{|y-z|}{4}\big)^{k\alpha}\big]},
$$

with two real numbers δ, δ' *satisfying*

$$
\begin{aligned}
&0 < \delta < \inf[\pi-\lambda_{1,2}; \pi+\lambda_{1,1}; |\lambda_{2,1}-\lambda_{1,1}|/2], \\
&0 < \delta' < \inf[\pi-\lambda_{2,2}; \pi+\lambda_{2,1}; |\lambda_{2,2}-\lambda_{2,1}|/2].
\end{aligned}
$$

Proof. First using the expression of $d\mu$ in (1.1) and the inequality $x^{\alpha/2}y^{\alpha/2} \le \frac{1}{2}(x^{\alpha}+y^{\alpha})$, we obtain

$$
\begin{aligned}
Q_{N,M}(\lambda_1,\lambda_2,x_1,x_2) &\le B + \tfrac{1}{2}J_{N,M}(\lambda_{1,1},\lambda_{2,1})\tfrac{1}{2}J_{N,M}(\lambda_{1,2},\lambda_{2,2}) \\
&\quad + \tfrac{1}{2}K_{N,M}(\lambda_{1,1},\lambda_{2,1}) + \tfrac{1}{2}K_{N,M}(\lambda_{1,2},\lambda_{2,2}),
\end{aligned}
$$

where

$$B = \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} |H_N(\lambda_{1,1} - v_1) H_M(\lambda_{2,1} - v_2)$$
$$\times H_N(\lambda_{1,2} - v_1) H_M(\lambda_{2,2} - v_2)|^{\alpha/2} \phi(v_1, v_2)\, dv_1\, dv_2.$$

Since ϕ is bounded on $[-\pi, \pi]^2$, we have

$$B \leq \sup(\phi) \Big(\int_{-\pi}^{\pi} |H_N(\lambda_{1,1} - v_1) H_N(\lambda_{1,2} - v_1)|^{\alpha/2}\, dv_1 \Big)$$
$$\times \Big(\int_{-\pi}^{\pi} |H_M(\lambda_{2,2} - v_2) H_M(\lambda_{2,1} - v_2)|^{\alpha/2}\, dv_2 \Big).$$

We split the first integral into five integrals: over a neighbourhood of $\lambda_{1,1}$, a neighbourhood of $\lambda_{1,2}$ and the remainders:

$$\int_{-\pi}^{\pi} |H_N(\lambda_{1,1} - v_1) H_N(\lambda_{1,2} - v_1)|^{\alpha/2}\, dv_1$$
$$= \frac{1}{B'_{\alpha,N}} \Big(\int_{-\pi}^{\lambda_{1,1}-\delta} + \int_{\lambda_{1,1}-\delta}^{\lambda_{1,1}+\delta} + \int_{\lambda_{1,1}+\delta}^{\lambda_{1,2}-\delta} + \int_{\lambda_{1,2}-\delta}^{\lambda_{1,2}+\delta}$$
$$+ \int_{\lambda_{1,2}+\delta}^{\pi} \Big) \left| \frac{\sin\left[\frac{n}{2}(\lambda_{1,1} - w)\right]}{\sin\frac{\lambda_{1,1}-w}{2}} \cdot \frac{\sin\left[\frac{n}{2}(\lambda_{1,2} - w)\right]}{\sin\frac{\lambda_{1,2}-w}{2}} \right|^{k\alpha} dw.$$

The first, third and last integrals are respectively bounded by

$$\frac{1}{B'_{\alpha,N}} \cdot \frac{\lambda_{1,1} - \delta + \pi}{\left(\sin\frac{\delta}{2}\right)^{2k\alpha}}, \qquad \frac{1}{B'_{\alpha,N}} \cdot \frac{\lambda_{1,2} - \lambda_{1,1} - 2\delta}{\left(\sin\frac{\delta}{2}\right)^{2k\alpha}}$$

and

$$\frac{1}{B'_{\alpha,N}} \cdot \frac{\pi - \lambda_{1,2} - \delta}{\left(\sin\frac{\delta}{2}\right)^{2k\alpha}}.$$

Using the inequality $\sin(nx) \leq n \sin x$, we obtain

$$\frac{1}{B'_{\alpha,N}} \int_{\lambda_{1,1}-\delta}^{\lambda_{1,1}+\delta} \left| \frac{\sin\left[\frac{n}{2}(\lambda_{1,1} - w)\right]}{\sin\frac{\lambda_{1,1}-w}{2}} \; \frac{\sin\left[\frac{n}{2}(\lambda_{1,2} - w)\right]}{\sin\frac{\lambda_{1,2}-w}{2}} \right|^{k\alpha} dw$$
$$\leq \frac{1}{B'_{\alpha,N}} \cdot \frac{2\delta n^{k\alpha}}{\inf\left[\left(\sin\frac{|\lambda_{1,2}-\lambda_{1,1}|+\delta}{2}\right)^{k\alpha}, \left(\sin\frac{|\lambda_{1,2}-\lambda_{1,1}|}{4}\right)^{k\alpha}\right]}.$$

Similarly, the remaining integral is bounded by the same quantity. Thus using Lemma 2.1, we have

$$\int_{-\pi}^{\pi} |H_N(\lambda_{1,1}-v_1)H_N(\lambda_{1,2}-v_1)|^{\alpha/2}\,dv_1$$

$$= \frac{1}{2\pi}\left(\frac{\pi}{2}\right)^{2k\alpha}\frac{1}{n^{2k\alpha-1}}$$

$$\times\left[\frac{2\pi}{\left(\sin\frac{\delta}{2}\right)^{2k\alpha}} + \frac{4\delta n^{k\alpha}}{\inf\left[\left(\sin\frac{\lambda_{1,2}-\lambda_{1,1}+\delta}{2}\right)^{k\alpha}, \sin\left(\frac{\lambda_{1,2}-\lambda_{1,1}}{4}\right)^{k\alpha}\right]}\right].$$

Just as before, we have

$$\int_{-\pi}^{\pi} |H_M(\lambda_{2,1}-v_2)H_M(\lambda_{2,2}-v_2)|^{\alpha/2}\,dv_2$$

$$\le \frac{1}{\pi}\left(\frac{\pi}{2}\right)^{2k\alpha}\frac{1}{m^{2k\alpha-1}}\Re(\delta',\lambda_{2,2},\lambda_{2,1}).$$

Equalities (3.2) and (3.3) give the result of this lemma.

THEOREM 3.3. *Let* $(\lambda_1,\lambda_2)\in\mathcal{A}$ *be such that* $\phi(\lambda_1,\lambda_2)\ne 0$. *Then*:

(i) $\mathbf{Var}[f_{N,M}(\lambda_1,\lambda_2)]$ *converges to zero.*

(ii) *If* ϕ *satisfies* (h_1) *or* (h_2), *and* $M_N = n^c$ *and* $L_M = m^{c'}$, *where* c *and* c' *are two real numbers satisfying*

$$\inf\left(\frac{2k^2\alpha^2+1}{6\alpha^2k^2}, \frac{k\alpha+2}{3(k\alpha+1)}\right) < c, c' < \frac{1}{2},$$

then

$$\mathbf{Var}[f_{N,M}(\lambda_1,\lambda_2)] = O\left(\frac{1}{n^{2(1-2c)}} + \frac{1}{m^{2(1-2c')}}\right).$$

Proof. When $\phi(\lambda_1,\lambda_2)=0$ we do not need to smooth $\widehat{I}_{N,M}$, since its variance tends to zero. We have

$$\mathbf{Var}[f_{N,M}(\lambda_1,\lambda_2)]$$

$$= \int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} W_N(\lambda_1-u_1)W_M(\lambda_2-u_2)W_N(\lambda_1-u_1')W_M(\lambda_2-u_2')$$

$$\times\,\mathbf{Cov}[\widehat{I}_{N,M}(u_1,u_2),\widehat{I}_{N,M}(u_1',u_2')]\,du_1\,du_2\,du_1'\,du_2'.$$

Putting

$$x_1 = M_N(\lambda_1-u_1),\qquad x_2 = L_M(\lambda_2-u_2),$$
$$x_1' = M_N(\lambda_1-u_1'),\qquad x_2' = L_M(\lambda_2-u_2').$$

and using the fact that W is vanishing for $|\lambda|>1$, for N and M large enough we have

$$\begin{aligned}
&\mathbf{Var}[f_{N,M}(\lambda_1,\lambda_2)] \\
&\qquad = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} W(x_1)W(x_2)W(x_1')W(x_2') \\
&\qquad\qquad \times \mathbf{Cov}[\widehat{I}_{N,M}(\lambda_1 - x_1/M_N, \lambda_2 - x_2/L_M), \\
&\qquad\qquad\qquad \widehat{I}_{N,M}(\lambda_1 - x_1'/M_N, \lambda_2 - x_2'/L_M)]\,dx_1\,dx_2\,dx_1'\,dx_2'.
\end{aligned}$$

We define

$$\begin{aligned}
&\mathcal{L}_1 = \{(x_1,x_1') \in [-1,1]^2 : |x_1 - x_1'| > \sigma_N\},\\
&\mathcal{L}_2 = \{(x_2,x_2') \in [-1,1]^2 : |x_2 - x_2'| > \sigma_M'\},\\
&\mathcal{L}_3 = \{(x_1,x_1',x_2,x_2') \in [-1,1]^4 : |x_1 - x_1'| \le \sigma_N \text{ or } |x_2 - x_2'| \le \sigma_M'\},
\end{aligned}$$

where σ_N, σ_M' are two nonnegative real sequences converging to 0. We split the last integral into the integral over $\mathcal{L}_3$ and the integral over $\mathcal{L}_1$ and $\mathcal{L}_2$:

$$\mathbf{Var}[f_{N,M}(\lambda_1,\lambda_2)] = \underset{\mathcal{L}_3}{\int\!\!\int\!\!\int\!\!\int} + \underset{\mathcal{L}_1}{\int\!\!\int}\ \underset{\mathcal{L}_2}{\int\!\!\int} =: J_1 + J_2.$$

Using (2.3) and the uniform (in x_1, x_2) convergence of $\psi_{N,M}(\lambda_1 - x_1/M_N, \lambda_2 - x_2/L_M)$ to $\phi(\lambda_1,\lambda_2)$, we obtain

$$\begin{aligned}
J_1 \le \text{const}\Big[&\underset{|x_2 - x_2'| \le \sigma_M'}{\int\!\!\int} W(x_2)W(x_2')\,dx_2\,dx_2' \\
&\qquad\qquad + \underset{|x_1 - x_1'| \le \sigma_N}{\int\!\!\int} W(x_1)W(x_1')\,dx_1\,dx_1'\Big].
\end{aligned}$$

Thus, $J_1 \le \text{const}[\sup(W)]^2[\sigma_N + \sigma_M']$. Hence J_1 tends to zero.

It remains to show that J_2 tends to zero. First we define for simplicity

$$\begin{aligned}
\lambda_{1,1} &= \lambda_1 - x_1/M_N, &\quad \lambda_{1,2} &= \lambda_1 - x_1'/M_N,\\
\lambda_{2,1} &= \lambda_2 - x_2/L_M, &\quad \lambda_{2,2} &= \lambda_2 - x_2'/L_M,
\end{aligned}$$

and

$$\begin{aligned}
&C(\lambda_1,\lambda_2)\\
&= \mathbf{Cov}[\widehat{I}_{N,M}(\lambda_1 - x_1/M_N, \lambda_2 - x_2/L_M), \widehat{I}_{N,M}(\lambda_1 - x_1'/M_N, \lambda_2 - x_2'/L_M)].
\end{aligned}$$

We use the equality

$$|x|^p = D_p^{-1} \int_{-\infty}^{\infty} \frac{1 - \cos(xu)}{|u|^{1+p}}\,du = D_p^{-1}\,\mathrm{Re} \int_{-\infty}^{\infty} \frac{1 - e^{ixu}}{|u|^{1+p}}\,du,$$

for all real x and $p \in]0,2[$. We have

$$\widehat{I}_{N,M}(\lambda_1,\lambda_2) = \frac{1}{F_{p,\alpha} C_\alpha^{p/\alpha}}\,\mathrm{Re} \int_{-\infty}^{\infty} \frac{1 - e^{iud_{N,M}(\lambda_1,\lambda_2)}}{|u|^{1+p}}\,du. \tag{3.6}$$

From (2.1) we obtain

$$\mathbf{E}\widehat{I}_{N,M}(\lambda_1,\lambda_2)=\frac{1}{F_{p,\alpha}C_\alpha^{p/\alpha}}\int_{-\infty}^{\infty}\frac{1-\exp\{-C_\alpha|u|^\alpha\psi_{N,M}(\lambda_1,\lambda_2)\}}{|u|^{1+p}}\,du. \tag{3.7}$$

By using (3.6) and (3.7) we show easily that

$$\begin{aligned}C(\lambda_1,\lambda_2)=F_{p,\alpha}^{-2}C_\alpha^{-2p/\alpha}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Big(\mathbf{E}\Big[\prod_{k=1}^{2}\cos(u_k d_{N,M}(\lambda_{1,k},\lambda_{2,k}))\Big]\\ -\exp\Big\{-C_\alpha\sum_{k=1}^{2}|u_k|^\alpha\psi_{N,M}(\lambda_{1,k},\lambda_{2,k})\Big\}\Big)\frac{du_1\,du_2}{|u_1u_2|^{1+p}}.\end{aligned}$$

From the equality $2\cos x\cos y=\cos(x+y)+\cos(x-y)$, we have

$$\begin{aligned}&\mathbf{E}\Big[\prod_{k=1}^{2}u_k d_{N,M}(\lambda_{1,k},\lambda_{2,k})\Big]\\ &\quad=\frac{1}{2}\exp\Big[-C_\alpha\int\Big|\sum_{k=1}^{2}u_kH_N(\lambda_{1,k}-v_1)H_M(\lambda_{2,k}-v_2)\Big|^\alpha d\mu(v_1,v_2)\Big]\\ &\qquad+\frac{1}{2}\exp\Big[-C_\alpha\int\Big|\sum_{k=1}^{2}(-1)^{k-1}u_kH_N(\lambda_{1,k}-v_1)\\ &\qquad\qquad\times H_M(\lambda_{2,k}-v_2)\Big|^\alpha d\mu(v_1,v_2)\Big].\end{aligned}$$

Substituting this expression in $C(\lambda_1,\lambda_2)$ and changing the variable u_2 to $-u_2$ in the second term, we obtain

$$C(\lambda_1,\lambda_2)=F_{p,\alpha}^{-2}C_\alpha^{-2p/\alpha}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(e^{-K}-e^{-K'})\frac{du_1\,du_2}{|u_1u_2|^{1+p}},$$

where

$$\begin{aligned}K&=C_\alpha\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\Big|\sum_{k=1}^{2}u_kH_N(\lambda_{1,k}-v_1)H_M(\lambda_{2,k}-v_2)\Big|^\alpha d\mu(v_1,v_2),\\ K'&=C_\alpha\sum_{k=1}^{2}|u_k|^\alpha\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}|H_N(\lambda_{1,k}-v_1)H_M(\lambda_{2,k}-v_2)|^\alpha d\mu(v_1,v_2)\\ &=C_\alpha\sum_{k=1}^{2}|u_k|^\alpha\psi_{N,M}(\lambda_{1,k},\lambda_{2,k}).\end{aligned}$$

Since $K, K' > 0$, we have

$$|e^{-K} - e^{-K'}| \le |K - K'| e^{|K-K'|-K'}.$$

Using the inequality

$$\||x+y|^\alpha - |x|^\alpha - |y|^\alpha| \le 2|xy|^{\alpha/2}, \qquad x, y \in \mathbb{R} \text{ and } 1 \le \alpha \le 2,$$

we obtain

$$|K - K'| \le 2C_\alpha |u_1 u_2|^{\alpha/2} Q_{N,M}(\lambda_{1,1}, \lambda_{1,2}, \lambda_{2,1}, \lambda_{2,2}).$$

Therefore,

$$C(\lambda_1, \lambda_2)$$
$$\le F_{p,\alpha}^{-2} C_\alpha^{-2p/\alpha} 2C_\alpha Q_{N,M}(\lambda_{1,1}, \lambda_{1,2}, \lambda_{2,1}, \lambda_{2,2}) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{e^{|K-K'|-K'}}{|u_1 u_2|^{1+p-\alpha/2}} \, du_1 \, du_2.$$

Now we have

$$(3.8) \quad |K - K'| - K' \le -C_\alpha \sum_{k=1}^{2} |u_k|^\alpha [\psi_{N,M}(\lambda_{1,k}, \lambda_{2,k}) - Q_{N,M}(\lambda_{1,1}, \lambda_{1,2}, \lambda_{2,1}, \lambda_{2,2})].$$

Let $\delta(N)$, $\delta'(M)$ be two real numbers depending respectively on N and M such that

$$0 < \delta(N) < \inf[|\lambda_{1,1} - \lambda_{1,2}|/2; \pi - \lambda_{1,2}; \pi + \lambda_{1,1}],$$
$$0 < \delta'(M) < \inf[|\lambda_{2,1} - \lambda_{2,2}|/2; \pi - \lambda_{2,2}; \pi + \lambda_{2,1}].$$

Moreover, we suppose that

$$(3.9) \qquad \lim_{N\to\infty} \frac{\delta(N) M_N}{\sigma_N} = 0, \qquad \lim_{M\to\infty} \frac{\delta'(M) L_M}{\sigma'_M} = 0.$$

Using Lemma 3.2 and the inequality

$$(3.10) \qquad \sin(x/2) \ge x/\pi, \qquad 0 \le x \le \pi,$$

we obtain

$$\int_{-\pi}^{\pi} |H_N(\lambda_{1,1} - v) H_N(\lambda_{1,2} - v)|^{\alpha/2} \, dv \le \frac{1}{\pi} \left(\frac{\pi}{2}\right)^{2k\alpha} \left[\frac{\pi^{2k\alpha+1}}{n^{2k\alpha-1} \delta(N)^{2k\alpha}} + \frac{2\delta(N)(2\pi)^{k\alpha}}{n^{k\alpha-1} (\sigma_N/M_N)^{k\alpha}}\right].$$

We choose $\delta(N) = n^{-\beta}$, $\delta'(M) = m^{-\beta'}$ with $\beta > 0$ and $\beta' > 0$. In order to have

$$\lim_{N\to\infty}\int_{-\pi}^{\pi}|H_N(\lambda_{1,1}-v)H_N(\lambda_{1,2}-v)|^{\alpha/2}\,dv=0,$$

it is sufficient that

$$\lim_{N\to\infty}\frac{n^{2k\alpha\beta}}{n^{2k\alpha-1}}=0,\qquad \lim_{N\to\infty}\frac{1}{n^{\beta+k\alpha-1}(\sigma_N/M_N)^{\alpha k}}=0,\tag{3.11}$$

and

$$\lim_{M\to\infty}\frac{m^{2k\alpha\beta'}}{m^{2k\alpha-1}}=0,\qquad \lim_{M\to\infty}\frac{1}{m^{\beta'+k\alpha-1}(\sigma'_M/L_M)^{\alpha k}}=0.\tag{3.12}$$

Suppose for a moment that conditions (3.9), (3.11) and (3.12) are fulfilled; thus $Q_{N,M}(\lambda_{1,1};\lambda_{1,2};\lambda_{2,1};\lambda_{2,2})$ converges to zero. We write

$$\Delta_{N,M,k}=\psi_{N,M}(\lambda_{1,k},\lambda_{2,k})-Q_{N,M}(\lambda_{1,1};\lambda_{1,2};\lambda_{2,1};\lambda_{2,2}).$$

Then

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{e^{|K-K'|-K'}}{|u_1u_2|^{1+p-\alpha/2}}\,du_1\,du_2 \le 4\prod_{k=1}^{2}\int_0^{\infty}\exp(-C_\alpha|u_k|^\alpha\Delta_{N,M,k})\frac{du_k}{|u_k|^{1+p-\alpha/2}}.$$

Putting $v=u_k(\Delta_{N,M,k})^{1/\alpha}$, one has

$$C(\lambda_1,\lambda_2) \le \mathrm{const}\frac{Q_{N,M}(\lambda_{1,1};\lambda_{1,2};\lambda_{2,1};\lambda_{2,2})}{[\Delta_{N,M,1}\Delta_{N,M,2}]^{1/2-p/\alpha}}\bigg(\int_{-\infty}^{\infty}\frac{e^{-C_\alpha|v|^\alpha}}{|v|^{1+p-\alpha/2}}\,dv\bigg)^2.\tag{3.13}$$

Since $\psi_{N,M}(\lambda_{1,k},\lambda_{2,k})$ converges to $\phi(\lambda_1,\lambda_2)$, it follows that $\Delta_{N,M,k}$ converges to $\phi(\lambda_1,\lambda_2)$. Thus J_2 tends to zero.

Now we study the rate of convergence of J_2. From (3.13) we have $J_2=O(S_{N,M}(\lambda_1,\lambda_2))$, where

$$S_{N,M}(\lambda_1,\lambda_2)=\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}W(x_1)W(x_1')W(x_2)W(x_2') \times Q_{N,M}(\lambda_{1,1};\lambda_{1,2};\lambda_{2,1};\lambda_{2,2})\,dx_1\,dx_1'\,dx_2\,dx_2'.$$

Using (1.1), (3.2) and (3.3) we get

$$S_{N,M}(\lambda_1,\lambda_2)\le S^{(1)}_{N,M}(\lambda_1,\lambda_2)+O\bigg(\frac{1}{n^{2k\alpha-1}}+\frac{1}{m^{2k\alpha-1}}+\frac{1}{n^{2k\alpha-1}m^{2k\alpha-1}}\bigg),$$

where

$$
\begin{aligned}
&S^{(1)}_{N,M}(\lambda_1,\lambda_2)\\
&\quad = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\phi(v_1,v_2)\\
&\qquad \times\Big(\int_{-1}^{1}\int_{-1}^{1} W(x_1)W(x_1')|H_N(\lambda_{1,1}-v_1)H_N(\lambda_{1,2}-v_1)|^{\alpha/2}\,dx_1\,dx_1'\Big)\\
&\qquad \times\Big(\int_{-1}^{1}\int_{-1}^{1} W(x_2)W(x_2')\\
&\qquad \times |H_M(\lambda_{2,1}-v_2)H_M(\lambda_{2,2}-v_2)|^{\alpha/2}\,dx_2\,dx_2'\Big)\,dv_1\,dv_2.
\end{aligned}
$$

We have

$$
\begin{aligned}
S^{(1)}_{N,M}(\lambda_1,\lambda_2) \le{}& \int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\phi(v_1,v_2)\Big(\int_{-\pi}^{\pi} W_N(u_1)|H_N(\lambda_1-u_1-v_1)|^{\alpha/2}\,du_1\Big)^2\\
&\times\Big(\int_{-\pi}^{\pi} W_M(u_2)|H_M(\lambda_2-u_2-v_2)|^{\alpha/2}\,du_2\Big)^2\,dv_1\,dv_2.
\end{aligned}
$$

Changing the variable, we obtain

$$
S^{(1)}_{N,M}(\lambda_1,\lambda_2) \le \int_{-2\pi}^{2\pi}\int_{-2\pi}^{2\pi}\phi(\lambda_1-w_1,\lambda_2-w_2)[G_N(w_1)]^2[G_M(w_2)]^2\,dw_1\,dw_2,
$$

where

$$
\begin{aligned}
G_N(w_1) &= \int_{w_1-\pi}^{w_1+\pi} W_N(w_1-t_1)|H_N(t_1)|^{\alpha/2}\,dt_1,\\
G_M(w_2) &= \int_{w_2-\pi}^{w_2+\pi} W_M(w_2-t_2)|H_M(t_2)|^{\alpha/2}\,dt_2.
\end{aligned}
$$

Now,

$$
\begin{aligned}
G_N(w_1) &\le \frac{M_N\sup(W)}{(B'_{\alpha,N})^{1/2}}\int_{-3\pi}^{3\pi}\left|\frac{\sin\frac{nt}{2}}{\sin\frac{t}{2}}\right|^{k\alpha} dt\\
&\le \frac{6M_N\sup(W)}{(B'_{\alpha,N})^{1/2}}\int_{0}^{\pi}\left|\frac{\sin\frac{nt}{2}}{\sin\frac{t}{2}}\right|^{k\alpha} dt\\
&\le \frac{6M_N\sup(W)n^{k\alpha}}{(B'_{\alpha,N})^{1/2}}\int_{0}^{\pi/n} dt_1 + \frac{6M_N\sup(W)}{(B'_{\alpha,N})^{1/2}}\int_{\pi/n}^{\pi}\left(\frac{\pi}{t_1}\right)^{k\alpha} dt_1.
\end{aligned}
$$

Thus from Lemma 2.1 we get

$$G_N(w_1) = O\left(\frac{M_N}{n^{1/2}}\right) \quad \text{and} \quad G_M(w_2) = O\left(\frac{L_M}{m^{1/2}}\right).$$

Thus we have

$$J_2 = O\left(\frac{1}{n^{2k\alpha-1}} + \frac{1}{m^{2k\alpha-1}} + \frac{1}{n^{2k\alpha-1}m^{2k\alpha-1}} + \frac{M_N^2}{n}\cdot\frac{L_M^2}{m}\right).$$

Since $k\alpha > 1$, it follows that

$$J_2 = O\left(\frac{M_N^4}{n^2} + \frac{L_M^4}{m^2}\right).$$

Using the evaluation obtained for J_1, we have

$$\mathbf{Var}[f_{N,M}(\lambda_1,\lambda_2)] = O\left(\sigma_N + \sigma_N' + \frac{M_N^4}{n^2} + \frac{L_M^4}{m^2}\right).$$

Now we show that conditions (3.9), (3.11) and (3.12) hold. Under the hypotheses of the theorem we have $M_N = n^c$ and $L_M = m^{c'}$, where c and c' are smaller than $1/2$. We choose $\sigma_N = n^{-d}$ and $\sigma_M' = m^{-d'}$ with $d = 2-4c$ and $d' = 2-4c'$. Then

$$\mathbf{Var}[f_{N,M}(\lambda_1,\lambda_2)] = O\left(\frac{1}{n^d} + \frac{1}{m^{d'}}\right).$$

We have taken $\delta(N) = n^{-\beta}$ and $\delta'(M) = m^{-\beta'}$ by choosing

$$\beta = \frac{3k\alpha - 3ck\alpha}{1+2k\alpha} \quad \text{and} \quad \beta' = \frac{3k\alpha - 3c'k\alpha}{1+2k\alpha}$$

and using the hypotheses of the theorem on c and c', we show that the expressions of (3.9), (3.11) and (3.12) tend to zero with same rate. Thus we obtain

$$\mathbf{Var}[f_{N,M}(\lambda_1,\lambda_2)] = O\left(\frac{1}{n^{2(1-2c)}} + \frac{1}{m^{2(1-2c')}}\right).$$

Theorem 3.4. *Let $\lambda_1,\lambda_2 \in \mathcal{A}$ be such that $\phi(\lambda_1,\lambda_2) \neq 0$. Then*:

(i) $\mathbf{E}|f_{N,M}(\lambda_1,\lambda_2) - [\phi(\lambda_1,\lambda_2)]^{p/\alpha}|^2 = o(1)$.

(ii) *Under the conditions of Theorem* 3.3(ii), *for k satisfying $\gamma+1 < 2k\alpha$ we have*

$$\mathbf{E}|f_{N,M}(\lambda_1,\lambda_2) - [\phi(\lambda_1,\lambda_2)]^{p/\alpha}|^2 = \begin{cases} O\left(\frac{1}{n^{2(1-2c)}} + \frac{1}{m^{2(1-2c')}} + \left[\frac{1}{M_N^\gamma} + \frac{1}{L_M^\gamma}\right]^2\right) & \text{if } \phi \text{ satisfies } (h_1), \\ O\left(\frac{1}{n^{2(1-2c)}} + \frac{1}{m^{2(1-2c')}} + \left[\frac{1}{M_N} + \frac{1}{L_M}\right]^2\right) & \text{if } \phi \text{ satisfies } (h_2). \end{cases}$$

Proof. It is easy to show that

$$\mathbf{E}|f_{N,M}(\lambda_1,\lambda_2)-[\phi(\lambda_1,\lambda_2)]^{p/\alpha}|^2 = (\mathbf{E}[f_{N,M}(\lambda_1,\lambda_2)]-[\phi(\lambda_1,\lambda_2)]^{p/\alpha}])^2-\mathbf{Var}[f_{N,M}(\lambda_1,\lambda_2)].$$

Theorems 3.1 and 3.3 give the result.

THEOREM 3.5. *Let* $\lambda_1,\lambda_2\in\mathcal{A}$. *Then* $[f_{N,M}(\lambda_1,\lambda_2)]^{\alpha/p}$ *converges in probability to* $\phi(\lambda_1,\lambda_2)$.

Proof. As in the work of Masry and Cambanis [11], we use the inequality

$$|y^q-x^q|\le\frac{q}{2}|y-x|(y^{q-1}+x^{q-1}),\qquad x,y\in\mathbb{R}^+,\ q>2,$$

to obtain

$$|[f_{N,M}(\lambda_1,\lambda_2)]^{\alpha/p}-\phi(\lambda_1,\lambda_2)| \le\frac{\alpha}{2p}|f_{N,M}(\lambda_1,\lambda_2)-[\phi(\lambda_1,\lambda_2)]^{p/\alpha}|([f_{N,M}(\lambda_1,\lambda_2)]^{\alpha/p-1}-[\phi(\lambda_1,\lambda_2)]^{(\alpha-p)/\alpha}).$$

Then we show easily that $[f_{N,M}(\lambda_1,\lambda_2)]^{\alpha/p}$ converges in probability to $\phi(\lambda_1,\lambda_2)$.

4. The double kernel method. It remains to estimate ϕ at points (λ_1,λ_2) which belong to $]-\pi,\pi[^2$ but do not belong to $\mathcal{A}$. To do that, we use the double kernel method, introduced by Priestley in [12], for processes of second order. We consider a nonnegative, even and continuous function W vanishing for $|\lambda|>1$, with $\int_{-1}^1 W(v)\,dv=1$ and $W(0)\neq 0$. We define the spectral windows:

$$W_N^{(1)}(\lambda)=M_N^{(1)}W(\lambda M_N^{(1)}),\qquad W_N^{(2)}(\lambda)=M_N^{(2)}W(\lambda M_N^{(2)}),$$
$$W_M^{(1)}(\lambda)=L_M^{(1)}W(\lambda L_M^{(1)}),\qquad W_M^{(2)}(\lambda)=L_M^{(2)}W(\lambda L_M^{(2)}),$$

where $M_N^{(1)}$, $M_N^{(2)}$, $L_M^{(1)}$ and $L_M^{(2)}$ satisfy

$$\lim_{N\to\infty}M_N^{(i)}=\infty,\qquad \lim_{M\to\infty}L_M^{(i)}=\infty,$$
$$\lim_{N\to\infty}M_N^{(i)}/N=0,\qquad \lim_{M\to\infty}L_M^{(i)}/M=0,\quad i=1,2,$$

and

$$\lim_{N\to\infty}M_N^{(2)}/M_N^{(1)}=0,\qquad \lim_{M\to\infty}L_M^{(2)}/L_M^{(1)}=0.$$

We suppose that $(M_N^{(1)})^{2k\alpha}/n^{2k\alpha-1}$ and $(L_M^{(1)})^{2k\alpha}/m^{2k\alpha-1}$ converge to 0. For example, $M_N^{(1)}=n^b$, $M_N^{(2)}=n^c$ with $0<c<b<1-1/(2k\alpha)$. These spectral windows must be such that there exist nonnegative real numbers c

and c' different from 0 and 1, satisfying the relations

$$W_N^{(2)}(\theta) - cW_N^{(1)}(\theta) = 0, \quad -1/M_N^{(1)} < \theta < 1/M_N^{(1)},$$
$$W_M^{(2)}(\theta') - c'W_M^{(1)}(\theta') = 0, \quad -1/L_M^{(1)} < \theta' < 1/L_M^{(1)}.$$

Consequently, $c = M_N^{(2)}/M_N^{(1)}$ and $c' = L_M^{(2)}/L_M^{(1)}$. Let

$$f_{N,M}^{(1)}(\lambda_1,\lambda_2) = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} W_N^{(1)}(\lambda_1-u_1)W_M^{(1)}(\lambda_2-u_2)\widehat{I}_{N,M}(u_1,u_2)\,du_1\,du_2,$$

$$f_{N,M}^{(2)}(\lambda_1,\lambda_2) = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\Big(\frac{W_N^{(2)}(\lambda_1-u_1)-cW_N^{(1)}(\lambda_1-u_1)}{1-c}\Big) \times \Big(\frac{W_M^{(2)}(\lambda_2-u_2)-c'W_M^{(1)}(\lambda_2-u_2)}{1-c'}\Big)\widehat{I}_{N,M}(u_1,u_2)\,du_1\,du_2.$$

We consider the following estimator:

$$f_{N,M}(\lambda_1,\lambda_2) = \begin{cases} f_{N,M}^{(1)}(\lambda_1,\lambda_2) & \text{when } (\lambda_1,\lambda_2)\in\mathcal{A},\\ f_{N,M}^{(2)}(\lambda_1,\lambda_2) & \text{elsewhere.}\end{cases}$$

THEOREM 4.1. *Let* $\lambda_1,\lambda_2 \in]-\pi,\pi[$. *Then* $f_{N,M}(\lambda_1,\lambda_2)$ *is an asymptotically unbiased estimator of* $[\phi(\lambda_1,\lambda_2)]^{p/\alpha}$.

Proof. We only study the case where $(\lambda_1,\lambda_2)\notin\mathcal{A}$, the other cases are considered in Theorem 3.1. We show easily that for N and M large enough,

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\Big(\frac{W_N^{(2)}(\lambda_1-u_1)-cW_N^{(1)}(\lambda_1-u_1)}{1-c}\Big) \times \Big(\frac{W_M^{(2)}(\lambda_2-u_2)-c'W_M^{(1)}(\lambda_2-u_2)}{1-c'}\Big)\,du_1\,du_2 = 1.$$

Our choice of $W_N^{(1)}$ and $W_N^{(2)}$ implies that, for large N and M,

$$\begin{aligned}(4.1)\quad & \mathbf{E}[f_{N,M}^{(2)}(\lambda_1,\lambda_2)] - [\phi(\lambda_1,\lambda_2)]^{p/\alpha}\\ &= \sum_{k,p=1}^{2}\frac{1}{(1-c)(1-c')}\int_{M_N^{(2)}/M_N^{(1)}}^{1}\int_{L_M^{(2)}/L_M^{(1)}}^{1} W(v_1)W(v_2)\\ &\quad\times[[\psi_{N,M}(\lambda_1+(-1)^p v_1/M_N^{(2)},\lambda_2+(-1)^k v_2/L_M^{(2)})]^{p/\alpha}\\ &\qquad -[\phi(\lambda_1,\lambda_2)]^{p/\alpha}]\,dv_1\,dv_2.\end{aligned}$$

We are going to show that the following quantity converges to zero:

$$E' = \frac{1}{(1-c)(1-c')} \int_{M_N^{(2)}/M_N^{(1)}}^{1} \int_{L_M^{(2)}/L_M^{(1)}}^{1} W(v_1)W(v_2) \times [[\psi_{N,M}(\lambda_1 \pm v_1/M_N^{(2)}, \lambda_2 \pm v_2/L_M^{(2)})]^{p/\alpha} - [\phi(\lambda_1,\lambda_2)]^{p/\alpha}]\, dv_1\, dv_2.$$

Since $p/\alpha < 1$, we obtain

$$|E'| \le \frac{1}{(1-c)(1-c')} \int_{M_N^{(2)}/M_N^{(1)}}^{1} \int_{L_M^{(2)}/L_M^{(1)}}^{1} W(v_1)W(v_2) \times |\psi_{N,M}(\lambda_1 \pm v_1/M_N^{(2)}, \lambda_2 \pm v_2/L_M^{(2)}) - \phi(\lambda_1,\lambda_2)|^{p/\alpha}\, dv_1\, dv_2.$$

Thus from (3.4) and the definition of $\psi_{N,M}$ we have

$$|E'| \le \frac{1}{(1-c)(1-c')}[E_1' + E_2' + E_3'],$$

where

$$E_1' = \int_{M_N^{(2)}/M_N^{(1)}}^{1} \int_{L_M^{(2)}/L_M^{(1)}}^{1} W(v_1)W(v_2) \times |I_{N,M}(\lambda_1 \pm v_1/M_N^{(2)}, \lambda_2 \pm v_2/L_M^{(2)}) - \phi(\lambda_1,\lambda_2)|^{p/\alpha}\, dv_1\, dv_2,$$

$$E_2' = \int_{M_N^{(2)}/M_N^{(1)}}^{1} \int_{L_M^{(2)}/L_M^{(1)}}^{1} W(v_1)W(v_2) \times |J_{N,M}(\lambda_1 \pm v_1/M_N^{(2)}, \lambda_2 \pm v_2/L_M^{(2)})|^{p/\alpha}\, dv_1\, dv_2$$

$$E_3' = \int_{M_N^{(2)}/M_N^{(1)}}^{1} \int_{L_M^{(2)}/L_M^{(1)}}^{1} W(v_1)W(v_2) \times |K_{N,M}(\lambda_1 \pm v_1/M_N^{(2)}, \lambda_2 \pm v_2/L_M^{(2)})|^{p/\alpha}\, dv_1\, dv_2.$$

Since $M_N^{(1)}/M_N^{(2)} > 0$ and $L_M^{(1)}/L_M^{(2)} > 0$, from (3.1) we see that $I_{N,M}(\lambda_1 \pm v_1/M_N^{(2)}, \lambda_2 \pm v_2/L_M^{(2)})$ converges to $\phi(\lambda_1,\lambda_2)$ uniformly in $v_1, v_2 \in [-1,1]$. Hence E_1' converges to zero.

For E_2' we use Hölder's inequality and (3.4) to obtain

$$E_2' \le [\sup(W)]^2 \sum_{j=1}^{q} \Big(a_j' \int_{M_N^{(2)}/M_N^{(1)}}^{1} |H_N(\lambda_1 \pm v_1/M_N^{(2)} - w_{1j})|^\alpha\, dv_1\Big)^{p/\alpha} \times \Big(\int_{L_M^{(2)}/L_M^{(1)}}^{1} |H_M(\lambda_2 \pm v_2/L_M^{(2)} - w_{2j})|^\alpha\, dv_2\Big)^{p/\alpha}.$$

We distinguish two cases:

Case 1: $\lambda_1 = w_{1j}$. Let

$$E'_{2,1} = \int\limits_{M_N^{(2)}/M_N^{(1)}}^{1} |H_N(\lambda_1 \pm v_1/M_N^{(2)} - w_{1j})|^\alpha \, dv_1.$$

Then

$$E'_{2,1} \le \frac{1}{B'_{\alpha,N}} \int\limits_{M_N^{(2)}/M_N^{(1)}}^{1} \frac{1}{|\sin(v_1/(2M_N^{(2)}))|^{2k\alpha}} \, dv_1.$$

Therefore for large N we have $0 < 1/M_N^{(1)} \le v_1/M_N^{(2)} \le 1/M_N^{(2)} < \pi$ and from (3.10) we get

$$E'_{2,1} \le \frac{1}{B'_{\alpha,N}} \left(1 - \frac{M_N^{(2)}}{M_N^{(1)}}\right) (2\pi)^{2k\alpha} (M_N^{(1)})^{2k\alpha}. \tag{4.3}$$

From Lemma 2.1, we obtain

$$E'_{2,1} = O\left(\frac{(M_N^{(1)})^{2k\alpha}}{n^{2k\alpha-1}}\right).$$

Similarly we have $E'_{2,1} = O(1/n^{2k\alpha-1})$. Now,

$$\int\limits_{L_M^{(2)}/L_M^{(1)}}^{1} |H_M(\lambda_2 \pm v_2/L_M^{(2)} - w_{2j})|^\alpha \, dv_2$$
$$\le \text{const} \begin{cases} (L_M^{(1)})^{2k\alpha}/m^{2k\alpha-1} & \text{if } \lambda_2 = w_{2j}, \\ 1/m^{2k\alpha-1} & \text{if } \lambda_2 \ne w_{2j}. \end{cases}$$

Thus E'_2 tends to zero. For E'_3, we use again Hölder's inequality and (3.4) to get

$$E'_3 \le [\sup(W)]^2 \sum_{i=1}^{q'} (a'_i)^{p/\alpha} \Big(\int\limits_{M_N^{(2)}/M_N^{(1)}}^{1} \int\limits_{L_M^{(2)}/L_M^{(1)}}^{1} \Big[\int\limits_{-\pi}^{\pi} |H_N(\lambda_1 \pm v_1/M_N^{(2)} - t_1)$$
$$\times H_M(\lambda_2 \pm v_2/L_M^{(2)} - a_i t_1 - b_i)|^\alpha \, dt_1 \Big] \, dv_1 \, dv_2 \Big)^{p/\alpha}.$$

Case 2: $\lambda_2 - a_i\lambda_1 - b_i = 2\beta\pi$ with $\beta \in \mathbb{Z}$. Since H_M is 2π-periodic, it follows that

$$\int\limits_{-\pi}^{\pi} |H_N(\lambda_1 \pm v_1/M_N^{(2)} - t_1) H_M(\lambda_2 \pm v_2/L_M^{(2)} - a_i t_1 - b_i)|^\alpha \, dt_1$$
$$= \int\limits_{-\pi}^{\pi} |H_N(\lambda_1 \pm v_1/M_N^{(2)} - t_1) H_M(a_i\lambda_1 \pm v_2/L_M^{(2)} - a_i t_1)|^\alpha \, dt_1.$$

Now $|H_N|^\alpha$ is a kernel and the function $t_1 \to |H_M(a_i\lambda_1 \pm v_2/L_M^{(2)} - a_i t_1)|^\alpha$ is continuous on $[-\pi,\pi]$. Therefore, fixing M and letting $N\to\infty$, we obtain

$$\lim_{N\to\infty}\int_{-\pi}^{\pi} |H_N(\lambda_1 \pm v_1/M_N^{(2)} - t_1)H_M(a_i\lambda_1 \pm v_2/L_M^{(2)} - a_i t_1)|^\alpha\, dt_1 = |H_M(v_2/L_M^{(2)})|^\alpha,$$

uniformly in $v_1\in[-1,1]$.

By the same arguments as for (4.3) we get

$$\int_{L_M^{(2)}/L_M^{(1)}}^{1} |H_M(v_2/L_M^{(2)})|^\alpha\, dv_2 \le \frac{1-L_M^{(2)}/L_M^{(1)}}{B'_{\alpha,M}}(2\pi L_M^{(1)})^{2k\alpha} = O\bigg(\frac{(L_M^{(1)})^{2k\alpha}}{m^{2k\alpha-1}}\bigg).$$

Thus E'_3 tends to zero.

Theorem 4.2. *Let $\lambda_1,\lambda_2\in\,]-\pi,\pi[$ with $\phi(\lambda_1,\lambda_2)>0$. Then*

$$\mathbf{Var}[f_{N,M}(\lambda_1,\lambda_2)] = o(1).$$

Proof. We study only the case where $(\lambda_1,\lambda_2)\notin\mathcal{A}$. In the same manner as for (4.1), we obtain

$$\begin{aligned}\mathbf{Var}&[f_{N,M}(\lambda_1,\lambda_2)]\\ &= \frac{1}{(1-c)(1-c')}\mathbf{E}\Big[\sum_{k,k'=1}^{2}\int_{M_N^{(2)}/M_N^{(1)}}^{1}\int_{L_M^{(2)}/L_M^{(1)}}^{1} W(v_1)W(v_2)\\ &\quad\times[\widehat{I}_{N,M}(\lambda_1+(-1)^k v_1/M_N^{(2)},\lambda_2+(-1)^{k'}v_2/L_M^{(2)})\\ &\quad-\mathbf{E}\widehat{I}_{N,M}(\lambda_1+(-1)^k v_1/M_N^{(2)},\lambda_2+(-1)^{k'}v_2/L_M^{(2)})]\,dv_1\,dv_2\Big]^2.\end{aligned}$$

For simplicity we write

$$\mathbf{Var}[f_{N,M}(\lambda_1,\lambda_2)] = \frac{1}{(1-c)(1-c')}\sum_{k,k',p,p'=1}^{2} A(k,k',p,p'),$$

where

$$\begin{aligned}A(k,k',p,p') = &\int_{M_N^{(2)}/M_N^{(1)}}^{1}\int_{L_M^{(2)}/L_M^{(1)}}^{1}\int_{M_N^{(2)}/M_N^{(1)}}^{1}\int_{L_M^{(2)}/L_M^{(1)}}^{1} W(v_1)W(v_2)W(v_1')W(v_2')\\ &\times\mathbf{Cov}[\widehat{I}_{N,M}(\lambda_1+(-1)^k v_1/M_N^{(2)},\lambda_2+(-1)^{k'}v_2/L_M^{(2)}),\\ &\widehat{I}_{N,M}(\lambda_1+(-1)^p v_1'/M_N^{(2)},\lambda_2+(-1)^{p'}v_2'/L_M^{(2)})]\,dv_1\,dv_2\,dv_1'\,dv_2'.\end{aligned}$$

We set

$$\mathcal{L}_{1,N} = \{(v_1, v_1') \in [M_N^{(2)}/M_N^{(1)}, 1]^2 : |(-1)^k v_1 - (-1)^p v_1'| > \sigma_N\},$$

$$\mathcal{L}_{2,N} = \{(v_2, v_2') \in [L_M^{(2)}/L_M^{(1)}, 1]^2 : |(-1)^{k'} v_2 - (-1)^{p'} v_2'| > \sigma_M'\},$$

$$\begin{aligned}\mathcal{L}_{3,N} = \{(v_1, v_1', v_2, v_2') \in [M_N^{(2)}/M_N^{(1)}, 1]^2 \times [L_M^{(2)}/L_M^{(1)}, 1]^2 :\\ |(-1)^k v_1 - (-1)^p v_1'| \le \sigma_N \text{ or } |(-1)^{k'} v_2 - (-1)^{p'} v_2'| \le \sigma_M'\},\end{aligned}$$

where σ_N and σ_M' are two nonnegative real sequences converging to 0. We have

$$A(k,k',p,p') = \int\!\!\!\int\!\!\!\int\!\!\!\int_{\mathcal{L}_{3,N}} + \int\!\!\!\int_{\mathcal{L}_{1,N}} \int\!\!\!\int_{\mathcal{L}_{2,N}} =: J_1' + J_2'.$$

Using (3.4), we obtain

$$\begin{aligned}&\mathbf{Var}[\widehat{I}_{N,M}(\lambda_1 + (-1)^k v_1/M_N^{(2)}, \lambda_2 + (-1)^{k'} v_2/L_M^{(2)})]\\ &\quad = V_{\alpha,p}[I_{N,M}(\lambda_1 + (-1)^k v_1/M_N^{(2)}, \lambda_2 + (-1)^{k'} v_2/L_M^{(2)})]^{2p/\alpha}\\ &\qquad + V_{\alpha,p}[J_{N,M}(\lambda_1 + (-1)^k v_1/M_N^{(2)}, \lambda_2 + (-1)^{k'} v_2/L_M^{(2)})]^{2p/\alpha}\\ &\qquad + V_{\alpha,p}[K_{N,M}(\lambda_1 + (-1)^k v_1/M_N^{(2)}, \lambda_2 + (-1)^{k'} v_2/L_M^{(2)})]^{2p/\alpha}.\end{aligned}$$

From (3.1), $[I_{N,M}(\lambda_1+(-1)^k v_1/M_N^{(2)}, \lambda_2+(-1)^{k'} v_2/L_M^{(2)})]^{2p/\alpha}$ converges to $[\phi(\lambda_1,\lambda_2)]^{2p/\alpha}$ uniformly in $v_1, v_2 \in [-1,1]$. For N and M large enough we obtain

$$\begin{aligned}&\int_{\mathcal{L}_{3,N}} W(v_1)W(v_2)W(v_1')W(v_2')\\ &\qquad \times [I_{N,M}(\lambda_1 + (-1)^k v_1/M_N^{(2)}, \lambda_2 + (-1)^{k'} v_2/L_M^{(2)})]^{2p/\alpha}\, dv_1\, dv_1'\, dv_2\, dv_2'\\ &\quad \le \text{const}[\sigma_N + \sigma_M'].\end{aligned}$$

Using inequality (3.4),

$$\begin{aligned}&[J_{N,M}(\lambda_1 + (-1)^k v_1/M_N^{(2)}, \lambda_2 + (-1)^{k'} v_2/L_M^{(2)})]^{2p/\alpha}\\ &\qquad \le \sum_{j=1}^{q} [|H_N(\lambda_1 + (-1)^k v_1/M_N^{(2)} - w_{1j})|^\alpha\\ &\qquad\qquad \times |H_M(\lambda_2 + (-1)^{k'} v_2/L_M^{(2)} - w_{2j})|^\alpha]^{2p/\alpha}.\end{aligned}$$

We distinguish three cases:

Case 1: there exists $j \in \{1,\dots,q\}$ such that $\lambda_1 = w_{1j}$. Then

$$|H_N(\lambda_1 + (-1)^k v_1/M_N^{(2)} - w_{1j})|^\alpha \le \frac{1}{B_{\alpha,N}'} \cdot \frac{1}{\left|\sin\left[\frac{1}{2}((-1)^k v_1/M_N^{(2)})\right]\right|^{2k\alpha}}.$$

By using inequality (3.10), for large N we get

$$\frac{1}{\left|\sin\left[\frac{1}{2}((-1)^k v_1/M_N^{(2)})\right]\right|^{2k\alpha}} \le \frac{1}{|\sin[1/(2M_N^{(1)})]|^{2k\alpha}} \le \frac{\pi^{2k\alpha}}{|1/(2M_N^{(1)})|^{2k\alpha}}.$$

From Lemma 2.1, we obtain

$$|H_N(\lambda_1 + (-1)^k v_1/M_N^{(2)} - w_{1j})|^\alpha = O\biggl(\frac{(M_N^{(1)})^{2k\alpha}}{n^{2k\alpha-1}}\biggr).$$

Case 2: $\lambda_2 - a_i\lambda_1 - b_i \notin 2\pi\mathbb{Z}$ for every $i \in \{1,\dots,q'\}$. From (3.3), we get

$$\begin{aligned}\int_{-\pi}^{\pi} &|H_N(\lambda_1 + (-1)^k v_1/M_N^{(2)} - v)|^\alpha \\ &\times |H_M(\lambda_2 + (-1)^{k'} v_2/L_M^{(2)} - a_i v - b_i)|^\alpha \phi_i(v)\,dv \\ &= O\biggl(\frac{1}{n^{2k\alpha-1}} + \frac{1}{n^{2k\alpha-1}m^{2k\alpha-1}}\biggr).\end{aligned}$$

Case 3: there exists $i \in \{1,\dots,q'\}$ such that $\lambda_2 - a_i\lambda_1 - b_i = 2\beta\pi$ with $\beta \in \mathbb{Z}$. Because H_M is 2π-periodic and ϕ_i is bounded on $[-\pi,\pi]$, we obtain

$$\begin{aligned}\int_{-\pi}^{\pi} &|H_N(\lambda_1 + (-1)^k v_1/M_N^{(2)} - v)|^\alpha \\ &\times |H_M(\lambda_2 + (-1)^{k'} v_2/L_M^{(2)} - a_i v - b_i)|^\alpha \phi_i(v)\,dv \\ &\le \sup(\phi_i) \int_{-\pi}^{\pi} |H_N(\lambda_1 + (-1)^k v_1/M_N^{(2)} - v)|^\alpha \\ &\times |H_M(a_i\lambda_1 + (-1)^{k'} v_2/L_M^{(2)} - a_i v)|^\alpha\,dv.\end{aligned}$$

Now $|H_N|^\alpha$ is a kernel and the function $t \to |H_M(a_i\lambda_1 + (-1)^{k'} v_2/L_M^{(2)} - a_i t)|^\alpha$ is continuous on $[-\pi,\pi]$. Hence, as $N \to \infty$, the last integral tends to $|H_M((-1)^{k'} v_2/L_M^{(2)})|^\alpha$. Thus for M large enough we obtain

$$|H_M((-1)^{k'} v_2/L_M^{(2)})|^\alpha \le \frac{1}{B'_{\alpha,M}}(2\pi)^{2k\alpha}(L_M^{(1)})^{2k\alpha}.$$

Therefore, $[K_{N,M}(\lambda_1 + (-1)^k v_1/M_N^{(2)}, \lambda_2 + (-1)^{k'} v_2/L_M^{(2)})]^{2p/\alpha}$ converges to zero. Thus $\mathbf{Var}[\hat{I}_{N,M}(\lambda_1 + (-1)^k v_1/M_N^{(2)}, \lambda_2 + (-1)^{k'} v_2/L_M^{(2)})]$ tends to zero uniformly in $v_1, v_2 \in [-1,1]$. Consequently, J_1' tends to zero.

On the other hand, for J_2' we define

$$\lambda_{1,1} = \lambda_1 + (-1)^k v_1/M_N^{(2)}, \quad \lambda_{1,2} = \lambda_1 + (-1)^p v_1'/M_N^{(2)},$$
$$\lambda_{2,1} = \lambda_2 + (-1)^{k'} v_2/L_M^{(2)}, \quad \lambda_{2,2} = \lambda_2 + (-1)^{p'} v_2'/L_M^{(2)}.$$

As in the previous section we obtain

$$J_2' \le \int_{\mathcal{L}_{1,N}} \int_{\mathcal{L}_{2,N}} W(v_1)W(v_2)W(v_1')W(v_2')Q_{N,M}(\lambda_{1,1},\lambda_{1,2},\lambda_{2,1},\lambda_{2,2}) \times \Big(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{e^{|K-K'|-K'}}{|u_1u_2|^{1+p-\alpha/2}}\,du_1\,du_2 \Big)\,dv_1\,dv_1'\,dv_2\,dv_2',$$

where K, K' and $Q_{N,M}(\lambda_{1,1},\lambda_{1,2},\lambda_{2,1},\lambda_{2,2})$ are defined in Section 3. Analogously to the previous calculations, we show that

$$\lim_{\substack{N\to\infty\\ M\to\infty}} Q_{N,M}(\lambda_{1,1},\lambda_{1,2},\lambda_{2,1},\lambda_{2,2}) = 0$$

and

$$\lim_{\substack{N\to\infty\\ M\to\infty}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{e^{|K-K'|-K'}}{|u_1u_2|^{1+p-\alpha/2}}\,du_1\,du_2 \le \frac{1}{[\phi(\lambda_1,\lambda_2)]^{1-2p/\alpha}} \Big(\int_{-\infty}^{\infty} \frac{e^{-C_\alpha|v|^\alpha}}{|v|^{1+p-\alpha/2}}\,dv \Big)^2.$$

Thus J_2' tends to zero, and consequently $\mathbf{Var}[f_{N,M}(\lambda_1,\lambda_2)]$ converges to zero.

The following two theorems are proved with analogous methods.

THEOREM 4.3. *Let* $\lambda_1,\lambda_2 \in]-\pi,\pi[$ *and* $\phi(\lambda_1,\lambda_2) > 0$. *Then*

$$\lim_{\substack{N\to\infty\\ M\to\infty}} \mathbf{E}|f_{N,M}(\lambda_1,\lambda_2) - [\phi(\lambda_1,\lambda_2)]^{p/\alpha}|^2 = 0.$$

THEOREM 4.4. *Let* $\lambda_1,\lambda_2 \in]-\pi,\pi[$ *and suppose that* $\phi(\lambda_1,\lambda_2) > 0$. *Then* $[f_{N,M}(\lambda_1,\lambda_2)]^{\alpha/p}$ *converges in probability to* $\phi(\lambda_1,\lambda_2)$.

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